

Question 2

Find the curvature of the circular helix

$$\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle, \quad a > 0,$$

where b is constant.

Tasks

1. Derive $\kappa(t)$ symbolically.
2. Explain why it is constant.
3. Check the special cases $b = 0$ and increasing $|b|$.

Question 2 – Solution

Strategy. Compute the speed and cross product in symbolic form before simplifying.

Step 1: Derivatives and speed

$$\mathbf{r}' = \langle -a \sin t, a \cos t, b \rangle, \quad \mathbf{r}'' = \langle -a \cos t, -a \sin t, 0 \rangle,$$

and

$$\|\mathbf{r}'\| = \sqrt{a^2 + b^2}.$$

Step 2: Cross product

$$\mathbf{r}' \times \mathbf{r}'' = \langle ab \sin t, -ab \cos t, a^2 \rangle.$$

Its magnitude is

$$\sqrt{a^2 b^2 (\sin^2 t + \cos^2 t) + a^4} = a \sqrt{a^2 + b^2}.$$

Thus

$$\kappa(t) = \frac{a \sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}.$$

Step 3: Interpretation No t remains, so curvature is constant. When $b = 0$, the helix becomes a circle of radius a and $\kappa = 1/a$. As $|b|$ grows, $a/(a^2 + b^2)$ decreases: the helix stretches vertically and bends less sharply.