

Question 5

The same parabola is parametrized by

$$\mathbf{r}(t) = \langle t, t^2, 0 \rangle \quad \text{and} \quad \mathbf{R}(u) = \langle u^3, u^6, 0 \rangle, \quad u > 0.$$

Tasks

1. Compute curvature from each parametrization.
2. Compare the answers at corresponding points $t = u^3$.
3. Explain why the restriction $u > 0$ is useful.

Question 5 – Solution

Strategy. Apply the cross-product formula twice and simplify all powers before comparing.

Step 1: First parametrization

$$\mathbf{r}' = \langle 1, 2t, 0 \rangle, \quad \mathbf{r}'' = \langle 0, 2, 0 \rangle.$$

Thus

$$\kappa_r(t) = \frac{2}{(1 + 4t^2)^{3/2}}.$$

Step 2: Second parametrization

$$\mathbf{R}' = \langle 3u^2, 6u^5, 0 \rangle, \quad \mathbf{R}'' = \langle 6u, 30u^4, 0 \rangle.$$

The cross product has k -component

$$(3u^2)(30u^4) - (6u^5)(6u) = 54u^6.$$

Also

$$\|\mathbf{R}'\| = 3u^2\sqrt{1 + 4u^6}$$

because $u > 0$. Hence

$$\kappa_R(u) = \frac{54u^6}{27u^6(1 + 4u^6)^{3/2}} = \frac{2}{(1 + 4u^6)^{3/2}}.$$

Step 3: Compare Substituting $t = u^3$ into κ_r gives exactly κ_R . Curvature depends on the geometric curve, not its regular parametrization. The restriction $u > 0$ ensures $\mathbf{R}' \neq \mathbf{0}$ and selects the positive-parameter branch. The speed simplification uses $u^2 \geq 0$, which also holds for negative u .