

Question 6

Among all points on the plane

$$\Pi : x + 2y + 2z = 9,$$

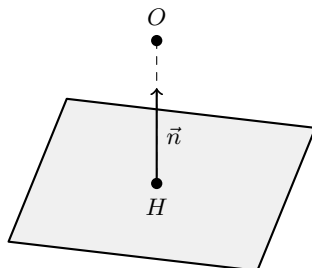
find the point closest to the origin.

Tasks

1. Determine the closest point and minimum distance.
2. Prove global minimality without using calculus.
3. Find the sphere centered at the origin tangent to Π .

Question 6 – Solution

Strategy The shortest segment from a point to a plane follows the plane's normal. Use Cauchy–Schwarz for a global proof.



Step 1: Follow the normal The normal is $\vec{n} = \langle 1, 2, 2 \rangle$ and

$$\|\vec{n}\| = \sqrt{1^2 + 2^2 + 2^2} = 3.$$

A point on the normal line through the origin has form $(t, 2t, 2t)$. Imposing the plane equation gives

$$t + 2(2t) + 2(2t) = 9 \implies 9t = 9 \implies t = 1.$$

Hence

$$\boxed{H = (1, 2, 2)}, \quad \boxed{d(O, \Pi) = \|H\| = 3}.$$

Step 2: Global proof For any $X = (x, y, z)$ on Π , $\vec{n} \cdot X = 9$. Cauchy–Schwarz gives

$$9 = |\vec{n} \cdot X| \leq \|\vec{n}\| \|X\| = 3\|X\|,$$

so $\|X\| \geq 3$. Equality occurs precisely when X is parallel to $\vec{n}$, as H is.

Step 3: Tangent sphere A sphere centered at the origin and reaching H has radius 3, so its equation is

$$\boxed{x^2 + y^2 + z^2 = 9}.$$

Its radius to H is normal to Π , so the plane is tangent there.