

**Question 10**

A plane cuts the positive coordinate axes at  $A$ ,  $B$ , and  $C$ . The centroid of the intercept triangle  $ABC$  is

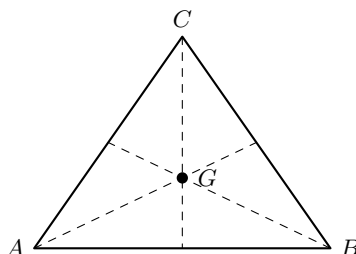
$$G = (2, 3, 4).$$

**Tasks**

1. Determine  $A$ ,  $B$ , and  $C$ .
2. Find the plane in intercept and Cartesian forms.
3. Find the area of triangle  $ABC$ .
4. Find the volume of the tetrahedron bounded by the plane and the coordinate planes.

## Question 10 – Solution

**Strategy** Write  $A = (a, 0, 0)$ ,  $B = (0, b, 0)$ , and  $C = (0, 0, c)$ . Their centroid directly determines the three intercepts.



**Step 1: Recover the intercepts** Write  $A = (a, 0, 0)$ ,  $B = (0, b, 0)$ , and  $C = (0, 0, c)$ . Then

$$G = \frac{A + B + C}{3} = \left( \frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right),$$

so  $a/3 = 2$ ,  $b/3 = 3$ , and  $c/3 = 4$ . Hence

$$A = (6, 0, 0), \quad B = (0, 9, 0), \quad C = (0, 0, 12).$$

**Step 2: Plane equation** A plane with intercepts  $a, b, c$  has equation  $x/a + y/b + z/c = 1$ . Therefore

$$\frac{x}{6} + \frac{y}{9} + \frac{z}{12} = 1, \quad 6x + 4y + 3z = 36.$$

**Step 3: Triangle area** Compute

$$\begin{aligned} \overrightarrow{AB} &= \langle -6, 9, 0 \rangle, & \overrightarrow{AC} &= \langle -6, 0, 12 \rangle, \\ \overrightarrow{AB} \times \overrightarrow{AC} &= \langle 9(12), 0(-6) - (-6)(12), (-6)(0) - 9(-6) \rangle = \langle 108, 72, 54 \rangle. \end{aligned}$$

Its length is

$$\sqrt{108^2 + 72^2 + 54^2} = \sqrt{19764} = 18\sqrt{61}.$$

Hence  $\text{Area}(ABC) = 9\sqrt{61}.$

**Step 4: Tetrahedron volume** The coordinate-axis edges from the origin are mutually perpendicular with lengths 6, 9, 12. A tri-rectangular tetrahedron has one-sixth the corresponding box volume, so

$$V = \frac{1}{6}(6)(9)(12) = 108.$$

**Verification** Substitution of each intercept into  $6x + 4y + 3z = 36$  succeeds, and their coordinate average is  $(2, 3, 4)$ .