

Question 4

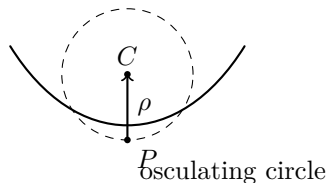
Find the osculating circle of $y = x^2$ at the vertex.

Tasks

1. Compute the curvature and principal-normal direction at $x = 0$.
2. Find the center and radius of the osculating circle.
3. Write its Cartesian equation and verify tangency.

Question 4 – Solution

Strategy. The center lies one radius of curvature from the point in the principal-normal direction.



Step 1: Curvature For $y = x^2$,

$$y' = 2x, \quad y'' = 2,$$

so

$$\kappa(0) = \frac{|2|}{(1+0)^{3/2}} = 2, \quad \rho(0) = \frac{1}{2}.$$

At the vertex the curve bends upward, so $\mathbf{N} = \langle 0, 1 \rangle$.

Step 2: Center With $P = (0, 0)$,

$$C = P + \rho \mathbf{N} = (0, 0) + \frac{1}{2}(0, 1) = \left(0, \frac{1}{2}\right).$$

Step 3: Circle equation Radius $1/2$ and center $(0, 1/2)$ give

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

The circle contains $(0, 0)$. Implicit differentiation gives

$$2x + 2\left(y - \frac{1}{2}\right)y' = 0,$$

so at $(0, 0)$, $y' = 0$, matching the parabola's horizontal tangent. The common curvature at contact gives second-order agreement.