

Question 3

For the graph $y = \ln x$, $x > 0$, determine where its curvature is greatest.

Tasks

1. Derive $\kappa(x)$ using the graph formula.
2. Maximize it exactly.
3. Give the point and maximum curvature.

Question 3 – Solution

Strategy. Use $\kappa = |y''|/[1 + (y')^2]^{3/2}$, then simplify before differentiating.

Step 1: Curvature function

$$y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}.$$

Therefore

$$\begin{aligned}\kappa(x) &= \frac{1/x^2}{(1 + 1/x^2)^{3/2}} \\ &= \frac{x}{(x^2 + 1)^{3/2}}, \quad x > 0.\end{aligned}$$

Step 2: Critical point

$$\begin{aligned}\kappa'(x) &= (x^2 + 1)^{-3/2} - 3x^2(x^2 + 1)^{-5/2} \\ &= \frac{1 - 2x^2}{(x^2 + 1)^{5/2}}.\end{aligned}$$

Thus $\kappa'(x) = 0$ when $x = 1/\sqrt{2}$. The numerator is positive before this point and negative after it, so this is the unique maximum.

Step 3: Point and value The point is

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2} \ln 2 \right).$$

At that value,

$$\kappa_{\max} = \frac{1/\sqrt{2}}{(3/2)^{3/2}} = \boxed{\frac{2}{3\sqrt{3}}}.$$

Also $\kappa \rightarrow 0$ as $x \rightarrow 0^+$ or $x \rightarrow \infty$, confirming the interior maximum.