

Question 10

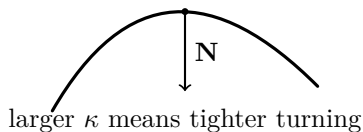
Let $\mathbf{r}(s)$ be a twice-differentiable curve parametrized by arc length, so $\|\mathbf{r}'(s)\| = 1$.

Tasks

1. Prove that $\mathbf{r}''(s)$ is perpendicular to $\mathbf{r}'(s)$.
2. Prove that $\kappa(s) = \|\mathbf{r}''(s)\|$.
3. Explain what $\mathbf{r}''(s) = \mathbf{0}$ means locally.

Question 10 – Solution

Strategy. Differentiate the unit-speed invariant, then compare the definition $\kappa = \|d\mathbf{T}/ds\|$ with $\mathbf{T} = \mathbf{r}'$.



Step 1: Orthogonality Unit speed means

$$\mathbf{r}'(s) \cdot \mathbf{r}'(s) = 1.$$

Differentiate both sides:

$$2\mathbf{r}'(s) \cdot \mathbf{r}''(s) = 0.$$

Therefore

$$\boxed{\mathbf{r}'(s) \cdot \mathbf{r}''(s) = 0}.$$

Thus the change of the unit tangent has no tangential component.

Step 2: Curvature formula Because the parameter is arc length,

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s).$$

Consequently

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''(s)\|,$$

so

$$\boxed{\kappa(s) = \|\mathbf{r}''(s)\|}.$$

Step 3: Zero second derivative If $\mathbf{r}'' = \mathbf{0}$ throughout a neighborhood, then $\mathbf{r}'$ is a constant unit vector there. Integrating gives $\mathbf{r}(s) = \mathbf{p} + s\mathbf{u}$ locally, a straight line traversed at unit speed. At a single point, $\mathbf{r}''(s_0) = \mathbf{0}$ means zero curvature at that point, but does not by itself force an entire neighboring segment to be straight.