

Question 7

Find the distance between $P = (3, \pi/6, \pi/3)$ and $Q = (5, 5\pi/6, 2\pi/3)$ in spherical coordinates without first converting both points to Cartesian form.

Tasks

1. Derive the angle between their radial directions.
2. Compute the distance exactly.
3. Check that the result obeys the triangle bounds.

Question 7 – Solution

Strategy. Take the dot product of the two unit radial directions and apply the three-dimensional law of cosines.

Step 1: Direction angle If γ is the angle between the radial directions, then

$$\cos \gamma = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2).$$

Here $\cos \phi_1 \cos \phi_2 = -1/4$, $\sin \phi_1 \sin \phi_2 = 3/4$, and $\cos(-2\pi/3) = -1/2$, so $\cos \gamma = -5/8$.

Step 2: Distance Therefore

$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos \gamma = 9 + 25 - 30(-5/8) = \frac{211}{4}.$$

Thus

$$d = \frac{\sqrt{211}}{2}.$$

Verification. The triangle inequality requires $|5-3| \leq d \leq 5+3$. Numerically $d \approx 7.26$, which lies between 2 and 8.