

Question 3

For a real parameter k , let

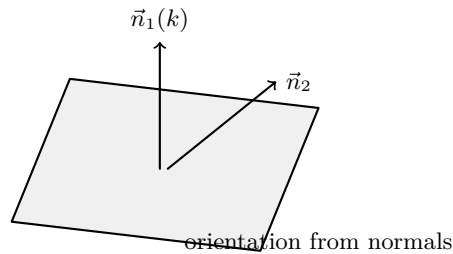
$$\Pi_k : kx + 2y - z = 5, \quad \Sigma : 2x - y + 2z = 7.$$

Tasks

1. Find k if the planes are perpendicular.
2. Determine whether any k makes the planes parallel.
3. Find every k for which the acute angle between the planes is 60° .

Question 3 – Solution

Strategy Plane orientation is controlled by the normals $\vec{n}_k = \langle k, 2, -1 \rangle$ and $\vec{m} = \langle 2, -1, 2 \rangle$.



Step 1: Perpendicularity Calculate

$$\vec{n}_k \cdot \vec{m} = k(2) + 2(-1) + (-1)(2) = 2k - 4.$$

Perpendicular normals require $2k - 4 = 0$, hence $\boxed{k = 2}$.

Step 2: Parallelism Parallelism would require $\langle k, 2, -1 \rangle = \lambda \langle 2, -1, 2 \rangle$. From $2 = -\lambda$, $\lambda = -2$; but the third coordinate would require $-1 = 2(-2) = -4$, a contradiction. Hence $\boxed{\text{no real } k \text{ makes them parallel}}$.

Step 3: The 60° condition The angle formula gives

$$\frac{|2k - 4|}{3\sqrt{k^2 + 5}} = \frac{1}{2}.$$

Cross-multiply and square:

$$2|2k - 4| = 3\sqrt{k^2 + 5} \implies 16(k - 2)^2 = 9(k^2 + 5).$$

Expanding gives $16k^2 - 64k + 64 = 9k^2 + 45$, so

$$7k^2 - 64k + 19 = 0.$$

The quadratic formula yields

$$k = \frac{64 \pm \sqrt{64^2 - 4(7)(19)}}{14} = \frac{64 \pm 18\sqrt{11}}{14},$$

and therefore

$$\boxed{k = \frac{32 \pm 9\sqrt{11}}{7}}.$$

Verification Neither root makes the original denominator zero, and substitution into the unsquared equation gives the positive value $1/2$, so no extraneous root remains.