

Question 1

For

$$\mathbf{F}(x, y, z) = \langle x^2y, yz^2, xz \rangle,$$

compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

Tasks

1. Compute the divergence term by term.
2. Compute all three components of the curl.
3. Evaluate both quantities at $(1, 2, -1)$.

Question 1 – Solution

Strategy. Label the components P, Q, R and apply the divergence and curl formulas in a fixed order.

Step 1: Divergence With $P = x^2y$, $Q = yz^2$, and $R = xz$,

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z = 2xy + z^2 + x.$$

Thus

$$\boxed{\nabla \cdot \mathbf{F} = 2xy + z^2 + x}.$$

Step 2: Curl

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle 0 - 2yz, 0 - z, 0 - x^2 \rangle.\end{aligned}$$

Therefore

$$\boxed{\nabla \times \mathbf{F} = \langle -2yz, -z, -x^2 \rangle}.$$

Step 3: Evaluate At $(1, 2, -1)$,

$$\boxed{\nabla \cdot \mathbf{F} = 6}, \quad \boxed{\nabla \times \mathbf{F} = \langle 4, 1, -1 \rangle}.$$

Verification The curl components follow the cyclic order $(R_y - Q_z, P_z - R_x, Q_x - P_y)$; recomputing from the determinant form gives the same signs.

Question 2

Consider the position field

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Tasks

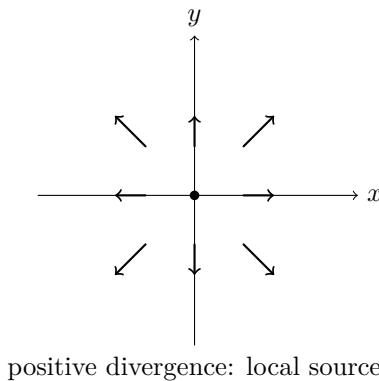
1. Compute its divergence and curl.
2. Interpret the divergence as local expansion or contraction.
3. Explain why the field has no local rotational tendency.

Question 2 – Solution

Strategy. Differentiate the matching coordinate in each divergence term and compare the cross derivatives for curl.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = \boxed{3}.$$



The positive constant divergence describes uniform local expansion: arrows point outward and grow with distance from the origin.

Step 2: Curl

Every cross derivative is zero, so

$$\nabla \times \mathbf{F} = \langle 0 - 0, 0 - 0, 0 - 0 \rangle = \boxed{\mathbf{0}}.$$

Step 3: Interpret Curl measures infinitesimal rotational tendency. This radial field expands directly away from the origin without circulating around any axis, consistent with zero curl.

Verification The field is $\nabla[(x^2 + y^2 + z^2)/2]$. A gradient field has zero curl where the second partial derivatives are continuous, confirming the calculation.

Question 3

Let

$$\mathbf{F}(x, y, z) = \langle -2y, 2x, z^2 \rangle.$$

At $P = (1, -1, 2)$, determine the local expansion rate and the curl vector.

Tasks

1. Compute and evaluate the divergence.
2. Compute and evaluate the curl.
3. Interpret the curl's axis, direction, and infinitesimal angular-speed relation.

Question 3 – Solution

Strategy. Use divergence for local volume change and curl for twice the infinitesimal angular-velocity vector.

Step 1: Expansion rate

$$\nabla \cdot \mathbf{F} = 0 + 0 + 2z = 2z.$$

At P , this is

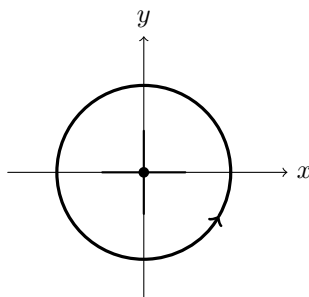
$$\boxed{(\nabla \cdot \mathbf{F})(P) = 4},$$

so the field has positive local expansion there.

Step 2: Curl

$$\nabla \times \mathbf{F} = \langle 0 - 0, 0 - 0, 2 - (-2) \rangle = \boxed{\langle 0, 0, 4 \rangle}.$$

This curl is constant, so its value at P is the same vector.



$\nabla \times \mathbf{F}$ points along $+z$

Step 3: Interpret The curl points along the positive z -axis, indicating counterclockwise rotation in horizontal planes when viewed from $+z$. The associated infinitesimal angular-velocity vector is half the curl:

$$\boxed{\vec{\omega} = \langle 0, 0, 2 \rangle}.$$

Verification The horizontal field $\langle -2y, 2x \rangle$ is rigid rotation with angular speed 2, which matches $|\nabla \times \mathbf{F}|/2 = 2$.

Question 4

For constants a, b, d , consider

$$\mathbf{F}(x, y, z) = \langle ax - by, bx - ay, dz \rangle.$$

Classify all parameter triples (a, b, d) for which the field is both divergence-free and curl-free on $\mathbb{R}^3$.

Tasks

1. Compute the divergence.
2. Compute the curl.
3. Solve the simultaneous parameter conditions and verify the resulting family.

Question 4 – Solution

Strategy. Treat the scalar divergence and all three curl components as simultaneous identities.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = a + (-a) + d = d.$$

Thus divergence-free requires $d = 0$.

Step 2: Curl With $P = ax - by$, $Q = bx - ay$, and $R = dz$,

$$\nabla \times \mathbf{F} = \langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, b - (-b) \rangle = \langle 0, 0, 2b \rangle.$$

Thus curl-free requires $b = 0$.

Step 3: Classify The parameter a is unrestricted, so

$$\boxed{(a, b, d) = (a, 0, 0), \quad a \in \mathbb{R}}.$$

The resulting field is

$$\mathbf{F} = \langle ax, -ay, 0 \rangle.$$

Verification Its divergence is $a - a + 0 = 0$, and all cross derivatives vanish. Hence every field in the boxed family satisfies both conditions, including the zero field when $a = 0$.

Question 5

Let

$$\mathbf{F}(x, y, z) = \langle xy^2, yz^2, zx^2 \rangle.$$

Verify directly that

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Tasks

1. Compute $\nabla \times \mathbf{F}$.
2. Take its divergence.
3. Explain why this is an instance of a general vector identity.

Question 5 – Solution

Strategy. Compute the curl componentwise, then differentiate each component with respect to its matching coordinate.

Step 1: Curl Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle 0 - 2yz, 0 - 2xz, 0 - 2xy \rangle \\ &= \boxed{\langle -2yz, -2xz, -2xy \rangle}.\end{aligned}$$

Step 2: Divergence of the curl

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}(-2yz) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(-2xy) \\ &= 0 + 0 + 0 = \boxed{0}.\end{aligned}$$

Step 3: General identity For a field with continuous second partial derivatives, expanding $\nabla \cdot (\nabla \times \mathbf{F})$ produces pairs of equal mixed partials with opposite signs. They cancel by equality of mixed partial derivatives.

Verification Each component of this particular curl omits the coordinate with respect to which divergence differentiates it, so all three terms vanish independently.

Question 6

For

$$f(x, y, z) = x^2y + yz^2 + xz,$$

verify directly that

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Tasks

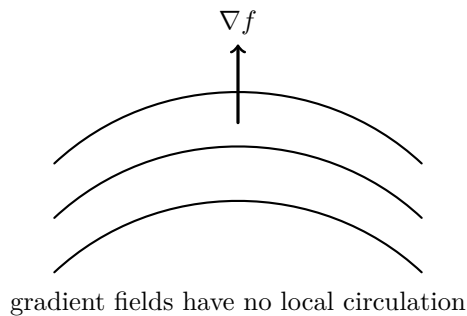
1. Compute ∇f .
2. Compute its curl and show each cancellation.
3. State the smoothness principle behind the identity.

Question 6 – Solution

Strategy. Differentiate once to obtain the gradient and again in the curl's cross-partial pairs.

Step 1: Gradient

$$\nabla f = \langle 2xy + z, x^2 + z^2, 2yz + x \rangle.$$



Step 2: Curl of the gradient Let the gradient components be P, Q, R . Then

$$R_y - Q_z = 2z - 2z = 0,$$

$$P_z - R_x = 1 - 1 = 0,$$

$$Q_x - P_y = 2x - 2x = 0.$$

Therefore

$$\boxed{\nabla \times (\nabla f) = \mathbf{0}}.$$

Step 3: General principle Each cancellation is equality of a mixed-partial pair:

$$f_{zy} = f_{yz}, \quad f_{xz} = f_{zx}, \quad f_{yx} = f_{xy}.$$

Continuous second partial derivatives guarantee these equalities.

Verification The polynomial f has continuous derivatives of every order on $\mathbb{R}^3$, so there are no domain or differentiability exceptions to the identity here.

Question 7

For the planar field

$$\mathbf{F}(x, y) = \langle x^2 - y, x + y^2 \rangle,$$

compute its planar divergence and scalar curl, then evaluate them at $(-1, 1)$.

Tasks

1. Compute $P_x + Q_y$.
2. Compute $Q_x - P_y$.
3. Interpret the two values at the specified point.

Question 7 – Solution

Strategy. In two dimensions, divergence is scalar and curl is represented by the z -component of the three-dimensional curl.

Step 1: Divergence With $P = x^2 - y$ and $Q = x + y^2$,

$$\nabla \cdot \mathbf{F} = P_x + Q_y = 2x + 2y.$$

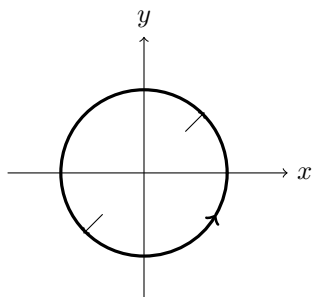
At $(-1, 1)$,

$$\boxed{\nabla \cdot \mathbf{F} = 0}.$$

Step 2: Scalar curl

$$\text{curl}_{2D} \mathbf{F} = Q_x - P_y = 1 - (-1) = \boxed{2}.$$

Equivalently, the three-dimensional embedding $\langle P, Q, 0 \rangle$ has curl $\langle 0, 0, 2 \rangle$.



zero divergence, positive scalar curl

Step 3: Interpret At the point there is no first-order net expansion or contraction, but there is positive counterclockwise rotational tendency. Zero divergence does not imply zero curl.

Verification The scalar curl is constant everywhere, while the divergence vanishes on the line $y = -x$; the point $(-1, 1)$ lies on that line.

Question 8

A linear planar field has the expansion-rotation form

$$\mathbf{F}(x, y) = \langle ax - by, bx + ay \rangle.$$

Its divergence is 4 and its scalar curl is 6 everywhere.

Tasks

1. Determine a and b .
2. Write the resulting field explicitly.
3. Separate it into pure expansion and pure rotation parts.

Question 8 – Solution

Strategy. Divergence isolates the expansion coefficient, while scalar curl isolates twice the rotation coefficient.

Step 1: Compute the invariants

$$\nabla \cdot \mathbf{F} = a + a = 2a, \quad \text{curl}_{2D} \mathbf{F} = b - (-b) = 2b.$$

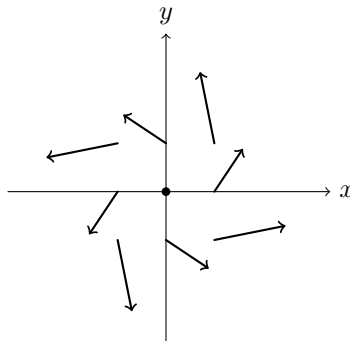
Thus

$$2a = 4, \quad 2b = 6,$$

so $\boxed{a = 2}$ and $\boxed{b = 3}$.

Step 2: Write the field

$$\boxed{\mathbf{F}(x, y) = \langle 2x - 3y, 3x + 2y \rangle}.$$



outward expansion plus counterclockwise rotation

Step 3: Decompose

$$\mathbf{F} = \underbrace{\langle 2x, 2y \rangle}_{\text{pure expansion}} + \underbrace{\langle -3y, 3x \rangle}_{\text{counterclockwise rotation}}.$$

The first part has divergence 4 and zero curl; the second has zero divergence and scalar curl 6.

Verification Recombining the two displayed parts returns both components of the boxed field and reproduces the prescribed invariants.

Question 9

On $D = \mathbb{R}^3 \setminus \{\mathbf{0}\}$, define

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Compute $\nabla \cdot \mathbf{F}$ on D .
2. Compute or otherwise justify $\nabla \times \mathbf{F}$ on D .
3. Explain why the excluded origin must be stated.

Question 9 – Solution

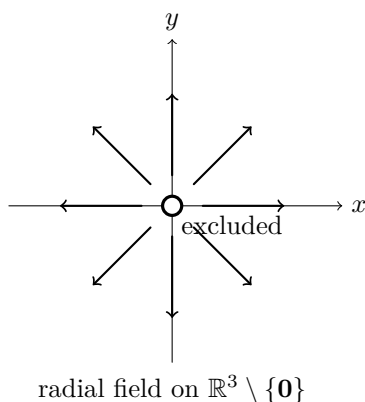
Strategy. Write $r = (x^2 + y^2 + z^2)^{1/2}$ and differentiate xr^{-3} , yr^{-3} , and zr^{-3} .

Step 1: Divergence Since

$$\frac{\partial}{\partial x}(xr^{-3}) = r^{-3} - 3x^2r^{-5},$$

with analogous formulas for y and z ,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= 3r^{-3} - 3(x^2 + y^2 + z^2)r^{-5} \\ &= 3r^{-3} - 3r^2r^{-5} = \boxed{0}, \quad r > 0.\end{aligned}$$



Step 2: Curl Because

$$\mathbf{F} = \nabla \left(-\frac{1}{r} \right)$$

on D , the curl-of-a-gradient identity gives

$$\boxed{\nabla \times \mathbf{F} = \mathbf{0}}.$$

Step 3: Domain caveat The formula is undefined at the origin, and the derivatives used above do not exist there. The conclusions are pointwise statements on D , not statements at $\mathbf{0}$.

Verification Direct cross differentiation gives, for example, $R_y - Q_z = -3yzzr^{-5} - (-3yzzr^{-5}) = 0$, and the other two components cancel similarly.

Question 10

Let

$$\mathbf{F}(x, y, z) = \langle -yz, xz, xy \rangle.$$

Tasks

1. Compute the divergence and curl everywhere.
2. Evaluate the curl at $P = (1, 2, 3)$ and give its direction.
3. Find every point at which the curl vanishes.

Question 10 – Solution

Strategy. Compute the global formulas first; then solve the component equations of the curl simultaneously.

Step 1: Divergence Each component omits its matching differentiation variable, so

$$\nabla \cdot \mathbf{F} = \frac{\partial(-yz)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z} = \boxed{0}.$$

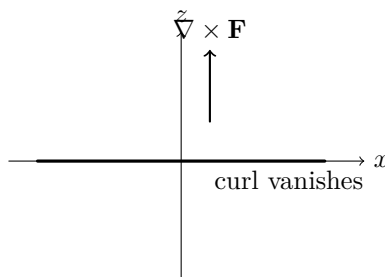
Step 2: Curl

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle x - x, -y - y, z - (-z) \rangle \\ &= \boxed{\langle 0, -2y, 2z \rangle}.\end{aligned}$$

At $P = (1, 2, 3)$,

$$(\nabla \times \mathbf{F})(P) = \boxed{\langle 0, -4, 6 \rangle},$$

which points in the direction $\boxed{\langle 0, -2, 3 \rangle / \sqrt{13}}$.



schematic $y = 0$ slice

Step 3: Vanishing locus The curl is zero exactly when

$$-2y = 0, \quad 2z = 0.$$

Thus $y = z = 0$ while x is arbitrary:

$$\boxed{\nabla \times \mathbf{F} = \mathbf{0} \text{ exactly on the } x\text{-axis}.}$$

Verification Substituting $(x, 0, 0)$ into the curl formula gives $\mathbf{0}$, and any point off the x -axis has $y \neq 0$ or $z \neq 0$, making at least one curl component nonzero.

Question 1

Consider the surface patch

$$\mathbf{r}(u, v) = \langle u, v, 1 + 2u - v \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 1.$$

Tasks

1. Identify the surface and its parameter-domain image.
2. Find two tangent vectors and a normal vector.
3. List the four corner points of the patch.

Question 1 – Solution

Strategy. Read $x = u$ and $y = v$, eliminate the parameters, and map the rectangle's four corners.

Step 1: Identify the surface Since $x = u$ and $y = v$,

$$z = 1 + 2x - y.$$

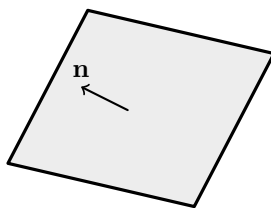
Thus the image is the portion of this plane above the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$.

Step 2: Tangent and normal vectors

$$\mathbf{r}_u = \langle 1, 0, 2 \rangle, \quad \mathbf{r}_v = \langle 0, 1, -1 \rangle.$$

Therefore

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v = \langle -2, 1, 1 \rangle}.$$



planar parameter rectangle

Step 3: Map the corners

$$\mathbf{r}(0, 0) = (0, 0, 1),$$

$$\mathbf{r}(2, 0) = (2, 0, 5),$$

$$\mathbf{r}(0, 1) = (0, 1, 0),$$

$$\mathbf{r}(2, 1) = (2, 1, 4).$$

These four points bound the parallelogram patch.

Verification The normal $\langle -2, 1, 1 \rangle$ is perpendicular to both tangent vectors, and each corner satisfies $z = 1 + 2x - y$.

Question 2

Parametrize the paraboloid $z = x^2 + y^2$ as a graph. At the point $P = (1, -1, 2)$, find a normal vector and the tangent plane.

Tasks

1. Give a two-parameter graph parametrization.
2. Compute the tangent vectors and their cross product at P .
3. Write and verify the tangent-plane equation.

Question 2 – Solution

Strategy. Use $x = u$, $y = v$, and let the defining graph supply the third component.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u, v, u^2 + v^2 \rangle, \quad (u, v) \in \mathbb{R}^2.$$

The point P corresponds to $(u, v) = (1, -1)$.

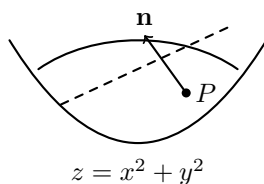
Step 2: Normal vector

$$\mathbf{r}_u = \langle 1, 0, 2u \rangle, \quad \mathbf{r}_v = \langle 0, 1, 2v \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2u, -2v, 1 \rangle.$$

At P , an upward normal is $\boxed{\langle -2, 2, 1 \rangle}$.



Step 3: Tangent plane

$$-2(x - 1) + 2(y + 1) + (z - 2) = 0,$$

or

$$\boxed{z = 2x - 2y - 2}.$$

Verification Substitution of $(1, -1, 2)$ gives $2 = 2 + 2 - 2$. The normal from the graph equation $x^2 + y^2 - z = 0$ is $\langle 2, -2, -1 \rangle$, the negative of the chosen normal.

Question 3

Parametrize the half-cylinder

$$x^2 + y^2 = 4, \quad y \geq 0, \quad -1 \leq z \leq 3,$$

and choose the orientation whose normal points outward.

Tasks

1. Give a parameter domain that covers the patch once, apart from boundary seams.
2. Compute a normal from the tangent vectors.
3. Verify that the normal is outward.

Question 3 – Solution

Strategy. Use an angular parameter for the circular direction and a height parameter for z .

Step 1: Parametrize

$$\mathbf{r}(\theta, s) = \langle 2 \cos \theta, 2 \sin \theta, s \rangle, \quad 0 \leq \theta \leq \pi, \quad -1 \leq s \leq 3.$$

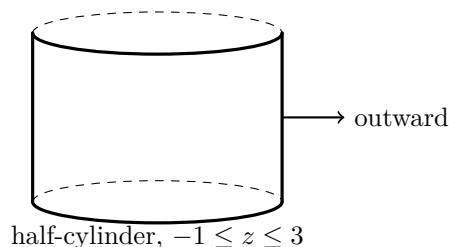
The interval $0 \leq \theta \leq \pi$ enforces $y \geq 0$.

Step 2: Tangent vectors

$$\mathbf{r}_\theta = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle, \quad \mathbf{r}_s = \langle 0, 0, 1 \rangle.$$

Their ordered cross product is

$$\boxed{\mathbf{r}_\theta \times \mathbf{r}_s = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle}.$$



Step 3: Check orientation At each point, the horizontal radial vector from the z -axis is $\langle x, y, 0 \rangle = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle$, exactly the cross product. Hence it points outward.

Verification Its magnitude is 2, never zero, so the parametrization is regular everywhere on the patch, including its boundary curves.

Question 4

Consider the standard spherical parametrization

$$\mathbf{r}(\phi, \theta) = \langle 3 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 3 \cos \phi \rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

Tasks

1. Show that the image is the sphere of radius 3.
2. Compute an outward normal vector from the tangent vectors.
3. Identify where the parametrization fails to be regular or one-to-one.

Question 4 – Solution

Strategy. Verify the implicit equation, then use the ordered product $\mathbf{r}_\phi \times \mathbf{r}_\theta$.

Step 1: Identify the image

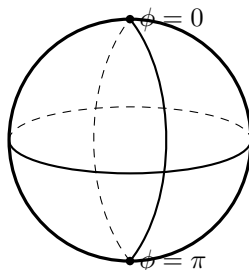
$$x^2 + y^2 + z^2 = 9 \sin^2 \phi + 9 \cos^2 \phi = 9.$$

Thus the image is $x^2 + y^2 + z^2 = 9$.

Step 2: Normal Differentiation and crossing give

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 9 \sin \phi \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle.$$

For $0 < \phi < \pi$, this is a positive multiple of the outward radial direction.



longitude collapses at the poles

Step 3: Degeneracies Its magnitude is $9 \sin \phi$, which vanishes at $\phi = 0$ and $\phi = \pi$. Every value of θ maps to the same pole there. Also $\theta = 0$ and $\theta = 2\pi$ describe the same meridian seam.

Verification Away from the poles, $9 \sin \phi > 0$, so the tangent vectors are independent and the patch is regular locally.

Question 5

Parametrize the conical frustum

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 2,$$

and find a normal pointing away from the z -axis.

Tasks

1. Choose radial and angular parameters with correct bounds.
2. Compute both possible normal orientations.
3. Select and verify the outward normal.

Question 5 – Solution

Strategy. On this cone the radius equals the height, so one parameter can serve both roles.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 1 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

Step 2: Cross the tangents

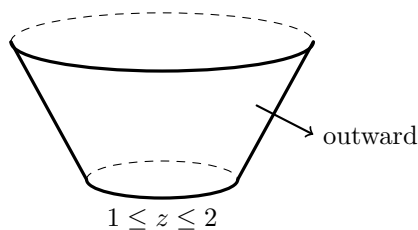
$$\mathbf{r}_u = \langle \cos v, \sin v, 1 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 0 \rangle.$$

Thus

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -u \cos v, -u \sin v, u \rangle.$$

This points inward horizontally. Reverse the order to obtain

$$\boxed{\mathbf{r}_v \times \mathbf{r}_u = \langle u \cos v, u \sin v, -u \rangle}.$$



Step 3: Verify Its horizontal component is $\langle x, y, 0 \rangle$, pointing away from the axis. It also agrees with the outward direction of $x^2 + y^2 - z^2 = 0$, whose gradient is $2\langle x, y, -z \rangle$.

Question 6

Analyze the parametrized surface

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

Tasks

1. Identify the surface geometrically.
2. Describe the coordinate curves $u = \text{constant}$ and $v = \text{constant}$.
3. Test regularity on the entire parameter domain.

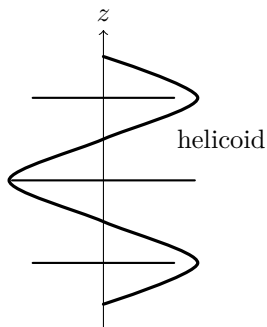
Question 6 – Solution

Strategy. Interpret radius, angle, and height directly, then test the tangent cross product.

Step 1: Identify In cylindrical terms,

$$\rho = u, \quad \theta = v, \quad z = v.$$

As the angle increases, height increases at the same rate; the image is a helicoid patch.



Step 2: Coordinate curves If $u = u_0$, then

$$\langle u_0 \cos v, u_0 \sin v, v \rangle$$

is a helix of radius u_0 ; for $u_0 = 0$ it is the z -axis. If $v = v_0$, varying u traces a radial line segment at height $z = v_0$.

Step 3: Regularity

$$\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 1 \rangle,$$

and

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \rangle}.$$

Its magnitude is $\sqrt{1 + u^2} > 0$, even at $u = 0$, so the parametrization is regular everywhere.

Verification The first two components of the cross product have squared sum 1, preventing the normal from vanishing.

Question 7

Consider

$$\mathbf{r}(u, v) = \langle u^2, v, u^3 \rangle, \quad -1 \leq u \leq 1, \quad -1 \leq v \leq 1.$$

Tasks

1. Eliminate u to find an implicit relation between x and z .
2. Determine where the parametrization is regular.
3. Explain the geometric significance of the failure set.

Question 7 – Solution

Strategy. Compare $x = u^2$ and $z = u^3$, then locate zeros of the tangent cross product.

Step 1: Eliminate the parameter

$$z^2 = u^6 = (u^2)^3 = x^3, \quad x \geq 0.$$

The image is the patch of the cuspidal cylinder $z^2 = x^3$ with $0 \leq x \leq 1$ and $-1 \leq y \leq 1$.

Step 2: Test regularity

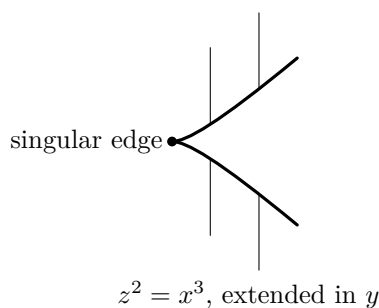
$$\mathbf{r}_u = \langle 2u, 0, 3u^2 \rangle, \quad \mathbf{r}_v = \langle 0, 1, 0 \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -3u^2, 0, 2u \rangle.$$

This vector vanishes exactly when $u = 0$. Therefore the parametrization is regular for $u \neq 0$ and singular along

$$\boxed{\mathbf{r}(0, v) = \langle 0, v, 0 \rangle}.$$



Step 3: Interpret That line is the sharp cuspidal edge. At it, the u -direction tangent collapses to zero, so the two tangent vectors cannot span a tangent plane.

Verification For $u \neq 0$, the third component $2u$ of the cross product is nonzero, proving regularity away from the edge.

Question 8

Two parametrizations are

$$\mathbf{r}(u, v) = \langle u, v, u + v \rangle, \quad \mathbf{s}(p, q) = \langle q, p, p + q \rangle.$$

Tasks

1. Show that they have the same image.
2. Compute the ordered normal from each parametrization.
3. Determine whether they induce the same or opposite orientation.

Question 8 – Solution

Strategy. Recognize that the second parametrization swaps the two parameter roles, then compare ordered cross products.

Step 1: Compare images Both satisfy

$$x + y = z.$$

Given (u, v) for $\mathbf{r}$, choosing $(p, q) = (v, u)$ gives $\mathbf{s}(v, u) = \mathbf{r}(u, v)$. Hence their images are the same plane.

Step 2: First normal

$$\mathbf{r}_u = \langle 1, 0, 1 \rangle, \quad \mathbf{r}_v = \langle 0, 1, 1 \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \boxed{\langle -1, -1, 1 \rangle}.$$

Step 3: Second normal

$$\mathbf{s}_p = \langle 0, 1, 1 \rangle, \quad \mathbf{s}_q = \langle 1, 0, 1 \rangle,$$

and therefore

$$\mathbf{s}_p \times \mathbf{s}_q = \boxed{\langle 1, 1, -1 \rangle}.$$

The normals are negatives, so the parametrizations induce opposite orientations.

Verification Swapping two parameters reverses the order in a cross product, and $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

Question 9

Let

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u^2 \rangle, \quad 1 \leq u \leq 2, \quad 0 \leq v \leq \pi.$$

Describe all boundary curves of the surface patch.

Tasks

1. Find the curves corresponding to $u = 1$ and $u = 2$.
2. Find the curves corresponding to $v = 0$ and $v = \pi$.
3. Identify the surface and verify that the four curves join correctly.

Question 9 – Solution

Strategy. Each edge of the rectangular parameter domain maps to one boundary curve.

Step 1: Constant- u edges

$$u = 1 : \langle \cos v, \sin v, 1 \rangle, \quad u = 2 : \langle 2 \cos v, 2 \sin v, 4 \rangle, \quad 0 \leq v \leq \pi.$$

These are semicircles at heights 1 and 4.

Step 2: Constant- v edges

$$v = 0 : \langle u, 0, u^2 \rangle, \quad v = \pi : \langle -u, 0, u^2 \rangle, \quad 1 \leq u \leq 2.$$

These are parabolic side curves in the plane $y = 0$.



half-annular paraboloid patch

Step 3: Identify and join Since $x^2 + y^2 = u^2$ and $z = u^2$, the patch lies on $z = x^2 + y^2$ above the half-annulus $1 \leq x^2 + y^2 \leq 4$, $y \geq 0$.

Verification The four corner images are $(1, 0, 1)$, $(-1, 0, 1)$, $(2, 0, 4)$, and $(-2, 0, 4)$; each occurs as an endpoint of exactly two adjacent boundary curves.

Question 10

A torus has major radius 3 and tube radius 1, centered about the z -axis.

Tasks

1. Construct a parametrization using two angles.
2. Compute an outward-pointing normal vector.
3. Prove the parametrization is regular everywhere and describe its seam identifications.

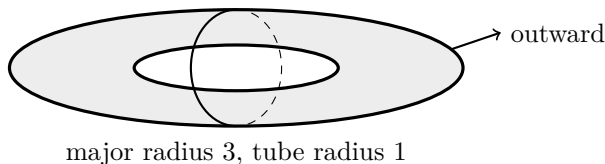
Question 10 – Solution

Strategy. First parametrize a radius-1 circle centered 3 units from the axis, then rotate it about the z -axis.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle (3 + \cos v) \cos u, (3 + \cos v) \sin u, \sin v \rangle,$$

with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.



Step 2: Compute a normal Let $A = 3 + \cos v$. Then

$$\mathbf{r}_u = \langle -A \sin u, A \cos u, 0 \rangle,$$

$$\mathbf{r}_v = \langle -\sin v \cos u, -\sin v \sin u, \cos v \rangle.$$

Their cross product is

$$\mathbf{r}_u \times \mathbf{r}_v = A \langle \cos u \cos v, \sin u \cos v, \sin v \rangle,$$

which points from the tube's center circle toward the surface, hence outward.

Step 3: Regularity and seams The unit vector in brackets has magnitude 1, while $A = 3 + \cos v \geq 2$. Thus the cross product never vanishes. The edges $u = 0, 2\pi$ are identified, as are $v = 0, 2\pi$.

Verification The distance from the z -axis is $3 + \cos v$ and the height is $\sin v$, so $(\sqrt{x^2 + y^2} - 3)^2 + z^2 = 1$, the intended torus.

Question 1

Let S be the portion of the plane $z = 2x + y$ above the rectangle

$$R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}.$$

Tasks

1. Find the scalar surface element dS .
2. Compute the area of S .
3. Evaluate $\iint_S (x + z) dS$ and find the average value of $x + z$ on S .

Question 1 – Solution

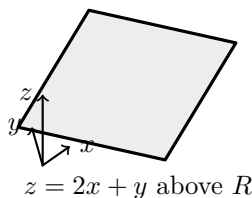
Strategy. Treat the plane as a graph, so the same constant area factor converts every integral over S into an integral over R .

Step 1: Surface element For $z = g(x, y) = 2x + y$,

$$g_x = 2, \quad g_y = 1,$$

and therefore

$$dS = \sqrt{1 + g_x^2 + g_y^2} dA = \sqrt{6} dA.$$



Step 2: Area

$$\text{area}(S) = \iint_R \sqrt{6} dA = \sqrt{6}(1)(2) = \boxed{2\sqrt{6}}.$$

Step 3: Weighted integral On S , $x + z = 3x + y$. Hence

$$\begin{aligned} \iint_S (x + z) dS &= \sqrt{6} \int_0^1 \int_0^2 (3x + y) dy dx \\ &= \sqrt{6} \int_0^1 (6x + 2) dx = \boxed{5\sqrt{6}}. \end{aligned}$$

The average value is

$$\frac{1}{\text{area}(S)} \iint_S (x + z) dS = \frac{5}{2}.$$

Verification The average $5/2$ lies between the minimum 0 and maximum 5 of $3x + y$ on R .

Question 2

Let S be the part of the paraboloid $z = x^2 + y^2$ above the unit disk $x^2 + y^2 \leq 1$.

Tasks

1. Express dS in polar coordinates.
2. Set up and evaluate $\iint_S z \, dS$.
3. Check the radial substitution and the scale of the result.

Question 2 – Solution

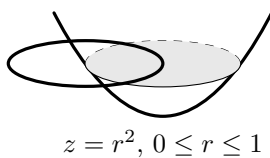
Strategy. Use the graph formula first, then exploit radial symmetry so the surface integral becomes one one-variable integral.

Step 1: Surface element With $g(x, y) = x^2 + y^2$,

$$dS = \sqrt{1 + 4x^2 + 4y^2} dA.$$

In polar coordinates, $z = r^2$ and $dA = r dr d\theta$, so

$$dS = \sqrt{1 + 4r^2} r dr d\theta.$$



$$z = r^2, 0 \leq r \leq 1$$

Step 2: Integrate

$$\iint_S z dS = 2\pi \int_0^1 r^3 \sqrt{1 + 4r^2} dr.$$

Let $t = 1 + 4r^2$. Then $r dr = dt/8$ and $r^2 = (t - 1)/4$, giving

$$\begin{aligned} \iint_S z dS &= \frac{\pi}{16} \int_1^5 (t^{3/2} - t^{1/2}) dt \\ &= \frac{\pi}{16} \left[\frac{2}{5} t^{5/2} - \frac{2}{3} t^{3/2} \right]_1^5 \\ &= \boxed{\frac{\pi(25\sqrt{5} + 1)}{60}}. \end{aligned}$$

Verification The integrand is nonnegative. Also $0 \leq z \leq 1$, so the answer must not exceed the surface area; numerically it is about 2.98, consistent with that bound.

Question 3

A thin surface occupies the lateral cylinder

$$S: x^2 + y^2 = 4, \quad 0 \leq z \leq 3,$$

with surface density $\delta(x, y, z) = 1 + z$.

Tasks

1. Parametrize S and compute its surface element.
2. Find the total mass.
3. Find the center of mass, using symmetry where appropriate.

Question 3 – Solution

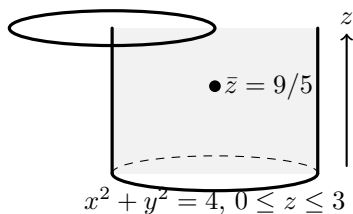
Strategy. Parametrize by angle and height. Rotational symmetry immediately locates two coordinates of the center of mass.

Step 1: Geometry

$$\mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3.$$

Since $\|\mathbf{r}_\theta \times \mathbf{r}_z\| = 2$,

$$dS = 2 \, d\theta \, dz.$$

**Step 2: Mass**

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^3 (1+z) 2 \, dz \, d\theta \\ &= 4\pi \left[z + \frac{z^2}{2} \right]_0^3 = 30\pi. \end{aligned}$$

Step 3: Center of mass Symmetry gives $\bar{x} = \bar{y} = 0$. The first moment about the xy -plane is

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^3 z(1+z) 2 \, dz \, d\theta \\ &= 4\pi \left[\frac{z^2}{2} + \frac{z^3}{3} \right]_0^3 = 54\pi. \end{aligned}$$

Thus

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{9}{5} \right).$$

Verification Because density increases with height, $\bar{z} = 1.8$ is correctly above the geometric midpoint 1.5 and remains inside $0 \leq z \leq 3$.

Question 4

Let S be the conical frustum

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 3.$$

Tasks

1. Parametrize S and derive dS .
2. Compute the surface area.
3. Evaluate $\iint_S \frac{1}{z} dS$ and its average value on S .

Question 4 – Solution

Strategy. Use height as the radius parameter. The factor of u in dS cancels against the weight $1/z$ in the second integral.

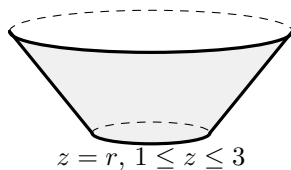
Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 1 \leq u \leq 3, \quad 0 \leq v \leq 2\pi.$$

The tangent vectors have cross-product magnitude

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{2},$$

so $\boxed{dS = u\sqrt{2} \, du \, dv}.$



Step 2: Area

$$\text{area}(S) = \int_0^{2\pi} \int_1^3 u\sqrt{2} \, du \, dv = \boxed{8\pi\sqrt{2}}.$$

Step 3: Weighted integral

Since $z = u$,

$$\iint_S \frac{1}{z} \, dS = \int_0^{2\pi} \int_1^3 \frac{1}{u} u\sqrt{2} \, du \, dv = \boxed{4\pi\sqrt{2}}.$$

Its average value is therefore

$$\boxed{\frac{4\pi\sqrt{2}}{8\pi\sqrt{2}} = \frac{1}{2}}.$$

Verification On the frustum, $1/3 \leq 1/z \leq 1$; the average $1/2$ lies in this interval.

Question 5

On the sphere $S : x^2 + y^2 + z^2 = 4$, evaluate the scalar surface integral

$$\iint_S z^2 dS.$$

Then find the average value of z^2 over the sphere.

Tasks

1. Choose spherical parameters and compute dS .
2. Evaluate the integral exactly.
3. Use symmetry to check the average value.

Question 5 – Solution

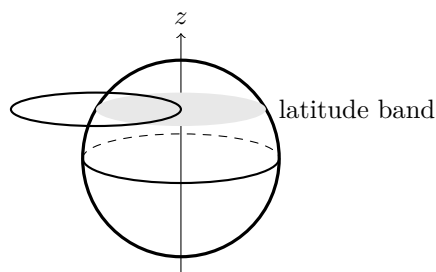
Strategy. In spherical parameters, z and dS separate into a polar factor and an azimuthal factor.

Step 1: Parametrize For radius 2, use

$$\mathbf{r}(\phi, \theta) = \langle 2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi \rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Then

$$z = 2 \cos \phi, \quad dS = 4 \sin \phi \, d\phi \, d\theta.$$



Step 2: Integrate

$$\begin{aligned} \iint_S z^2 \, dS &= \int_0^{2\pi} \int_0^\pi 16 \cos^2 \phi \sin \phi \, d\phi \, d\theta \\ &= 32\pi \int_0^\pi \cos^2 \phi \sin \phi \, d\phi \\ &= 32\pi \left[-\frac{\cos^3 \phi}{3} \right]_0^\pi = \boxed{\frac{64\pi}{3}}. \end{aligned}$$

Because $\text{area}(S) = 4\pi(2^2) = 16\pi$, the average is

$$\boxed{\text{Avg}_S(z^2) = \frac{4}{3}}.$$

Verification Rotational symmetry gives equal averages of x^2 , y^2 , z^2 . Their sum is always 4, so each average must be $4/3$.

Question 6

Let S be the upper hemisphere

$$z = \sqrt{9 - x^2 - y^2}, \quad z \geq 0.$$

Evaluate $\iint_S z \, dS$ by viewing S as a graph, and find the average value of z on S .

Tasks

1. Derive the graph surface element and address its boundary singularity.
2. Explain the cancellation that simplifies the integrand.
3. Evaluate the integral and average value.

Question 6 – Solution

Strategy. The graph factor becomes unbounded at the equator, but multiplication by z cancels it exactly in the interior.

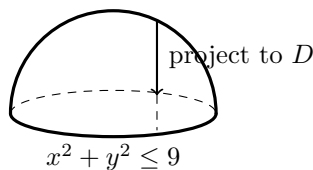
Step 1: Graph factor Let $g(x, y) = \sqrt{9 - x^2 - y^2}$ over the disk $D : x^2 + y^2 \leq 9$. Since

$$g_x = -\frac{x}{z}, \quad g_y = -\frac{y}{z},$$

we have, for $z > 0$,

$$\sqrt{1 + g_x^2 + g_y^2} = \sqrt{1 + \frac{x^2 + y^2}{z^2}} = \frac{3}{z}.$$

Thus $dS = (3/z)dA$ away from the boundary.



Step 2: Cancel and integrate The weighted element is simply

$$z dS = z \frac{3}{z} dA = 3 dA.$$

Therefore

$$\boxed{\iint_S z dS = 3 \text{ area}(D) = 27\pi.}$$

The hemisphere has area $2\pi(3^2) = 18\pi$, so

$$\boxed{\text{Avg}_S(z) = \frac{27\pi}{18\pi} = \frac{3}{2}.}$$

Verification Integrating over disks of radius $R < 3$ gives $3\pi R^2$; taking $R \rightarrow 3^-$ justifies the boundary limit and yields 27π .

Question 7

Let S be the entire boundary of the unit cube $0 \leq x, y, z \leq 1$.

Tasks

1. Decompose S into its six planar faces.
2. Evaluate $\iint_S (x + y + z) \, dS$.
3. Find the average value of $x + y + z$ on the cube's boundary and verify it by symmetry.

Question 7 – Solution

Strategy. Pair opposite faces. Each pair produces the same contribution, avoiding six unrelated computations.

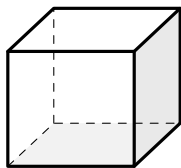
Step 1: Pair the x -faces On $x = 0$, $dS = dy dz$ and

$$\int_0^1 \int_0^1 (y + z) dy dz = 1.$$

On $x = 1$,

$$\int_0^1 \int_0^1 (1 + y + z) dy dz = 2.$$

Thus the pair perpendicular to the x -axis contributes 3.



six faces of the unit cube

Step 2: Use coordinate symmetry The cube and integrand are unchanged by permuting x, y, z . Consequently, the two y -faces and two z -faces also contribute 3 apiece. Hence

$$\boxed{\iint_S (x + y + z) dS = 3 + 3 + 3 = 9}.$$

Step 3: Average The six unit-square faces give $\text{area}(S) = 6$, so

$$\boxed{\text{Avg}_S(x + y + z) = \frac{9}{6} = \frac{3}{2}}.$$

Verification The map $(x, y, z) \mapsto (1 - x, 1 - y, 1 - z)$ preserves the boundary and pairs values whose sum is 3. Therefore their average is $3/2$, confirming the computation.

Question 8

Find the area of the helicoid patch

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.$$

Tasks

1. Compute the tangent vectors and surface element.
2. Evaluate the resulting nonpolynomial integral.
3. Check regularity and the scale of the area.

Question 8 – Solution

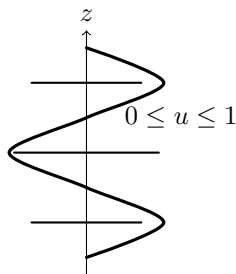
Strategy. Compute the cross product in the given parameters; its magnitude depends only on u , so the v -integration separates.

Step 1: Surface element

$$\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 1 \rangle.$$

Thus

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \rangle, \quad \boxed{dS = \sqrt{1 + u^2} \, du \, dv}.$$



Step 2: Integrate

$$\begin{aligned} \text{area}(S) &= 2\pi \int_0^1 \sqrt{1 + u^2} \, du \\ &= 2\pi \left[\frac{1}{2} \left(u\sqrt{1 + u^2} + \ln(u + \sqrt{1 + u^2}) \right) \right]_0^1 \\ &= \boxed{\pi \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)}. \end{aligned}$$

Step 3: Check The cross product never vanishes because its first two components have squared sum 1. Also $1 \leq \sqrt{1 + u^2} \leq \sqrt{2}$, so

$$2\pi \leq \text{area}(S) \leq 2\pi\sqrt{2},$$

and the exact result lies between these bounds.

Question 9

Let S be the entire unbounded paraboloid $z = x^2 + y^2$. Determine whether the improper scalar surface integral

$$\iint_S \frac{e^{-z}}{\sqrt{1+4z}} dS$$

converges, and evaluate it if it does.

Tasks

1. Truncate the surface above disks $x^2 + y^2 \leq R^2$.
2. Reduce the truncated surface integral to one variable.
3. Take the improper limit and justify convergence.

Question 9 – Solution

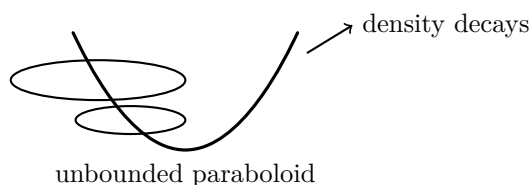
Strategy. On this paraboloid, the denominator in the density is exactly the graph-area factor, leaving a Gaussian radial integral.

Step 1: Truncate Let S_R be the part above $0 \leq r \leq R$. For $z = r^2$,

$$dS = \sqrt{1 + 4r^2} r dr d\theta.$$

Therefore the density times the area element simplifies to

$$\frac{e^{-r^2}}{\sqrt{1 + 4r^2}} \sqrt{1 + 4r^2} r dr d\theta = e^{-r^2} r dr d\theta.$$



Step 2: Evaluate the truncation

$$\begin{aligned} \iint_{S_R} \frac{e^{-z}}{\sqrt{1 + 4z}} dS &= 2\pi \int_0^R e^{-r^2} r dr \\ &= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^R \\ &= \pi(1 - e^{-R^2}). \end{aligned}$$

Step 3: Take the limit The integrand is nonnegative and the truncated values increase to a finite limit:

$$\boxed{\iint_S \frac{e^{-z}}{\sqrt{1 + 4z}} dS = \lim_{R \rightarrow \infty} \pi(1 - e^{-R^2}) = \pi.}$$

Verification The omitted tail equals πe^{-R^2} , which tends to zero and quantifies the convergence.

Question 10

The same planar patch is described over the unit square by

$$\mathbf{r}(u, v) = \langle u, v, u + 2v \rangle, \quad \mathbf{s}(p, q) = \langle 1 - p, q, 1 - p + 2q \rangle.$$

Tasks

1. Show that the parametrizations have the same image but opposite orientations.
2. Evaluate $\iint_S (x + y) \, dS$ using each parametrization.
3. Explain why reversing orientation cannot change a scalar surface integral.

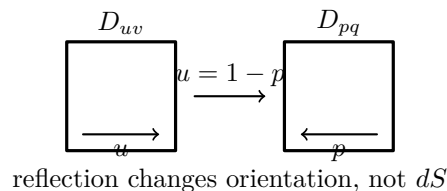
Question 10 – Solution

Strategy. Recognize $\mathbf{s}(p, q) = \mathbf{r}(1 - p, q)$, then compare cross products by magnitude rather than sign.

Step 1: Image and orientation The reflection $(u, v) = (1 - p, q)$ maps the unit square onto itself, so the images agree. Moreover,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1, -2, 1 \rangle, \quad \mathbf{s}_p \times \mathbf{s}_q = \langle 1, 2, -1 \rangle.$$

The normals are negatives, so the orientations are opposite; both magnitudes are $\sqrt{6}$.



Step 2: First parametrization

$$\iint_S (x + y) dS = \int_0^1 \int_0^1 (u + v) \sqrt{6} du dv = \boxed{\sqrt{6}}.$$

Step 3: Second parametrization

$$\iint_S (x + y) dS = \int_0^1 \int_0^1 (1 - p + q) \sqrt{6} dp dq = \boxed{\sqrt{6}}.$$

Verification A scalar surface integral uses $\|\mathbf{r}_u \times \mathbf{r}_v\|$, so reversing the cross product does not change dS . Here the reflection has Jacobian determinant -1 , whose absolute value is 1, confirming the equality under change of parameters.

Question 1

Let S be the part of the plane $z = 1 + x + y$ above the unit square $0 \leq x, y \leq 1$, oriented upward. Let

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Tasks

1. Find the upward-oriented vector surface element.
2. Compute the flux $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$.
3. State how the result changes under downward orientation.

Question 1 – Solution

Strategy. Parametrize the graph by (x, y) and retain the cross product itself, not merely its magnitude.

Step 1: Oriented element Use

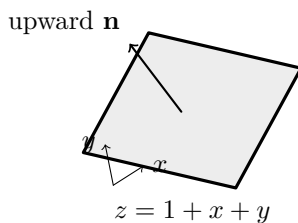
$$\mathbf{r}(x, y) = \langle x, y, 1 + x + y \rangle.$$

Then

$$\mathbf{r}_x = \langle 1, 0, 1 \rangle, \quad \mathbf{r}_y = \langle 0, 1, 1 \rangle,$$

so the upward vector area element is

$$\mathbf{n} \, dS = (\mathbf{r}_x \times \mathbf{r}_y) \, dx \, dy = \langle -1, -1, 1 \rangle \, dx \, dy.$$



Step 2: Dot product On the surface, $z = 1 + x + y$, and hence

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -x - y + z = 1.$$

Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^1 \int_0^1 1 \, dx \, dy = 1.$$

Step 3: Reverse orientation A downward normal is the negative of the chosen one, so the downward flux is $\boxed{-1}$.

Verification The cross product has positive third component, confirming the upward orientation; changing only the orientation must negate a flux integral.

Question 2

Let S be the paraboloid patch $z = x^2 + y^2$ above $x^2 + y^2 \leq 1$, oriented upward. For

$$\mathbf{F}(x, y, z) = \langle x, y, 1 \rangle,$$

compute the flux across S .

Tasks

1. Derive the upward vector surface element.
2. Evaluate the flux in polar coordinates.
3. Explain how positive and negative contributions cancel.

Question 2 – Solution

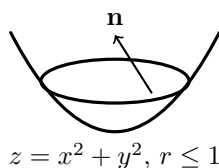
Strategy. Use the upward graph element $\langle -g_x, -g_y, 1 \rangle dA$ and then exploit radial symmetry.

Step 1: Vector area element For $g(x, y) = x^2 + y^2$,

$$\mathbf{n} dS = \langle -2x, -2y, 1 \rangle dA.$$

Thus

$$\mathbf{F} \cdot \mathbf{n} dS = (-2x^2 - 2y^2 + 1) dA = (1 - 2r^2) dA.$$



Step 2: Integrate

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^1 (1 - 2r^2) r dr d\theta \\ &= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{2} \right]_0^1 = \boxed{0}. \end{aligned}$$

Step 3: Locate the cancellation The normal component is positive for $r < 1/\sqrt{2}$ and negative for $r > 1/\sqrt{2}$. The inner contribution is

$$2\pi \left[\frac{r^2}{2} - \frac{r^4}{2} \right]_0^{1/\sqrt{2}} = \frac{\pi}{4},$$

while the outer annulus contributes $-\pi/4$.

Verification The exact cancellation explains why zero flux does not mean the field is tangent everywhere.

Question 3

Let S be only the curved side of the cylinder

$$x^2 + y^2 = 4, \quad 0 \leq z \leq 3,$$

oriented away from the z -axis. For $\mathbf{F} = \langle x, y, z^2 \rangle$, find the outward flux.

Tasks

1. Parametrize the curved side with the correct orientation.
2. Compute the oriented vector area element.
3. Evaluate and interpret the flux.

Question 3 – Solution

Strategy. Use angle and height parameters. The outward cross product is horizontal, so the vertical component of the field contributes nothing.

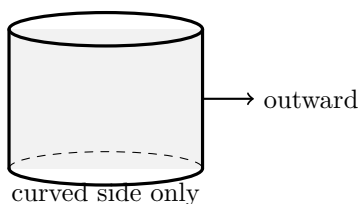
Step 1: Parametrize

$$\mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3.$$

The ordered cross product

$$\mathbf{r}_\theta \times \mathbf{r}_z = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle$$

points away from the axis.



Step 2: Form the integrand On S ,

$$\mathbf{F} = \langle 2 \cos \theta, 2 \sin \theta, z^2 \rangle,$$

and therefore

$$\mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) = 4 \cos^2 \theta + 4 \sin^2 \theta = 4.$$

Step 3: Integrate

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^3 4 \, dz \, d\theta = 24\pi.$$

Verification The flux density is constantly positive, so the field points outward through every point of the curved side. The z^2 component is tangent to that side and correctly disappears.

Question 4

For $a > 0$, let S_a be the sphere $x^2 + y^2 + z^2 = a^2$, oriented outward. Consider the inverse-square radial field

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Find $\mathbf{F} \cdot \mathbf{n}$ on S_a .
2. Compute the outward flux directly.
3. Explain why the result is independent of a .

Question 4 – Solution

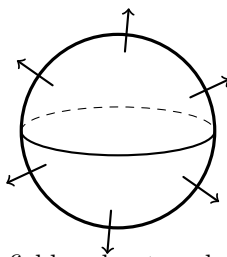
Strategy. On a centered sphere, the field and unit normal are parallel; only their magnitudes need to be compared.

Step 1: Normal component Write $\mathbf{x} = \langle x, y, z \rangle$. On S_a ,

$$\mathbf{n} = \frac{\mathbf{x}}{a}, \quad \mathbf{F} = \frac{\mathbf{x}}{a^3}.$$

Consequently,

$$\mathbf{F} \cdot \mathbf{n} = \frac{\mathbf{x} \cdot \mathbf{x}}{a^4} = \frac{1}{a^2}.$$



radial field and outward normal

Step 2: Integrate Since the sphere has area $4\pi a^2$,

$$\iint_{S_a} \mathbf{F} \cdot \mathbf{n} \, dS = \frac{1}{a^2} (4\pi a^2) = 4\pi.$$

Step 3: Interpret the scale Increasing the radius weakens the normal component by the factor a^{-2} while increasing spherical area by a^2 . These factors cancel exactly.

Verification The field is everywhere outward on S_a , so the positive sign is correct. The calculation is direct and uses no theorem about closed surfaces.

Question 5

Let S be the lateral cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \leq z \leq 2,$$

oriented away from the z -axis. For $\mathbf{F} = \langle 0, 0, z \rangle$, compute the flux across S .

Tasks

1. Parametrize the cone and choose the requested orientation.
2. Evaluate the oriented surface integral.
3. Address the singular parameter point at the cone's vertex.

Question 5 – Solution

Strategy. The normal pointing horizontally away from the axis also has a downward vertical component, which determines the sign of the flux.

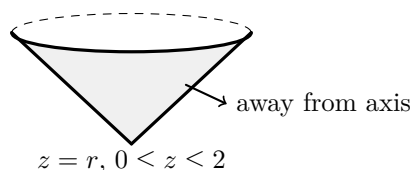
Step 1: Oriented cross product Use

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

The ordered product

$$\mathbf{r}_v \times \mathbf{r}_u = \langle u \cos v, u \sin v, -u \rangle$$

has a radial component pointing away from the axis.



Step 2: Integrate On the cone, $\mathbf{F} = \langle 0, 0, u \rangle$, so

$$\mathbf{F} \cdot (\mathbf{r}_v \times \mathbf{r}_u) = -u^2.$$

Therefore

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^2 -u^2 \, du \, dv \\ &= \boxed{-\frac{16\pi}{3}}. \end{aligned}$$

Step 3: Vertex The parametrization is singular at $u = 0$, but that single surface point has zero area. Equivalently, integrate over $\varepsilon \leq u \leq 2$ and let $\varepsilon \rightarrow 0^+$.

Verification The chosen normal has negative z -component while the field points upward, so a negative flux is required.

Question 6

Let E be the tetrahedron in the first octant bounded by the coordinate planes and $x + y + z = 1$. Let $S = \partial E$ carry the outward orientation, and let

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Compute the total outward flux directly by summing the four faces.

Tasks

1. Determine the flux through the three coordinate-plane faces.
2. Parametrize the slanted face with its outward orientation.
3. Sum the face contributions and verify the sign.

Question 6 – Solution

Strategy. The field has zero normal component on each coordinate face, leaving only the slanted triangular face.

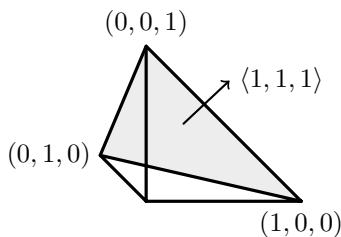
Step 1: Coordinate faces On $x = 0$, the outward normal is $-\mathbf{i}$ and $\mathbf{F} \cdot (-\mathbf{i}) = -x = 0$. Similarly, the flux densities on $y = 0$ and $z = 0$ vanish. These three fluxes are all zero.

Step 2: Slanted face Write $z = 1 - x - y$ over

$$D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}.$$

The outward side points away from the origin, and its vector element is

$$\mathbf{n} dS = \langle 1, 1, 1 \rangle dx dy.$$



On this face,

$$\mathbf{F} \cdot \langle 1, 1, 1 \rangle = x + y + z = 1.$$

Step 3: Integrate and sum

$$\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} dS = 0 + \iint_D 1 dA = \boxed{\frac{1}{2}}.$$

Verification The field points away from the origin and is tangent to the coordinate faces, so all nonzero flux must leave through the slanted face and must be positive.

Question 7

Let S be the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented outward. Compute the flux of the constant field $\mathbf{F} = \langle 0, 0, 1 \rangle$ across S .

Tasks

1. Express the hemisphere as a graph and find its upward vector element.
2. Evaluate the flux using projection onto the xy -plane.
3. State the result for the opposite orientation.

Question 7 – Solution

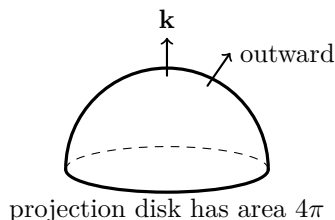
Strategy. A constant vertical field dotted with the upward graph element leaves exactly the projected area element.

Step 1: Graph orientation Write

$$z = g(x, y) = \sqrt{4 - x^2 - y^2}$$

over the disk $D : x^2 + y^2 \leq 4$. The outward orientation on the upper hemisphere is upward, so

$$\mathbf{n} dS = \langle -g_x, -g_y, 1 \rangle dA.$$



Step 2: Project Since $\mathbf{F} = \mathbf{k}$,

$$\mathbf{F} \cdot \mathbf{n} dS = \mathbf{k} \cdot \langle -g_x, -g_y, 1 \rangle dA = dA.$$

Thus

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D dA = \pi(2^2) = 4\pi.}$$

Step 3: Reverse orientation With the inward orientation the vector element changes sign, so the flux is $\boxed{-4\pi}$.

Verification The graph derivatives become unbounded at the equator, but their components are annihilated by the dot product with $\mathbf{k}$. Truncating to radius $R < 2$ and taking $R \rightarrow 2^-$ yields the same result.

Question 8

The Möbius band can be parametrized by

$$\mathbf{r}(u, v) = \left\langle \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \cos u, \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \sin u, \frac{v}{2} \sin \frac{u}{2} \right\rangle,$$

where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.

Tasks

1. Verify the seam identification $\mathbf{r}(0, v) = \mathbf{r}(2\pi, -v)$.
2. Compare the transverse tangent direction at the two identified edges.
3. Explain why a globally oriented flux integral is not defined on the band.

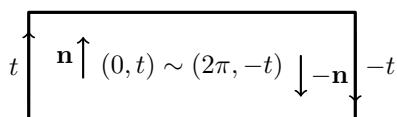
Question 8 – Solution

Strategy. Follow a normal direction once around the band and test whether it returns consistently at the identified seam.

Step 1: Identify the seam Direct substitution gives

$$\mathbf{r}(0, v) = \left\langle 1 + \frac{v}{2}, 0, 0 \right\rangle = \mathbf{r}(2\pi, -v).$$

Thus the left edge at height v is glued to the right edge at height $-v$.



the seam reverses the transverse direction

Step 2: Compare transverse tangents Along the centerline $v = 0$,

$$\mathbf{r}_v(0, 0) = \left\langle \frac{1}{2}, 0, 0 \right\rangle, \quad \mathbf{r}_v(2\pi, 0) = \left\langle -\frac{1}{2}, 0, 0 \right\rangle.$$

Also $\mathbf{r}_u(0, 0) = \mathbf{r}_u(2\pi, 0) = \langle 0, 1, 0 \rangle$. Hence

$$(\mathbf{r}_u \times \mathbf{r}_v)(0, 0) = \left\langle 0, 0, -\frac{1}{2} \right\rangle, \quad (\mathbf{r}_u \times \mathbf{r}_v)(2\pi, 0) = \left\langle 0, 0, \frac{1}{2} \right\rangle.$$

Step 3: Conclude The two parameter points represent the same surface point, yet continuation around the strip reverses the normal. Therefore no continuous global unit normal exists, and

a global oriented flux integral over the Möbius band is not defined.

Verification Cutting the band at the seam permits a local choice of normal, but gluing the seam back forces opposite choices, exposing the obstruction.

Question 9

A plane patch over the unit square has two parametrizations

$$\mathbf{r}(u, v) = \langle u, v, u + 2v \rangle, \quad \mathbf{s}(p, q) = \langle q, p, q + 2p \rangle.$$

For $\mathbf{F} = \langle 1, 0, 0 \rangle$, compare the flux values produced by the ordered cross products of the two parametrizations.

Tasks

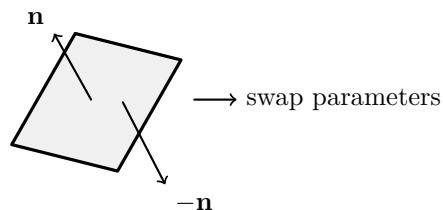
1. Show that $\mathbf{s}$ swaps the parameters of $\mathbf{r}$.
2. Compute both signed flux integrals.
3. Explain how to make the two computations represent the same geometric orientation.

Question 9 – Solution

Strategy. Swapping parameters reverses the cross product, so a vector surface integral detects a sign change that a scalar surface integral would not.

Step 1: Compare cross products Because $\mathbf{s}(p, q) = \mathbf{r}(q, p)$, the images are the same. However,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1, -2, 1 \rangle, \quad \mathbf{s}_p \times \mathbf{s}_q = \langle 1, 2, -1 \rangle.$$



Step 2: Compute signed fluxes Since $\mathbf{F} = \langle 1, 0, 0 \rangle$ and both parameter domains have area 1,

$$\iint \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) du dv = \boxed{-1},$$

whereas

$$\iint \mathbf{F} \cdot (\mathbf{s}_p \times \mathbf{s}_q) dp dq = \boxed{1}.$$

Step 3: Match orientation If the orientation from $\mathbf{r}_u \times \mathbf{r}_v$ is prescribed geometrically, then the $\mathbf{s}$ calculation must use $-(\mathbf{s}_p \times \mathbf{s}_q)$. It then also gives -1 .

Verification The two raw answers are negatives, exactly as required because the ordered normals are negatives. Flux is invariant under reparametrization only after the geometric orientation is held fixed.

Question 10

Let S be the closed cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 2$, including its top and bottom disks, with outward orientation. For

$$\mathbf{F}(x, y, z) = \langle x, y, -2z \rangle,$$

compute the net outward flux directly, without using a general flux theorem.

Tasks

1. Compute the flux through the curved side.
2. Compute the flux through each cap with the correct outward normal.
3. Sum the contributions and interpret the cancellation.

Question 10 – Solution

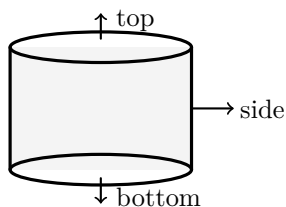
Strategy. Decompose the closed surface into three oriented pieces and track the outward normal on each one.

Step 1: Curved side Parametrize by

$$\mathbf{r}(\theta, z) = \langle \cos \theta, \sin \theta, z \rangle.$$

The outward vector element is $\langle \cos \theta, \sin \theta, 0 \rangle d\theta dz$. Its dot product with $\mathbf{F}$ is 1, so

$$\Phi_{\text{side}} = \int_0^{2\pi} \int_0^2 1 \, dz \, d\theta = 4\pi.$$



Step 2: Caps On the top disk $z = 2$, the outward normal is $\mathbf{k}$, and

$$\mathbf{F} \cdot \mathbf{k} = -2z = -4.$$

Thus $\Phi_{\text{top}} = -4\pi$. On the bottom disk $z = 0$, the outward normal is $-\mathbf{k}$, but the vertical field component is zero, so $\Phi_{\text{bottom}} = 0$.

Step 3: Net flux

$$\Phi_{\text{net}} = 4\pi - 4\pi + 0 = 0.$$

Verification The side carries 4π units outward while the top carries the same amount inward; the signed balance is zero even though neither contribution vanishes.

Question 1

Let C be the unit circle $x^2 + y^2 = 1$ in the plane $z = 0$, oriented counterclockwise when viewed from above. For

$$\mathbf{F}(x, y, z) = \langle -y, x, z \rangle,$$

evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem.

Tasks

1. Choose an oriented spanning surface and compute $\nabla \times \mathbf{F}$.
2. Apply Stokes' Theorem.
3. Verify the answer by direct parametrization of C .

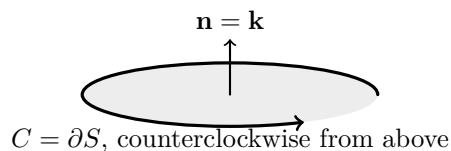
Question 1 – Solution

Strategy. Span the circle by its flat disk; the prescribed counterclockwise direction selects the upward normal.

Step 1: Curl and orientation

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & z \end{vmatrix} = \langle 0, 0, 2 \rangle.$$

Let S be the unit disk with $\mathbf{n} = \mathbf{k}$.



Step 2: Apply Stokes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_S 2 dA = 2\pi.$$

Step 3: Direct check With $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$,

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t, 0 \rangle = \mathbf{r}'(t).$$

Therefore $\int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}' dt = \int_0^{2\pi} 1 dt = 2\pi$.

Verification The boundary is counterclockwise from the tip of $\mathbf{k}$, so the orientation pairing is correct.

Question 2

Let C be the triangular boundary of the first-octant portion of the plane $x + y + z = 1$. Orient C consistently with the normal having positive components. For

$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Compute the curl and an oriented vector area element.
2. Evaluate the surface integral over the projected triangle.
3. Verify the result by integrating along the three edges.

Question 2 – Solution

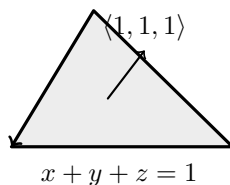
Strategy. Use the triangular plane itself; its graph vector element is constant and points in the prescribed direction.

Step 1: Curl and surface element

$$\nabla \times \mathbf{F} = \langle 1, 1, 1 \rangle.$$

Writing $z = 1 - x - y$ over $D = \{x, y \geq 0 : x + y \leq 1\}$ gives

$$\boxed{\mathbf{n} dS = \langle 1, 1, 1 \rangle dx dy}.$$



Step 2: Apply Stokes

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle dA \\ &= 3 \text{ area}(D) = \boxed{\frac{3}{2}}. \end{aligned}$$

Step 3: Edge check The orientation follows

$$(1, 0, 0) \rightarrow (0, 1, 0) \rightarrow (0, 0, 1) \rightarrow (1, 0, 0).$$

Parametrizing each edge by $0 \leq t \leq 1$ gives integrands $1 - t$, $1 - t$, and $1 - t$, respectively. Hence the three line integrals are each $1/2$, totaling $3/2$.

Verification Two successive directed edge vectors have cross product parallel to $\langle 1, 1, 1 \rangle$, confirming the boundary orientation.

Question 3

Let C be the equator of the upper hemisphere $x^2 + y^2 + z^2 = 9$, oriented as the boundary of the outward-oriented hemisphere. For

$$\mathbf{F} = \langle -y, x, z \rangle,$$

find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ without parametrizing the hemisphere.

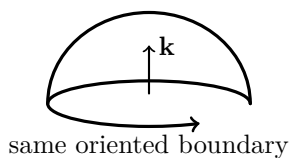
Tasks

1. Determine the induced direction of C as viewed from above.
2. Replace the hemisphere by a simpler spanning surface.
3. Evaluate the circulation and explain why the replacement is valid.

Question 3 – Solution

Strategy. Stokes' Theorem permits any smooth oriented spanning surface with the same oriented boundary; use the equatorial disk.

Step 1: Boundary orientation The outward normal on the upper hemisphere induces counterclockwise motion around the equator when viewed from $+z$.



Step 2: Choose the disk Let D be $x^2 + y^2 \leq 9$, $z = 0$, with upward normal. It has the same oriented boundary as the hemisphere. Since

$$\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle,$$

Stokes' Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 2 \, dA = 2(9\pi) = 18\pi.$$

Step 3: Explain the replacement Both the hemisphere and disk are orientable, share C , and lie where $\mathbf{F}$ has continuous derivatives. Each curl flux therefore equals the same boundary circulation.

Verification On the equator, $\mathbf{F} = \langle -y, x, 0 \rangle$ is tangent in the counterclockwise direction and has magnitude 3; multiplying by the circumference 6π also gives 18π .

Question 4

Let C be the intersection of the cylinder $x^2 + y^2 = 4$ with the plane $z = x + y$. Orient C consistently with the upward normal to the planar region it bounds. For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Identify a convenient spanning surface and its vector element.
2. Compute the curl flux.
3. Explain why the tilt of the curve does not alter the result.

Question 4 – Solution

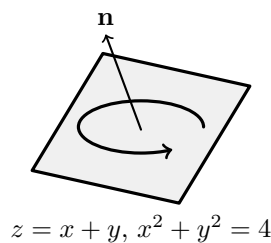
Strategy. Use the planar patch $z = x + y$ over the radius-2 disk; its vertical vector-area component is especially simple.

Step 1: Curl and vector area We have

$$\nabla \times \mathbf{F} = \langle 0, 0, 1 \rangle.$$

For $z = x + y$, the upward vector element is

$$\mathbf{n} dS = \langle -1, -1, 1 \rangle dA.$$



Step 2: Apply Stokes

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_{x^2+y^2 \leq 4} \langle 0, 0, 1 \rangle \cdot \langle -1, -1, 1 \rangle dA \\ &= \iint_{x^2+y^2 \leq 4} 1 dA = \boxed{4\pi}. \end{aligned}$$

Step 3: Interpret The curl is vertical, so only the vertical component of the oriented area vector matters. That component equals the area element of the horizontal projection, whose area is 4π .

Verification Reversing the curve would force the downward normal and change the answer to -4π .

Question 5

Let S be the annulus $1 \leq x^2 + y^2 \leq 4$ in the plane $z = 0$, oriented upward. Its boundary consists of an outer circle C_o and inner circle C_i . For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

evaluate the circulation around the fully oriented boundary ∂S .

Tasks

1. Determine the induced direction on each boundary component.
2. Apply Stokes' Theorem to the annulus.
3. Verify the result from the two separate circle integrals.

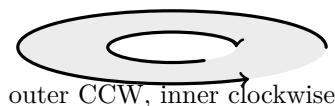
Question 5 – Solution

Strategy. For an upward annulus, the outer boundary is counterclockwise and the inner boundary is clockwise.

Step 1: Orient the boundary The right-hand rule gives

$$\partial S = C_o - C_i^{\text{CCW}}.$$

Equivalently, traverse the outer circle counterclockwise and the inner circle clockwise.



Step 2: Apply Stokes Since $\nabla \times \mathbf{F} = \mathbf{k}$,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S 1 \, dA = \pi(2^2 - 1^2) = 3\pi.$$

Step 3: Check components For a counterclockwise circle of radius R ,

$$\oint \mathbf{F} \cdot d\mathbf{r} = \pi R^2.$$

Thus the outer circle contributes 4π , while the clockwise inner circle contributes $-\pi$. Their sum is 3π .

Verification Assigning both circles counterclockwise would give 5π and violate the induced-boundary convention.

Question 6

Use Stokes' Theorem to derive an area formula for a positively oriented simple closed curve C in the xy -plane. Then apply it to the ellipse

$$C : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0.$$

Tasks

1. Choose a vector field with constant vertical curl.
2. Derive $A = \frac{1}{2} \oint_C (x \, dy - y \, dx)$.
3. Evaluate the formula for the ellipse.

Question 6 – Solution

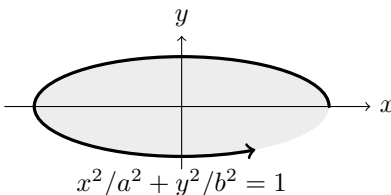
Strategy. Encode planar area as the curl flux of a field whose vertical curl is 1.

Step 1: Choose the field Let

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle.$$

Then $\nabla \times \mathbf{F} = \mathbf{k}$. If D is the region enclosed by positively oriented C , Stokes gives

$$\text{area}(D) = \iint_D 1 \, dA = \oint_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\frac{1}{2} \oint_C (x \, dy - y \, dx)}.$$



Step 2: Parametrize the ellipse Use

$$x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi.$$

Then

$$x \, dy - y \, dx = ab(\cos^2 t + \sin^2 t) \, dt = ab \, dt.$$

Therefore

$$\boxed{\text{area}(D) = \frac{1}{2} \int_0^{2\pi} ab \, dt = \pi ab}.$$

Verification Setting $a = b = R$ recovers the circle area πR^2 , including the correct scale and orientation sign.

Question 7

Let C be the unit circle in $z = 0$, counterclockwise from above. Two surfaces span it: the upper hemisphere S_1 and the paraboloid $S_2 : z = x^2 + y^2 - 1, x^2 + y^2 \leq 1$, both oriented to induce the given direction. For

$$\mathbf{F} = \langle -y, x, z \rangle,$$

compare the curl fluxes through S_1 and S_2 .

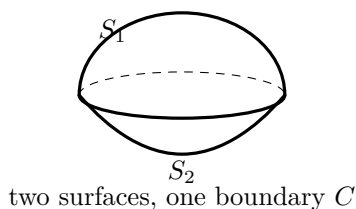
Tasks

1. Determine compatible orientations on both surfaces.
2. Evaluate one common boundary circulation or curl flux.
3. Explain why the two fluxes must agree.

Question 7 – Solution

Strategy. Use Stokes' Theorem as a surface-independence statement: each compatible curl flux equals the same circulation around C .

Step 1: Orient the surfaces Use the outward orientation on the upper hemisphere and the upward orientation on the paraboloid. Both induce counterclockwise motion on C viewed from above.



Step 2: Compute the common value Since

$$\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle,$$

we may use the upward unit disk D spanning C :

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 2 \, dA = 2\pi.$$

Consequently,

$$\boxed{\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 \, dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, dS = 2\pi.}$$

Step 3: Explain Stokes' Theorem ties curl flux to the oriented boundary, not to the particular spanning surface, provided the field is smooth throughout the relevant surfaces.

Verification Both surfaces project once onto the unit disk with positive vertical orientation, so integrating the vertical curl 2 directly would also give 2π on each.

Question 8

Let C be the boundary of the part of the plane $z = 2x - y$ above

$$0 \leq x \leq 1, \quad 0 \leq y \leq 2,$$

oriented consistently with the normal having positive z -component. Let

$$\mathbf{F} = \frac{1}{2} \langle 2z - 3y, 3x - z, y - 2x \rangle.$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Compute the curl of $\mathbf{F}$.
2. Find the oriented vector surface element for the plane.
3. Apply Stokes' Theorem and verify the orientation.

Question 8 – Solution

Strategy. The field was chosen to have constant curl, while the plane has a constant vector area element.

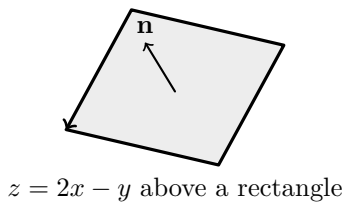
Step 1: Curl Direct differentiation gives

$$\nabla \times \mathbf{F} = \langle 1, 2, 3 \rangle.$$

Step 2: Surface element With $z = g(x, y) = 2x - y$,

$$\mathbf{n} dS = \langle -g_x, -g_y, 1 \rangle dA = \langle -2, 1, 1 \rangle dA.$$

Its third component is positive, as required.



Step 3: Apply Stokes The curl flux density is constant:

$$\langle 1, 2, 3 \rangle \cdot \langle -2, 1, 1 \rangle = 3.$$

The projected rectangle has area 2, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R 3 dA = 6.$$

Verification Reversing the traversal of all four edges would correspond to the negative normal and would change the value to -6 .

Question 9

Let S consist of the top and four side faces of the unit cube $0 \leq x, y, z \leq 1$, but not the bottom face. Give S the outward orientation. Its boundary C is the rim of the missing bottom square. For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem on the composite surface.

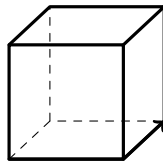
Tasks

1. Determine the induced direction of C as viewed from above.
2. Compute the curl flux through the five faces.
3. Verify the result using the missing bottom square.

Question 9 – Solution

Strategy. The curl is vertical, so the four vertical faces contribute zero; only the top face remains.

Step 1: Boundary direction The outward orientation on the four sides induces counterclockwise traversal of the bottom rim when viewed from above.



C : rim of the missing bottom

Step 2: Flux through S Since

$$\nabla \times \mathbf{F} = \mathbf{k},$$

its dot product with every horizontal side-face normal is zero. On the top, the outward normal is $\mathbf{k}$, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0 + \iint_{[0,1]^2} 1 \, dA = 1.$$

Step 3: Replace the surface The upward-oriented bottom square has the same counterclockwise boundary C . Applying Stokes to that single square also gives $\iint 1 \, dA = 1$.

Verification Using the outward normal $-\mathbf{k}$ on the missing bottom face would induce the opposite boundary direction; it is not the compatible replacement for the stated C .

Question 10

On $\mathbb{R}^3$ minus the z -axis, define

$$\mathbf{F}(x, y, z) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right\rangle.$$

Let C be the unit circle in $z = 0$, counterclockwise from above.

Tasks

1. Compute $\nabla \times \mathbf{F}$ where the field is defined.
2. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly.
3. Explain why applying Stokes to the full disk is invalid, and reconcile the result using a punctured disk.

Question 10 – Solution

Strategy. The apparent contradiction “zero curl but nonzero circulation” signals a failed hypothesis: the field is singular on every full disk spanning C .

Step 1: Curl and direct integral Differentiation gives

$$\nabla \times \mathbf{F} = \mathbf{0} \quad (x^2 + y^2 > 0).$$

On C , let $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t, 0 \rangle = \mathbf{r}'(t),$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 \, dt = 2\pi.$$



Step 2: Locate the failed hypothesis The field is undefined at the origin, which lies on the full spanning disk. Thus $\mathbf{F}$ does not have continuous derivatives on an open set containing that disk, and Stokes' Theorem cannot be applied there.

Step 3: Use an annulus Remove a radius- ε disk. On the resulting annulus the curl is zero, so Stokes gives

$$0 = \oint_{C_{\text{outer}}} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_{\text{inner, clockwise}}} \mathbf{F} \cdot d\mathbf{r} = 2\pi - 2\pi.$$

Verification The inner boundary carries exactly the circulation needed to account for the puncture; no contradiction remains.

Question 1

Let S be the sphere $x^2 + y^2 + z^2 = 4$, oriented outward, and let

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Tasks

1. Compute $\nabla \cdot \mathbf{F}$.
2. Use the Divergence Theorem to find the outward flux through S .
3. Verify the result directly from the geometry of the sphere.

Question 1 – Solution

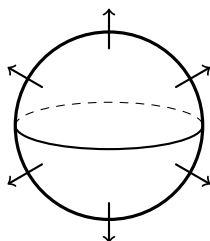
Strategy. Replace the closed-surface flux by a volume integral over the radius-2 ball.

Step 1: Divergence

$$\boxed{\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3}.$$

If E is the ball bounded by S , then

$$\text{vol}(E) = \frac{4}{3}\pi(2^3) = \frac{32\pi}{3}.$$



outward flux through $r = 2$

Step 2: Apply the theorem

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E 3 \, dV = 3 \left(\frac{32\pi}{3} \right) = 32\pi}.$$

Step 3: Direct check On the sphere, the outward unit normal is $\mathbf{n} = \langle x, y, z \rangle / 2$. Therefore

$$\mathbf{F} \cdot \mathbf{n} = \frac{x^2 + y^2 + z^2}{2} = 2.$$

Multiplying by the surface area $4\pi(2^2) = 16\pi$ gives $2(16\pi) = 32\pi$.

Verification The radial field points outward everywhere, so the positive sign is required.

Question 2

Let $E = [0, 1] \times [0, 2] \times [0, 3]$ and let $S = \partial E$ have outward orientation. For

$$\mathbf{F}(x, y, z) = \langle x^2, yz, z \rangle,$$

compute the total outward flux through the six faces.

Tasks

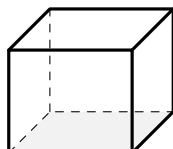
1. Find the divergence.
2. Evaluate the resulting triple integral by separating its terms.
3. Explain why this is preferable to six direct surface integrals.

Question 2 – Solution

Strategy. The box is closed and the field is smooth, so one volume integral replaces six oriented face calculations.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = 2x + z + 1.$$



$1 \times 2 \times 3$ box

Step 2: Integrate term by term The three contributions over E are

$$\begin{aligned}\iiint_E 2x \, dV &= \left(\int_0^1 2x \, dx \right) (2)(3) = 6, \\ \iiint_E z \, dV &= (1)(2) \int_0^3 z \, dz = 9, \\ \iiint_E 1 \, dV &= (1)(2)(3) = 6.\end{aligned}$$

Hence

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E (2x + z + 1) \, dV = 21.$$

Step 3: Interpret A direct method would require a different outward normal and parameter domain on every face. The divergence automatically combines all six signed contributions.

Verification The divergence is positive throughout E , so a positive net outward flux is consistent.

Question 3

Let E be the solid cylinder $x^2 + y^2 \leq 4$, $0 \leq z \leq 3$, and let $S = \partial E$ be oriented outward. For

$$\mathbf{F}(x, y, z) = \langle xz, yz, z^2 \rangle,$$

find the net outward flux and verify it by splitting S into its side and caps.

Tasks

1. Apply the Divergence Theorem in cylindrical coordinates.
2. Compute the flux through the curved side and both caps directly.
3. Compare the totals.

Question 3 – Solution

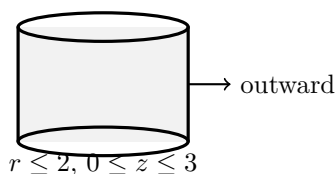
Strategy. First obtain the net flux from the divergence, then use the simple normals of a cylinder as an independent check.

Step 1: Volume integral Since

$$\nabla \cdot \mathbf{F} = z + z + 2z = 4z,$$

we obtain

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^2 \int_0^3 4z \, r \, dz \, dr \, d\theta \\ &= \boxed{72\pi}. \end{aligned}$$



Step 2: Curved side On $r = 2$, the outward vector element is $\langle 2 \cos \theta, 2 \sin \theta, 0 \rangle \, d\theta \, dz$. Thus the dot product is $4z$, and

$$\Phi_{\text{side}} = \int_0^{2\pi} \int_0^3 4z \, dz \, d\theta = 36\pi.$$

Step 3: Caps On the top $z = 3$, $\mathbf{F} \cdot \mathbf{k} = 9$, so $\Phi_{\text{top}} = 9(4\pi) = 36\pi$. On the bottom, $z = 0$, so $\Phi_{\text{bottom}} = 0$.

$$\boxed{\Phi_{\text{side}} + \Phi_{\text{top}} + \Phi_{\text{bottom}} = 72\pi}.$$

Verification The direct total matches the volume-integral result exactly.

Question 4

Let E be the spherical shell $a \leq \sqrt{x^2 + y^2 + z^2} \leq b$, where $0 < a < b$, with outward orientation on its entire boundary. For

$$\mathbf{F} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}},$$

find the net flux through the shell boundary.

Tasks

1. Compute the divergence in the shell.
2. Track the outward normal on both spherical boundary components.
3. Reconcile the separate nonzero fluxes with the net result.

Question 4 – Solution

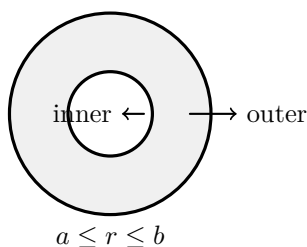
Strategy. The origin is excluded from the shell, so the field is smooth there and has zero divergence; the inner normal points toward the origin.

Step 1: Apply the theorem For $r > 0$,

$$\nabla \cdot \left(\frac{\langle x, y, z \rangle}{r^3} \right) = 0.$$

Therefore

$$\boxed{\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E 0 \, dV = 0}.$$



Step 2: Outer sphere At $r = b$, $\mathbf{F}$ points in the outward-normal direction with normal component $1/b^2$. Hence

$$\Phi_{r=b} = \frac{1}{b^2}(4\pi b^2) = 4\pi.$$

Step 3: Inner sphere The outward normal for the shell at $r = a$ points into the cavity, namely $-\mathbf{e}_r$. Thus

$$\Phi_{r=a} = -\frac{1}{a^2}(4\pi a^2) = -4\pi.$$

The two boundary components sum to zero.

Verification The cancellation depends on reversing the normal at the inner boundary; using the radial normal on both spheres would not describe the shell's outward orientation.

Question 5

Let S be the curved upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented outward. For

$$\mathbf{F}(x, y, z) = \langle x, y, z + 1 \rangle,$$

find the flux through S by closing it with its base disk.

Tasks

1. Apply the Divergence Theorem to the closed upper half-ball.
2. Compute the flux through the base disk with its outward normal.
3. Isolate and verify the curved-surface flux.

Question 5 – Solution

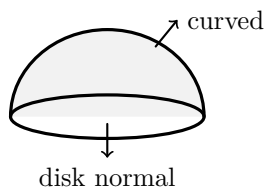
Strategy. Add the base disk, compute the total closed flux, and subtract the disk contribution with careful attention to its downward normal.

Step 1: Closed flux The divergence is

$$\nabla \cdot \mathbf{F} = 3.$$

The upper half-ball has volume $\frac{1}{2} \cdot \frac{4}{3}\pi(2^3) = 16\pi/3$. Therefore

$$\Phi_{\text{closed}} = 3 \left(\frac{16\pi}{3} \right) = 16\pi.$$



Step 2: Base disk On $z = 0$, the outward normal for the half-ball is $-\mathbf{k}$. Since $\mathbf{F} = \langle x, y, 1 \rangle$ there,

$$\Phi_{\text{disk}} = \iint_D (-1) dA = -4\pi.$$

Step 3: Curved part Because $\Phi_{\text{closed}} = \Phi_S + \Phi_{\text{disk}}$,

$$\boxed{\Phi_S = 16\pi - (-4\pi) = 20\pi}.$$

Verification On the sphere, $\mathbf{F} = \langle x, y, z \rangle + \mathbf{k}$. The radial part contributes $2(8\pi) = 16\pi$ over the hemisphere, and the constant vertical part contributes its projected area 4π , totaling 20π .

Question 6

Let E be the first-octant tetrahedron $x + y + z \leq 1$, and let $S = \partial E$ be outward oriented. For

$$\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle,$$

compute the total outward flux.

Tasks

1. Find the divergence.
2. Evaluate the needed coordinate moments over the tetrahedron.
3. Check the answer directly on the slanted face.

Question 6 – Solution

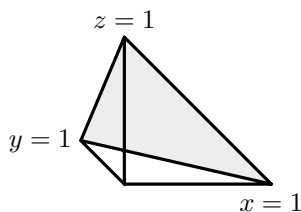
Strategy. Symmetry makes the three first moments equal; the coordinate-plane faces also provide a short direct check.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z.$$

The tetrahedron has volume $1/6$ and centroid $(1/4, 1/4, 1/4)$, so

$$\iiint_E x \, dV = \iiint_E y \, dV = \iiint_E z \, dV = \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}.$$

**Step 2: Apply the theorem**

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = 2 \left(\frac{1}{24} + \frac{1}{24} + \frac{1}{24} \right) = \frac{1}{4}.}$$

Step 3: Direct check On each coordinate face the corresponding field component is zero, so those fluxes vanish. On $z = 1 - x - y$, the outward vector element is $\langle 1, 1, 1 \rangle dA$, giving

$$\iint_D (x^2 + y^2 + (1 - x - y)^2) \, dA = \frac{1}{4}.$$

Verification The direct slanted-face integral agrees with the divergence result and is positive, as expected in the first octant.

Question 7

Let E be the solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Its closed boundary is oriented outward. For

$$\mathbf{F}(x, y, z) = \langle 0, 0, z \rangle,$$

compute the net outward flux and verify the contributions from the two boundary pieces.

Tasks

1. Describe the cylindrical-coordinate bounds for E .
2. Apply the Divergence Theorem.
3. Compute the plane and paraboloid fluxes separately.

Question 7 – Solution

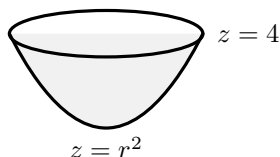
Strategy. The divergence equals 1, so the net flux is the volume; the direct check tests the downward orientation on the paraboloid.

Step 1: Bounds and volume The surfaces meet at $r = 2$, and

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad r^2 \leq z \leq 4.$$

Since $\nabla \cdot \mathbf{F} = 1$,

$$\Phi_{\text{net}} = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dz \, dr \, d\theta = \boxed{8\pi}.$$



Step 2: Top plane On $z = 4$, the outward normal is $\mathbf{k}$, so

$$\Phi_{\text{top}} = \iint_{r \leq 2} 4 \, dA = 16\pi.$$

Step 3: Paraboloid The solid lies above the paraboloid, so its outward vector element is downward:

$$\mathbf{n} \, dS = \langle 2x, 2y, -1 \rangle \, dA.$$

Thus

$$\Phi_{\text{paraboloid}} = \iint_{r \leq 2} -r^2 \, dA = -2\pi \int_0^2 r^3 \, dr = -8\pi.$$

The direct total is $16\pi - 8\pi = 8\pi$.

Verification The sign of the lower contribution is negative because the field points upward while the lower outward normal points downward.

Question 8

For a constant a , let

$$\mathbf{F}_a(x, y, z) = \langle ax, 2y, 3z \rangle.$$

The measured outward flux through the unit sphere is 24π . Determine a .

Tasks

1. Express the divergence in terms of a .
2. Use the flux measurement and the unit-ball volume.
3. Verify the recovered value directly on the sphere.

Question 8 – Solution

Strategy. The divergence is constant, so the measured flux gives a linear equation for the unknown coefficient.

Step 1: Apply the theorem

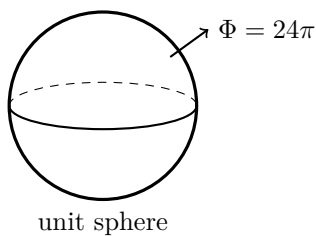
$$\nabla \cdot \mathbf{F}_a = a + 2 + 3 = a + 5.$$

The unit ball has volume $4\pi/3$, so

$$24\pi = (a + 5) \frac{4\pi}{3}.$$

Therefore $a + 5 = 18$, and

$$\boxed{a = 13}.$$



Step 2: Direct verification On the unit sphere, $\mathbf{n} = \langle x, y, z \rangle$, so with $a = 13$,

$$\mathbf{F}_{13} \cdot \mathbf{n} = 13x^2 + 2y^2 + 3z^2.$$

By spherical symmetry,

$$\iint_S x^2 dS = \iint_S y^2 dS = \iint_S z^2 dS = \frac{4\pi}{3}.$$

Hence the direct flux is

$$(13 + 2 + 3) \frac{4\pi}{3} = 24\pi.$$

Verification The recovered parameter reproduces the stated measurement by both methods.

Question 9

On $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, consider

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Let S be the outward-oriented unit sphere.

Tasks

1. Compute the flux through S directly.
2. Explain why zero divergence away from the origin does not permit applying the Divergence Theorem to the full unit ball.
3. Reconcile the result using a punctured ball.

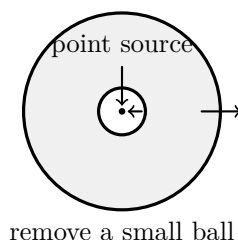
Question 9 – Solution

Strategy. The nonzero flux is caused by a singularity inside the sphere; remove a small ball so the theorem's hypotheses hold.

Step 1: Direct flux On the unit sphere, $\mathbf{F} = \mathbf{n}$, so

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S 1 \, dS = 4\pi.$$

Although $\nabla \cdot \mathbf{F} = 0$ for $r > 0$, the field is undefined at the origin and is not smooth throughout the full ball.



Step 2: Puncture the ball Remove the ball $r < \varepsilon$. On the shell $\varepsilon \leq r \leq 1$, the divergence is zero, so the total outward flux is zero.

Step 3: Track both boundaries The outer flux is 4π . On $r = \varepsilon$, the shell's outward normal is $-\mathbf{e}_r$, while $\mathbf{F} = \mathbf{e}_r/\varepsilon^2$. Thus

$$\Phi_{r=\varepsilon} = -\frac{1}{\varepsilon^2}(4\pi\varepsilon^2) = -4\pi.$$

Therefore $4\pi - 4\pi = 0$, consistent with the theorem on the punctured region.

Verification The inner flux remains -4π as $\varepsilon \rightarrow 0^+$, recording the enclosed point source rather than disappearing.

Question 10

Let E be a bounded solid with piecewise smooth closed boundary S and outward unit normal $\mathbf{n}$.

Tasks

1. Use the Divergence Theorem with a suitable vector field to prove $\text{vol}(E) = \frac{1}{3} \iint_S \langle x, y, z \rangle \cdot \mathbf{n} \, dS$.
2. Apply the identity to a right circular cone of base radius R and height h , with vertex at the origin and base in the plane $z = h$.
3. Explain why the lateral surface contributes zero.

Question 10 – Solution

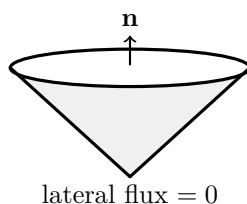
Strategy. Choose a field with divergence 1, then exploit the fact that the cone's radial position vector is tangent to its lateral generators.

Step 1: Derive the identity Let

$$\mathbf{G} = \frac{1}{3} \langle x, y, z \rangle.$$

Then $\nabla \cdot \mathbf{G} = 1$, so

$$\text{vol}(E) = \iiint_E 1 \, dV = \iint_S \mathbf{G} \cdot \mathbf{n} \, dS = \boxed{\frac{1}{3} \iint_S \langle x, y, z \rangle \cdot \mathbf{n} \, dS}.$$



Step 2: Lateral surface Every ray from the origin along the cone is tangent to the lateral surface. Hence $\langle x, y, z \rangle$ is tangent there and

$$\langle x, y, z \rangle \cdot \mathbf{n} = 0.$$

Step 3: Base On the base, $z = h$ and $\mathbf{n} = \mathbf{k}$, so the dot product equals h . Therefore

$$\boxed{\text{vol}(E) = \frac{1}{3} \iint_{\text{base}} h \, dA = \frac{1}{3} h (\pi R^2) = \frac{1}{3} \pi R^2 h}.$$

Verification The formula has the expected “one-third base times height” form and correct cubic units.