

Question 1

For the plane curve $\mathbf{r}(t) = \langle t, t^2, 0 \rangle$, find the curvature at $t = 1$.

Tasks

1. Compute $\mathbf{r}'$ and $\mathbf{r}''$.
2. Use $\kappa = \|\mathbf{r}' \times \mathbf{r}''\| / \|\mathbf{r}'\|^3$.
3. State the radius of curvature at the point.

Question 1 – Solution

Strategy. The cross-product formula avoids differentiating the unit tangent.

Step 1: Derivatives

$$\mathbf{r}'(t) = \langle 1, 2t, 0 \rangle, \quad \mathbf{r}''(t) = \langle 0, 2, 0 \rangle.$$

At $t = 1$, $\mathbf{r}'(1) = \langle 1, 2, 0 \rangle$ and $\|\mathbf{r}'(1)\| = \sqrt{5}$.

Step 2: Cross product

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 0 \rangle \times \langle 0, 2, 0 \rangle = \langle 0, 0, 2 \rangle,$$

so its magnitude is 2. Therefore

$$\kappa(1) = \frac{2}{(\sqrt{5})^3} = \frac{2}{5\sqrt{5}}.$$

Step 3: Radius of curvature Since $\rho = 1/\kappa$,

$$\rho(1) = \frac{5\sqrt{5}}{2}.$$

Both values are positive, and their product is 1, verifying the reciprocal relationship.