

Question 10

Let a regular motion have velocity $\mathbf{v}$, speed $v = \|\mathbf{v}\|$, unit tangent $\mathbf{T}$, curvature κ , and principal normal $\mathbf{N}$.

Tasks

1. Derive $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$.
2. Derive formulas for the tangential and normal scalar components using only $\mathbf{v}$ and $\mathbf{a}$.
3. Explain the physical meaning of each component.

Question 10 – Solution

Strategy. Begin with $\mathbf{v} = v\mathbf{T}$, differentiate, and convert $d\mathbf{T}/dt$ through arc length.

Step 1: Differentiate velocity

$$\mathbf{a} = \frac{d}{dt}(v\mathbf{T}) = v'\mathbf{T} + v\frac{d\mathbf{T}}{dt}.$$

By the chain rule,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} = (\kappa\mathbf{N})v.$$

Therefore

$$\boxed{\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}}.$$

Step 2: Tangential component Dot with $\mathbf{T} = \mathbf{v}/v$:

$$a_T = \mathbf{a} \cdot \mathbf{T} = \boxed{\frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}}.$$

It also equals v' , the rate of change of speed.

Step 3: Normal component Orthogonality gives

$$\|\mathbf{a}\|^2 = a_T^2 + a_N^2.$$

Thus

$$\boxed{a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}} = \boxed{\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}} = \kappa v^2.$$

The tangential component changes speed; the normal component changes direction. A motion can therefore accelerate even at constant speed if its path bends.