

Question 8

All planes containing the intersection of

$$\Pi_1 : x + y + z = 1 \quad \text{and} \quad \Pi_2 : x - y + 2z = 3$$

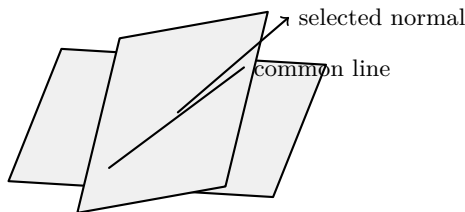
form a pencil of planes.

Tasks

1. Write a one-parameter equation for this pencil.
2. Find the member perpendicular to $\Sigma : x + 2y - z = 4$.
3. Verify that the selected plane contains the entire common line of Π_1 and Π_2 .

Question 8 – Solution

Strategy Every combination $F_1 + \lambda F_2 = 0$ vanishes wherever both original plane expressions vanish. Choose λ using normal-vector perpendicularity.



Step 1: Build the pencil Move all terms to the left:

$$F_1 = x + y + z - 1, \quad F_2 = x - y + 2z - 3.$$

At every point on the common line, $F_1 = F_2 = 0$. Therefore every equation

$$F_1 + \lambda F_2 = 0 \ (\lambda \in \mathbb{R}), \quad \text{together with } F_2 = 0$$

contains that line. The member $F_2 = 0$ is also needed for the full pencil; its normal has dot product -3 with $\langle 1, 2, -1 \rangle$, so it is not the requested perpendicular member. Collecting coefficients shows its normal is

$$\vec{n}_\lambda = \langle 1 + \lambda, 1 - \lambda, 1 + 2\lambda \rangle.$$

Step 2: Impose perpendicularity The normal of Σ is $\vec{m} = \langle 1, 2, -1 \rangle$. Perpendicular planes have perpendicular normals, so

$$\vec{n}_\lambda \cdot \langle 1, 2, -1 \rangle = 2 - 3\lambda = 0,$$

because $(1 + \lambda) + 2(1 - \lambda) - (1 + 2\lambda) = 2 - 3\lambda$. Thus $\lambda = 2/3$. Multiplying by 3 gives

$$3F_1 + 2F_2 = 0,$$

and expanding yields

$$5x + y + 7z = 9.$$

Verification Any point common to Π_1 and Π_2 has $F_1 = F_2 = 0$, so it satisfies $3F_1 + 2F_2 = 0$. Its normal $\langle 5, 1, 7 \rangle$ dots with $\langle 1, 2, -1 \rangle$ to 0.