

Question 7

For the space curve

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle,$$

find all real parameters at which the curvature is zero.

Tasks

1. Compute $\mathbf{r}' \times \mathbf{r}''$.
2. Decide whether its magnitude can vanish.
3. Interpret the conclusion geometrically.

Question 7 – Solution

Strategy. Since the curve is regular, curvature is zero exactly when $\mathbf{r}'$ and $\mathbf{r}''$ are parallel.

Step 1: Derivatives

$$\mathbf{r}' = \langle 1, 2t, 3t^2 \rangle, \quad \mathbf{r}'' = \langle 0, 2, 6t \rangle.$$

The first component of $\mathbf{r}'$ is always 1, so the curve is regular.

Step 2: Cross product

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= \langle 1, 2t, 3t^2 \rangle \times \langle 0, 2, 6t \rangle \\ &= \langle 12t^2 - 6t^2, -6t, 2 \rangle \\ &= \langle 6t^2, -6t, 2 \rangle.\end{aligned}$$

Its squared magnitude is

$$36t^4 + 36t^2 + 4.$$

Every term is nonnegative and the constant term is $4 > 0$. Therefore the cross product never vanishes.

Step 3: Conclusion

$\kappa(t) > 0$ for every real t ; there are no zero-curvature points.

Geometrically, the tangent direction is changing everywhere. In particular, no point of this twisted cubic is locally straight in the curvature sense.