

Question 5

Let $f(x, y) = \sqrt{1 - x^2 - y^2}$ and define $g(u, v) = f(u + v, u - v)$.

Tasks

1. Find a simplified formula and exact domain for g .
2. Describe its level curves.
3. Explain the geometric effect of the coordinate change.

Question 5 – Solution

Strategy Substitute first and expand carefully.

Step 1: Simplify Since

$$(u + v)^2 + (u - v)^2 = 2u^2 + 2v^2,$$

we get $\boxed{g(u, v) = \sqrt{1 - 2u^2 - 2v^2}}$.

Step 2: Domain The radicand must be nonnegative:

$$1 - 2u^2 - 2v^2 \geq 0 \implies \boxed{u^2 + v^2 \leq \frac{1}{2}}.$$

Step 3: Levels For $0 \leq c \leq 1$, $g = c$ implies

$$u^2 + v^2 = \frac{1 - c^2}{2},$$

circles centered at the origin. The map $(u, v) \mapsto (x, y) = (u + v, u - v)$ reflects v to $-v$, then rotates by 45° and scales lengths by $\sqrt{2}$. Its inverse image of the unit disk is the disk of radius $1/\sqrt{2}$. At $c = 1$ the level is just the origin; levels outside $[0, 1]$ are empty.