

Question 7

Let

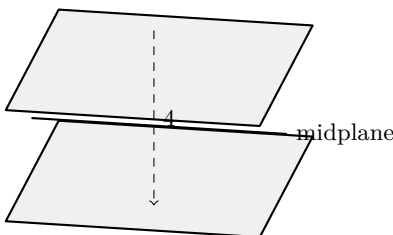
$$\Pi_1 : 2x - 2y + z = 7, \quad \Pi_2 : 2x - 2y + z = -5.$$

Tasks

1. Find the distance between the parallel planes.
2. Derive the plane exactly halfway between them.
3. Find endpoints of one shortest segment joining the planes and verify its midpoint lies on the halfway plane.

Question 7 – Solution

Strategy Put the equations over the same normal. Their constant difference divided by the normal length is the separation.



Step 1: Distance Both planes have normal $\vec{n} = \langle 2, -2, 1 \rangle$, whose length is

$$\|\vec{n}\| = \sqrt{2^2 + (-2)^2 + 1^2} = 3.$$

Because their left sides are identical, the distance is the absolute difference of constants divided by $\|\vec{n}\|$:

$$d(\Pi_1, \Pi_2) = \frac{|7 - (-5)|}{3} = 4.$$

Step 2: Halfway plane A parallel plane has form $2x - 2y + z = c$. Equal distance from c to 7 and -5 requires

$$7 - c = c - (-5) \implies 2c = 2 \implies c = 1.$$

Thus

$$2x - 2y + z = 1.$$

Step 3: Shortest connector Choose $A = (3, 0, 1)$; indeed $2(3) - 2(0) + 1 = 7$. The unit normal is $\vec{n}/3$, so moving four units toward Π_2 gives

$$B = A - \frac{4}{3} \langle 2, -2, 1 \rangle = \left(\frac{1}{3}, \frac{8}{3}, -\frac{1}{3} \right).$$

Substitution gives $2(1/3) - 2(8/3) - 1/3 = -5$, so $B \in \Pi_2$. Its midpoint with A is

$$M = \left(\frac{5}{3}, \frac{4}{3}, \frac{1}{3} \right).$$

Verification $2(5/3) - 2(4/3) + 1/3 = 1$. Also $B - A$ is parallel to the common normal, so AB is perpendicular to both planes and is therefore a shortest connector.