

**Question 10**

Consider

$$5x^2 - 6xy + 5y^2 - z^2 = 0, \quad u = \frac{x+y}{\sqrt{2}}, \quad v = \frac{x-y}{\sqrt{2}}.$$

**Tasks**

1. Rewrite the equation in  $u, v, z$ .
2. Classify the surface and identify its axis.
3. Derive the traces in  $z = c$ ,  $u = 0$ , and  $v = 0$ .
4. Relate the principal directions to the original  $xy$ -axes.

## Question 10 – Solution

**Strategy** Express  $x^2 + y^2$  and  $xy$  in rotated coordinates so the cross term disappears.

**Step 1: Invert and substitute** We have

$$x = \frac{u+v}{\sqrt{2}}, \quad y = \frac{u-v}{\sqrt{2}}, \quad x^2 + y^2 = u^2 + v^2, \quad xy = \frac{u^2 - v^2}{2}.$$

Therefore

$$5(u^2 + v^2) - 3(u^2 - v^2) - z^2 = 0, \quad \boxed{2u^2 + 8v^2 - z^2 = 0}.$$

**Step 2: Classify** This is an elliptic cone with vertex at the origin and axis the  $z$ -axis.

**Step 3: Traces** For  $z = c \neq 0$ ,

$$\boxed{\frac{u^2}{c^2/2} + \frac{v^2}{c^2/8} = 1}.$$

For  $v = 0$ ,  $z = \pm\sqrt{2}u$ ; for  $u = 0$ ,  $z = \pm 2\sqrt{2}v$ . At  $z = 0$ , only the vertex remains.

**Step 4: Interpret rotation** The  $u$ -axis is  $x = y$  and the  $v$ -axis is  $x = -y$ , so the principal horizontal directions are rotated  $45^\circ$  from the original axes.