

Question 6

Determine the exact range of

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad (x, y) \neq (0, 0).$$

Tasks

1. Prove upper and lower bounds.
2. Show every intermediate value occurs.
3. Describe the level sets $f = c$.

Question 6 – Solution

Strategy Compare $|x^2 - y^2|$ with $x^2 + y^2$, then solve the level equation.

Step 1: Bounds Since $x^2, y^2 \geq 0$,

$$|x^2 - y^2| \leq x^2 + y^2.$$

The denominator is positive off the origin, so $-1 \leq f \leq 1$.

Step 2: Endpoints and intermediate values On the x -axis, $f = 1$; on the y -axis, $f = -1$. For $-1 < c < 1$,

$$(1 - c)x^2 = (1 + c)y^2.$$

Choosing $y = 1$ and $x = \sqrt{(1 + c)/(1 - c)}$ realizes c . Hence

$$\boxed{\text{range}(f) = [-1, 1]}.$$

Step 3: Levels For $-1 < c < 1$, the level set is the pair of lines

$$\boxed{y = \pm \sqrt{\frac{1 - c}{1 + c}} x},$$

with the origin removed. At $c = 1$ and $c = -1$, the levels are the punctured x - and y -axes.