

Question 1

Consider the planar vector field

$$\mathbf{F}(x, y) = \langle x, -y \rangle.$$

Tasks

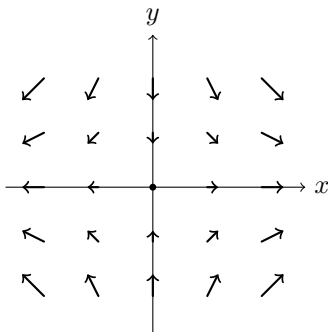
1. Evaluate $\mathbf{F}$ at $(1, 2)$, $(-2, 1)$, and $(0, 0)$.
2. Find the magnitude and direction of $\mathbf{F}(1, 2)$.
3. Describe the field on the coordinate axes and identify every zero vector.

Question 1 – Solution

Strategy. Evaluate the two components pointwise, then use the component signs to understand the field globally.

Step 1: Sample vectors

$$\mathbf{F}(1, 2) = \langle 1, -2 \rangle, \quad \mathbf{F}(-2, 1) = \langle -2, -1 \rangle, \quad \mathbf{F}(0, 0) = \langle 0, 0 \rangle.$$



$$\mathbf{F}(x, y) = \langle x, -y \rangle$$

Step 2: Magnitude and direction At $(1, 2)$,

$$|\mathbf{F}| = \sqrt{1^2 + (-2)^2} = \boxed{\sqrt{5}}, \quad \hat{\mathbf{F}} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle.$$

Step 3: Axis behavior and zeros On the x -axis, $\mathbf{F}(x, 0) = \langle x, 0 \rangle$, so arrows point away from the origin. On the y -axis, $\mathbf{F}(0, y) = \langle 0, -y \rangle$, so arrows point toward the origin. Finally,

$$\mathbf{F}(x, y) = \mathbf{0} \iff x = 0 \text{ and } y = 0.$$

Thus the origin is the field's only zero.

Verification The plotted arrows reverse horizontally when x changes sign and vertically when y changes sign, exactly matching the formula.

Question 2

Determine the domain of the vector field

$$\mathbf{F}(x, y) = \left\langle \ln(x - y), \sqrt{4 - x^2 - y^2} \right\rangle.$$

Tasks

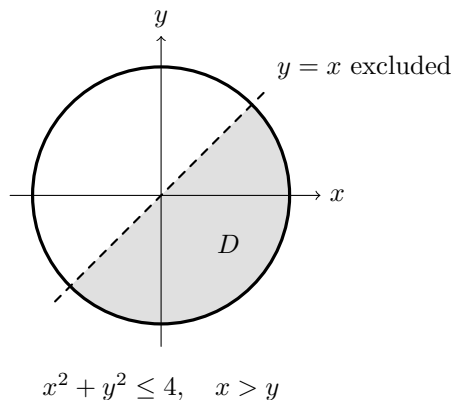
1. Express the domain as simultaneous inequalities.
2. Decide which parts of the line $y = x$ and circle $x^2 + y^2 = 4$ are included.
3. State where the field is continuous and where both components are differentiable.

Question 2 – Solution

Strategy. Intersect the natural domains of the logarithm and square root, keeping track of strict versus non-strict inequalities.

Step 1: Component restrictions The logarithm requires $x - y > 0$, while the square root requires $4 - x^2 - y^2 \geq 0$. Therefore

$$D = \{(x, y) : x > y, x^2 + y^2 \leq 4\}.$$



Step 2: Boundary decisions No point of $y = x$ is included because $\ln 0$ is undefined. The circular boundary is included only where $x > y$; there the second component equals 0 and the first remains defined.

Step 3: Regularity Both component functions are continuous wherever they are defined, so $\mathbf{F}$ is continuous on all of D . Both are differentiable where their defining inequalities are strict:

$$x > y, \quad x^2 + y^2 < 4.$$

At the included circular arc, the square-root component is continuous but its derivatives become singular.

Verification The shaded half-disk excludes its straight diagonal edge but includes the appropriate circular arc, reflecting $>$ versus $\leq$.

Question 3

For

$$\mathbf{F}(x, y) = \frac{\langle x, y \rangle}{x^2 + y^2},$$

analyze the field geometrically.

Tasks

1. State its domain and find its magnitude at a point of radius r .
2. Determine its direction and compare vectors at $(1, 0)$, $(2, 0)$, and $(0, -2)$.
3. Describe what happens as r approaches 0 and as r grows.

Question 3 – Solution

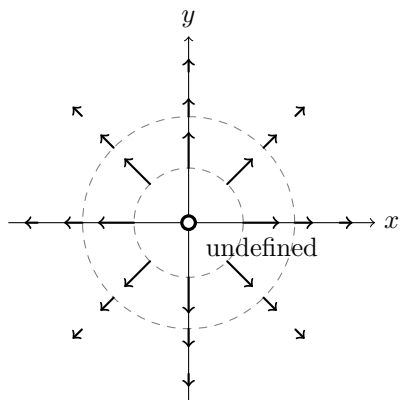
Strategy. Factor the position vector into its length and radial unit vector.

Step 1: Polar form With $r = \sqrt{x^2 + y^2}$ and $\mathbf{e}_r = \langle x/r, y/r \rangle$,

$$\mathbf{F} = \frac{r\mathbf{e}_r}{r^2} = \frac{1}{r}\mathbf{e}_r.$$

Hence the domain is $\mathbb{R}^2 \setminus \{(0, 0)\}$, the direction is radially outward, and

$$|\mathbf{F}| = \frac{1}{r}.$$



outward direction; magnitude $1/r$

Step 2: Samples

$$\mathbf{F}(1, 0) = \langle 1, 0 \rangle, \quad \mathbf{F}(2, 0) = \left\langle \frac{1}{2}, 0 \right\rangle, \quad \mathbf{F}(0, -2) = \left\langle 0, -\frac{1}{2} \right\rangle.$$

Doubling the radius halves the arrow length while preserving radial direction.

Step 3: Limiting behavior As $r \rightarrow 0^+$, $|\mathbf{F}| = 1/r \rightarrow \infty$, explaining the missing origin. As $r \rightarrow \infty$, the magnitude tends to 0.

Verification Multiplying any vector by its radius gives $\mathbf{e}_r$, a unit vector, so both the direction and inverse-radius magnitude are confirmed.

Question 4

A three-dimensional gravitational field is modeled by

$$\mathbf{G}(x, y, z) = -12 \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Evaluate $\mathbf{G}$ at $P = (2, -1, 2)$.
2. Find its magnitude and a unit vector in its direction.
3. Explain the minus sign and state the field's domain.

Question 4 – Solution

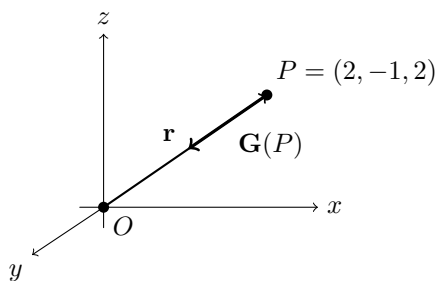
Strategy. Compute the distance from the origin first; it controls both the denominator and the inverse-square magnitude.

Step 1: Evaluate at the point At $P = (2, -1, 2)$,

$$r = \sqrt{2^2 + (-1)^2 + 2^2} = 3, \quad r^3 = 27.$$

Therefore

$$\mathbf{G}(P) = -\frac{12}{27}\langle 2, -1, 2 \rangle = \boxed{\left\langle -\frac{8}{9}, \frac{4}{9}, -\frac{8}{9} \right\rangle}.$$



Step 2: Magnitude and unit direction In general, $|\mathbf{G}| = 12r/r^3 = 12/r^2$. Thus

$$|\mathbf{G}(P)| = \boxed{\frac{4}{3}}, \quad \widehat{\mathbf{G}(P)} = \boxed{\left\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle}.$$

Step 3: Interpretation The negative sign reverses the outward position direction, so gravity points toward the origin. The denominator vanishes only at the origin; hence the domain is

$$\boxed{\mathbb{R}^3 \setminus \{(0, 0, 0)\}}.$$

Verification The displayed unit vector has length 1, and multiplying it by $4/3$ reproduces $\mathbf{G}(P)$.

Question 5

Let

$$\mathbf{F}(x, y) = \langle -y, x \rangle.$$

Tasks

1. Prove that $\mathbf{F}$ is tangent to every circle $x^2 + y^2 = c^2$ centered at the origin.
2. Find its magnitude and orientation.
3. At $P = (3, 4)$, find the vector and the corresponding unit tangent direction.

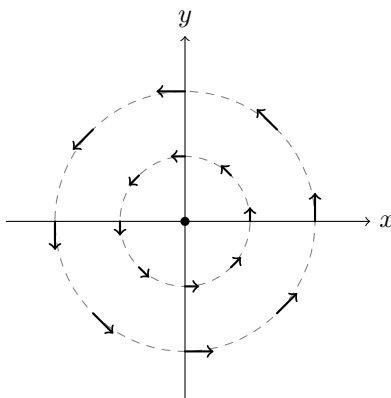
Question 5 – Solution

Strategy. A circle's radial vector is normal to the circle, so orthogonality to that vector proves tangency.

Step 1: Tangency At (x, y) the radial normal is $\mathbf{r} = \langle x, y \rangle$. Since

$$\mathbf{F} \cdot \mathbf{r} = \langle -y, x \rangle \cdot \langle x, y \rangle = -xy + xy = 0,$$

$\mathbf{F}$ is tangent to every centered circle.



vectors are tangent to centered circles

Step 2: Magnitude and orientation

$$|\mathbf{F}| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = r.$$

At $(r, 0)$ the vector is $\langle 0, r \rangle$, which points upward; therefore the orientation is counterclockwise.

Step 3: The point $(3, 4)$

$$\mathbf{F}(3, 4) = \langle -4, 3 \rangle, \quad |\mathbf{F}(3, 4)| = 5,$$

so the counterclockwise unit tangent is

$$\left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle.$$

Verification The dot product $\langle -4, 3 \rangle \cdot \langle 3, 4 \rangle = -12 + 12 = 0$, confirming tangency at P .

Question 6

A temperature distribution is

$$T(x, y) = 100 - x^2 - 4y^2.$$

The heat-flux vector field is $\mathbf{H} = -\frac{1}{2}\nabla T$.

Tasks

1. Find a formula for $\mathbf{H}(x, y)$.
2. Evaluate its magnitude and unit direction at $P = (2, 1)$.
3. Show that the heat flux is normal to the isotherm through P and identify that isotherm.

Question 6 – Solution

Strategy. The gradient is normal to level curves; multiplying it by a negative constant reverses direction but preserves normality.

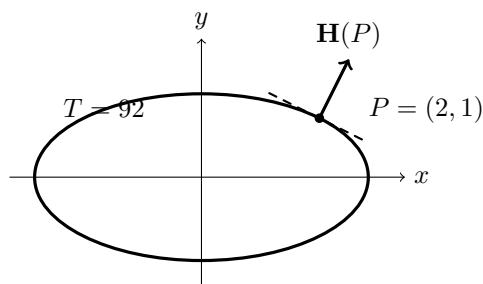
Step 1: Construct the field

$$\nabla T = \langle -2x, -8y \rangle, \quad \mathbf{H} = -\frac{1}{2}\nabla T = \boxed{\langle x, 4y \rangle}.$$

Step 2: Evaluate at P

$$\mathbf{H}(2, 1) = \langle 2, 4 \rangle, \quad |\mathbf{H}(2, 1)| = \boxed{2\sqrt{5}},$$

$$\widehat{\mathbf{H}}(2, 1) = \boxed{\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle}.$$



heat flux is normal to the isotherm

Step 3: Isotherm and normality Since $T(2, 1) = 100 - 4 - 4 = 92$, the isotherm is

$$x^2 + 4y^2 = 8.$$

Its normal direction is $\nabla(x^2 + 4y^2) = \langle 2x, 8y \rangle$. At P this is $\langle 4, 8 \rangle = 2\mathbf{H}(P)$, so the flux is normal to the ellipse and points toward lower temperature.

Verification Moving a short distance along $\mathbf{H}(P)$ increases $x^2 + 4y^2$, so it decreases $T = 100 - (x^2 + 4y^2)$, as heat flow should.

Question 7

A linear planar vector field has the form

$$\mathbf{F}(x, y) = \langle ax + by, cx + dy \rangle.$$

Measurements give

$$\mathbf{F}(1, 0) = \langle 2, -1 \rangle, \quad \mathbf{F}(0, 1) = \langle 3, 4 \rangle.$$

Tasks

1. Determine a, b, c, d and write the field explicitly.
2. Evaluate $\mathbf{F}(-1, 2)$.
3. Find every point where $\mathbf{F} = \mathbf{0}$ and explain uniqueness.

Question 7 – Solution

Strategy. The values on the coordinate basis vectors are exactly the columns of the matrix representing a linear field.

Step 1: Recover the coefficients Substitution at $(1, 0)$ gives $\langle a, c \rangle = \langle 2, -1 \rangle$, while substitution at $(0, 1)$ gives $\langle b, d \rangle = \langle 3, 4 \rangle$. Hence

$$\boxed{\mathbf{F}(x, y) = \langle 2x + 3y, -x + 4y \rangle}.$$

Equivalently,

$$\mathbf{F}(x, y) = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Step 2: Evaluate

$$\mathbf{F}(-1, 2) = \langle -2 + 6, 1 + 8 \rangle = \boxed{\langle 4, 9 \rangle}.$$

Step 3: Find zero vectors Solve

$$2x + 3y = 0, \quad -x + 4y = 0.$$

The second equation gives $x = 4y$; the first then gives $11y = 0$. Thus

$$\boxed{(x, y) = (0, 0)}$$

is the only zero. Equivalently, the coefficient determinant is $2(4) - 3(-1) = 11 \neq 0$, so the linear map has a trivial nullspace.

Verification The recovered formula returns both measured vectors when the two basis points are substituted, so all four coefficients satisfy the data.

Question 8

For a real parameter λ , consider

$$\mathbf{F}_\lambda(x, y) = \langle x - y, \lambda x + y \rangle.$$

Tasks

1. Find all equilibrium points satisfying $\mathbf{F}_\lambda = \mathbf{0}$.
2. Identify the exceptional value of λ and describe its equilibrium set.
3. For that exceptional value, describe the field on either side of the equilibrium set.

Question 8 – Solution

Strategy. Solve the two component equations and separate the parameter value that makes them dependent.

Step 1: Solve generally The first component gives $x = y$. Substituting into the second gives

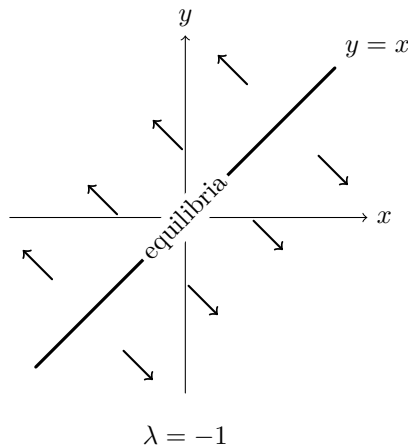
$$\lambda x + y = (\lambda + 1)x = 0.$$

If $\lambda \neq -1$, then $x = 0$ and $y = 0$, so

$$\lambda \neq -1 : \text{the origin is the only equilibrium}.$$

Step 2: Exceptional value If $\lambda = -1$, the second component is $-x + y$, the negative of the first. Every point with $x = y$ is then a zero:

$$\lambda = -1 : \{(x, y) : y = x\} \text{ is the equilibrium set}.$$



Step 3: Behavior off the line For $\lambda = -1$, set $d = x - y$. Then

$$\mathbf{F}_{-1} = \langle d, -d \rangle = d\langle 1, -1 \rangle.$$

This vector is normal to $y = x$. When $x > y$, it points southeast; when $x < y$, it points northwest. Both directions lead away from the equilibrium line.

Verification The coefficient determinant is $1(1) - (-1)\lambda = 1 + \lambda$, which vanishes exactly at $\lambda = -1$, confirming the case split.

Question 9

Let

$$\mathbf{F}(x, y, z) = \langle y, z, x \rangle$$

and let the helix be

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle.$$

Tasks

1. Express the field along the helix as $\mathbf{F}(\mathbf{r}(t))$.
2. Find the unit tangent $\mathbf{T}(t)$.
3. At $t = 0$, decompose $\mathbf{F}(\mathbf{r}(0))$ into components parallel and perpendicular to the helix.

Question 9 – Solution

Strategy. Substitute the curve into the field, then use vector projection onto the unit tangent.

Step 1: Restrict the field to the curve

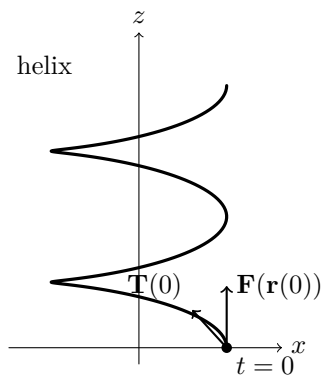
$$\mathbf{F}(\mathbf{r}(t)) = \langle \sin t, t, \cos t \rangle.$$

Also,

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle, \quad |\mathbf{r}'(t)| = \sqrt{2},$$

so

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle.$$



Step 2: Data at $t = 0$

$$\mathbf{F}(\mathbf{r}(0)) = \langle 0, 0, 1 \rangle, \quad \mathbf{T}(0) = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle.$$

The scalar tangential component is

$$\mathbf{F} \cdot \mathbf{T} = \frac{1}{\sqrt{2}}.$$

Step 3: Vector decomposition

$$\mathbf{F}_{\parallel} = (\mathbf{F} \cdot \mathbf{T})\mathbf{T} = \left\langle 0, \frac{1}{2}, \frac{1}{2} \right\rangle,$$

$$\mathbf{F}_{\perp} = \mathbf{F} - \mathbf{F}_{\parallel} = \left\langle 0, -\frac{1}{2}, \frac{1}{2} \right\rangle.$$

Verification The two boxed vectors sum to $\langle 0, 0, 1 \rangle$, and $\mathbf{F}_{\perp} \cdot \mathbf{T}(0) = 0$.

Question 10

On the unit circle $x^2 + y^2 = 1$, consider

$$\mathbf{F}(x, y) = \langle 2x - y, x + 2y \rangle.$$

Tasks

1. Decompose $\mathbf{F}$ into outward radial and counterclockwise tangential components.
2. Find its magnitude and its angle from the outward radial direction.
3. Evaluate the decomposition at $(0, 1)$ and determine whether the field has any zero away from the origin.

Question 10 – Solution

Strategy. Recognize the field as a fixed combination of the radial vector and its 90° counterclockwise rotation.

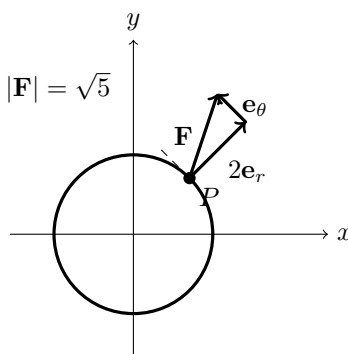
Step 1: Decompose On the unit circle,

$$\mathbf{e}_r = \langle x, y \rangle, \quad \mathbf{e}_\theta = \langle -y, x \rangle.$$

Therefore

$$\mathbf{F} = 2\langle x, y \rangle + \langle -y, x \rangle = \boxed{2\mathbf{e}_r + \mathbf{e}_\theta}.$$

The radial scalar component is 2 and the counterclockwise tangential scalar component is 1.



Step 2: Magnitude and angle Because the two unit vectors are perpendicular,

$$\boxed{|\mathbf{F}| = \sqrt{2^2 + 1^2} = \sqrt{5}}.$$

If α is measured counterclockwise from $\mathbf{e}_r$, then

$$\boxed{\alpha = \tan^{-1}\left(\frac{1}{2}\right)}.$$

Step 3: Point check and zeros At $(0, 1)$,

$$\mathbf{e}_r = \langle 0, 1 \rangle, \quad \mathbf{e}_\theta = \langle -1, 0 \rangle, \quad \mathbf{F} = \boxed{\langle -1, 2 \rangle}.$$

For the full plane, $\mathbf{F} = \mathbf{0}$ gives $2x - y = 0$ and $x + 2y = 0$. Their coefficient determinant is $5 \neq 0$, so only the origin is a zero; in particular, there are none on the unit circle.

Verification The sample vector $\langle -1, 2 \rangle$ has length $\sqrt{5}$ and equals $2\mathbf{e}_r + \mathbf{e}_\theta$, confirming all parts.

Question 1

Evaluate the scalar line integral

$$\int_C (x + y) \, ds,$$

where C is the line segment from $(0, 0)$ to $(3, 4)$.

Tasks

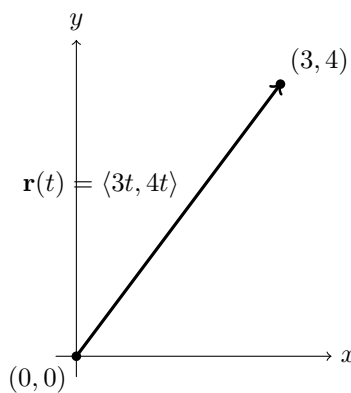
1. Parametrize C on $0 \leq t \leq 1$.
2. Compute ds and evaluate the integral.
3. Verify the constant speed and total length of the segment.

Question 1 – Solution

Strategy. Use linear interpolation between the endpoints, then replace ds by $|\mathbf{r}'(t)| dt$.

Step 1: Parametrize the segment

$$\mathbf{r}(t) = \langle 3t, 4t \rangle, \quad 0 \leq t \leq 1.$$



Along the curve,

$$x + y = 3t + 4t = 7t.$$

Step 2: Arc-length element

$$\mathbf{r}'(t) = \langle 3, 4 \rangle, \quad |\mathbf{r}'(t)| = 5, \quad ds = 5 dt.$$

Therefore

$$\int_C (x + y) ds = \int_0^1 (7t)(5) dt = 35 \left[\frac{t^2}{2} \right]_0^1 = \boxed{\frac{35}{2}}.$$

Verification The speed is constant and the parameter interval has length 1, so the curve length is 5, agreeing with the distance formula $\sqrt{3^2 + 4^2} = 5$. The integrand rises linearly from 0 to 7, whose average is 7/2; multiplying by length 5 again gives 35/2.

Question 2

Evaluate

$$\int_C x \, ds,$$

where C is the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Tasks

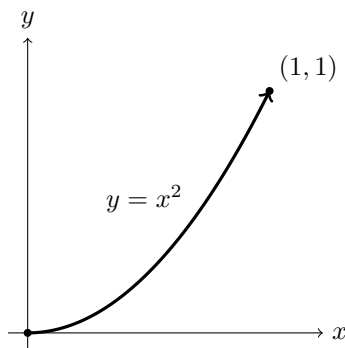
1. Use x itself as a parameter.
2. Derive the arc-length element and evaluate the integral.
3. Check the sign and size of the result.

Question 2 – Solution

Strategy. Parametrize the graph by $\mathbf{r}(x) = \langle x, x^2 \rangle$ so the given scalar function is already the parameter.

Step 1: Parameter and speed

$$\begin{aligned}\mathbf{r}(x) &= \langle x, x^2 \rangle, & 0 \leq x \leq 1, \\ \mathbf{r}'(x) &= \langle 1, 2x \rangle, & ds = \sqrt{1 + 4x^2} \, dx.\end{aligned}$$



Step 2: Evaluate Let $u = 1 + 4x^2$, so $du = 8x \, dx$. Then

$$\begin{aligned}\int_C x \, ds &= \int_0^1 x \sqrt{1 + 4x^2} \, dx \\ &= \frac{1}{8} \int_1^5 u^{1/2} \, du = \frac{1}{12} \left[u^{3/2} \right]_1^5 = \boxed{\frac{5\sqrt{5} - 1}{12}}.\end{aligned}$$

Verification Both x and ds are nonnegative, so the answer must be positive. Also $0 \leq x \leq 1$ implies the integral is no larger than the curve length; numerically the result is about 0.848, consistent with that bound.

Question 3

Let C be the upper semicircle

$$x^2 + y^2 = 4, \quad y \geq 0,$$

oriented from $(2, 0)$ to $(-2, 0)$. Evaluate

$$\int_C (x^2 + y^2) \, ds.$$

Tasks

1. Parametrize the semicircle with its stated orientation.
2. Evaluate the line integral.
3. Give a geometric verification without integration.

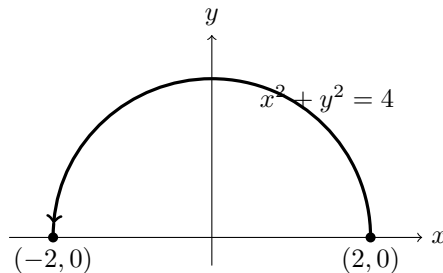
Question 3 – Solution

Strategy. On a circle, the scalar integrand $x^2 + y^2$ is constant, so the integral is that constant times arc length.

Step 1: Parametrization

$$\mathbf{r}(\theta) = \langle 2 \cos \theta, 2 \sin \theta \rangle, \quad 0 \leq \theta \leq \pi.$$

This begins at $(2, 0)$ and ends at $(-2, 0)$, as required.

**Step 2: Evaluate** Along C ,

$$x^2 + y^2 = 4, \quad |\mathbf{r}'(\theta)| = | \langle -2 \sin \theta, 2 \cos \theta \rangle | = 2.$$

Thus

$$\int_C (x^2 + y^2) ds = \int_0^\pi (4)(2) d\theta = \boxed{8\pi}.$$

Verification The upper semicircle has length $\frac{1}{2}(2\pi \cdot 2) = 2\pi$. Since the integrand is constantly 4, the integral must be $4(2\pi) = 8\pi$.

Question 4

Let C be the helix

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle, \quad 0 \leq t \leq \pi.$$

Evaluate

$$\int_C (x^2 + y^2 + z) \, ds.$$

Tasks

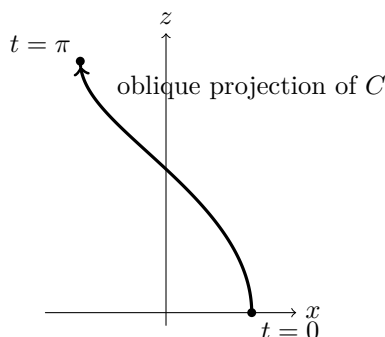
1. Express the scalar function along the helix.
2. Compute the speed and evaluate the integral.
3. Verify the result by separating the constant and variable parts.

Question 4 – Solution

Strategy. Substitute the parametrization into both the scalar function and the arc-length factor.

Step 1: Restrict the integrand Along C ,

$$x^2 + y^2 + z = 4 \cos^2 t + 4 \sin^2 t + t = 4 + t.$$



Step 2: Arc length and integral

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 1 \rangle, \quad |\mathbf{r}'(t)| = \sqrt{4 + 1} = \sqrt{5}.$$

Therefore

$$\begin{aligned} \int_C (x^2 + y^2 + z) ds &= \sqrt{5} \int_0^\pi (4 + t) dt \\ &= \sqrt{5} \left[4t + \frac{t^2}{2} \right]_0^\pi = \boxed{\sqrt{5} \left(4\pi + \frac{\pi^2}{2} \right)}. \end{aligned}$$

Verification The constant part 4 contributes $4L = 4\pi\sqrt{5}$ because $L = \pi\sqrt{5}$. The $z = t$ part has average $\pi/2$ and contributes $(\pi/2)L = \pi^2\sqrt{5}/2$, matching the two terms above.

Question 5

The piecewise smooth curve $C = C_1 \cup C_2$ travels from

$$A = (0, 0) \quad \text{to} \quad B = (1, 0) \quad \text{to} \quad D = (1, 2)$$

along straight segments. Evaluate

$$\int_C (xy + 1) \, ds.$$

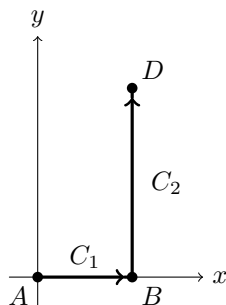
Tasks

1. Parametrize each segment.
2. Evaluate the integral on each piece and add the results.
3. Check the contribution using the integrand along each segment.

Question 5 – Solution

Strategy. A scalar line integral over a piecewise curve is the sum of the integrals over its smooth pieces.

Step 1: First segment



For C_1 , use $\mathbf{r}_1(t) = \langle t, 0 \rangle$, $0 \leq t \leq 1$. Then $ds = dt$ and $xy + 1 = 1$, so

$$\int_{C_1} (xy + 1) ds = \int_0^1 1 dt = 1.$$

Step 2: Second segment For C_2 , use $\mathbf{r}_2(u) = \langle 1, u \rangle$, $0 \leq u \leq 2$. Then $ds = du$ and $xy + 1 = u + 1$, giving

$$\int_{C_2} (xy + 1) ds = \int_0^2 (u + 1) du = \left[\frac{u^2}{2} + u \right]_0^2 = 4.$$

Hence

$$\boxed{\int_C (xy + 1) ds = 1 + 4 = 5}.$$

Verification On C_1 the constant value 1 times length 1 gives 1. On C_2 the integrand rises linearly from 1 to 3, so its average 2 times length 2 gives 4.

Question 6

Let C be the unit circle. Compute

$$\int_C (x + 2) \, ds$$

once with a counterclockwise parametrization and once with a clockwise parametrization.

Tasks

1. Write parametrizations for both orientations.
2. Show directly that both integrals have the same value.
3. Explain why every scalar line integral with respect to ds is orientation-independent.

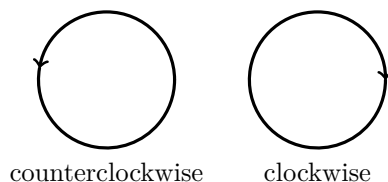
Question 6 – Solution

Strategy. Reversing direction changes the parameter derivative's sign but not its magnitude, so ds is unchanged.

Step 1: Counterclockwise Let $\mathbf{r}_+(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Since $|\mathbf{r}'_+(t)| = 1$,

$$I_+ = \int_0^{2\pi} (\cos t + 2) dt = 0 + 4\pi = 4\pi.$$

same unit circle, opposite orientations



Step 2: Clockwise Let $\mathbf{r}_-(t) = \langle \cos t, -\sin t \rangle$, $0 \leq t \leq 2\pi$. Again the speed is 1, and $x = \cos t$, so

$$I_- = \int_0^{2\pi} (\cos t + 2) dt = \boxed{4\pi}.$$

Step 3: General reason Under a reversed parametrization, $\tilde{\mathbf{r}}(t) = \mathbf{r}(a+b-t)$, the derivative is $-\mathbf{r}'(a+b-t)$, but

$$|\tilde{\mathbf{r}}'(t)| = |\mathbf{r}'(a+b-t)|.$$

After substitution, the scalar values and arc-length factors are identical. Therefore $\int_C f ds$ does not depend on orientation.

Verification Here the x -term cancels by symmetry, while the constant 2 contributes 2 times the circumference 2π , confirming 4π .

Question 7

A wire follows the first-quadrant astroid arc

$$\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad 0 \leq t \leq \frac{\pi}{2},$$

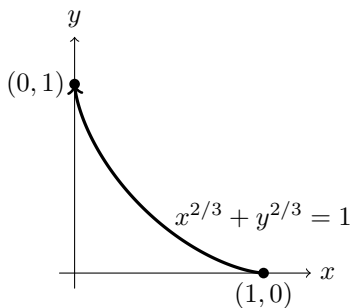
and has linear density $\delta(x, y) = x + y$. Find its mass.

Tasks

1. Derive and simplify the speed on the given interval.
2. Express the density in terms of t and evaluate the mass.
3. Check positivity and endpoint behavior.

Question 7 – Solution

Strategy. Differentiate the astroid parametrization; in the first quadrant the absolute values in the speed simplify.

Step 1: Speed

$$\mathbf{r}'(t) = \langle -3 \cos^2 t \sin t, 3 \sin^2 t \cos t \rangle.$$

Because $\sin t, \cos t \geq 0$,

$$\begin{aligned} |\mathbf{r}'(t)| &= 3 \sin t \cos t \sqrt{\cos^2 t + \sin^2 t} \\ &= 3 \sin t \cos t. \end{aligned}$$

Step 2: Mass Along the wire, $\delta = \cos^3 t + \sin^3 t$. Hence

$$\begin{aligned} M &= \int_0^{\pi/2} 3(\cos^3 t + \sin^3 t) \sin t \cos t \, dt \\ &= 3 \left(\int_0^{\pi/2} \cos^4 t \sin t \, dt + \int_0^{\pi/2} \sin^4 t \cos t \, dt \right) \\ &= 3 \left(\frac{1}{5} + \frac{1}{5} \right) = \boxed{\frac{6}{5}}. \end{aligned}$$

Verification Density and speed are nonnegative. The speed vanishes only at the two cusps, which does not affect the integral; the finite positive mass $6/5$ is therefore consistent.

Question 8

Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$. Find the average value of

$$f(x, y) = x^2$$

along C with respect to arc length.

Tasks

1. Compute $\int_C x^2 ds$ and the length of C .
2. Find the arc-length average value.
3. Identify all points on C where x^2 equals that average.

Question 8 – Solution

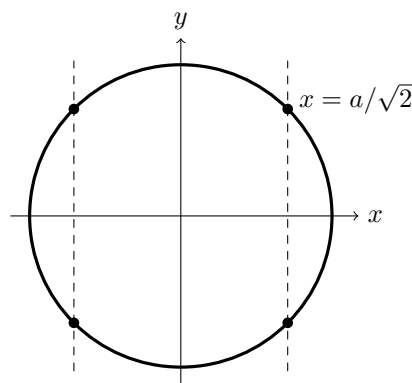
Strategy. Parametrize the circle and divide the scalar line integral by total arc length.

Step 1: Integral and length Use

$$\mathbf{r}(\theta) = \langle a \cos \theta, a \sin \theta \rangle, \quad 0 \leq \theta \leq 2\pi.$$

Then $ds = a d\theta$, so

$$\int_C x^2 ds = a^3 \int_0^{2\pi} \cos^2 \theta d\theta = \pi a^3, \quad L = 2\pi a.$$



$$x^2 + y^2 = a^2 \text{ (scaled sketch)}$$

Step 2: Average value

$$f_{\text{avg}} = \frac{1}{L} \int_C f ds = \frac{\pi a^3}{2\pi a} = \boxed{\frac{a^2}{2}}.$$

Step 3: Points attaining the average We need $x^2 = a^2/2$, so $x = \pm a/\sqrt{2}$. The circle equation then gives $y = \pm a/\sqrt{2}$, independently. Thus the four points are

$$\boxed{\left(\frac{\pm a}{\sqrt{2}}, \frac{\pm a}{\sqrt{2}} \right)},$$

with all four sign combinations.

Verification Symmetry gives equal averages for x^2 and y^2 . Their sum is constantly a^2 on C , so each average must be $a^2/2$.

Question 9

For $a > 0$, a straight wire runs from $(0, 0)$ to (a, a) and has density

$$\delta(x, y) = x + y.$$

Its mass is $8\sqrt{2}$. Determine a and the wire's length.

Tasks

1. Parametrize the wire and derive its mass as a function of a .
2. Solve the inverse condition, respecting $a > 0$.
3. Compute the resulting length and verify the mass.

Question 9 – Solution

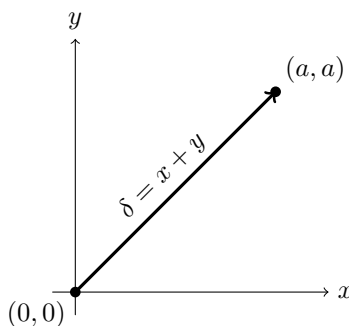
Strategy. Express the line integral entirely in terms of the unknown endpoint coordinate a , then solve the resulting quadratic equation.

Step 1: Mass formula Use

$$\mathbf{r}(t) = \langle at, at \rangle, \quad 0 \leq t \leq 1.$$

Then $\delta = 2at$ and $ds = |\langle a, a \rangle| dt = a\sqrt{2} dt$. Therefore

$$M(a) = \int_0^1 (2at)(a\sqrt{2}) dt = 2a^2\sqrt{2} \left[\frac{t^2}{2} \right]_0^1 = a^2\sqrt{2}.$$



Step 2: Recover a The given mass implies

$$a^2\sqrt{2} = 8\sqrt{2}, \quad a^2 = 8.$$

Because $a > 0$,

$$\boxed{a = 2\sqrt{2}}.$$

Step 3: Length and verification The diagonal length is

$$L = \sqrt{a^2 + a^2} = a\sqrt{2} = (2\sqrt{2})(\sqrt{2}) = \boxed{4}.$$

Substitution into the mass formula gives $M = (2\sqrt{2})^2\sqrt{2} = 8\sqrt{2}$, exactly the specified mass.

Question 10

A wire occupies the upper semicircle

$$x^2 + y^2 = R^2, \quad y \geq 0,$$

and has density $\delta(x, y) = 1 + y/R$, where $R > 0$.

Tasks

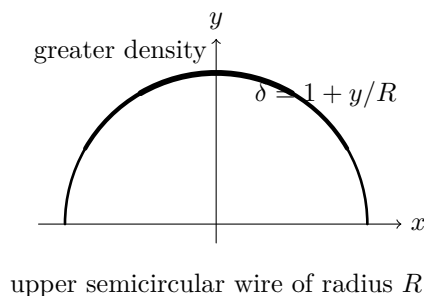
1. Find the wire's mass.
2. Compute its moments about the x - and y -axes.
3. Determine its center of mass and check the result by symmetry and bounds.

Question 10 – Solution

Strategy. Parametrize by polar angle; symmetry immediately predicts that the center of mass lies on the y -axis.

Step 1: Mass Let $x = R \cos \theta$, $y = R \sin \theta$, $0 \leq \theta \leq \pi$, so $ds = R d\theta$ and $\delta = 1 + \sin \theta$. Then

$$M = R \int_0^\pi (1 + \sin \theta) d\theta = \boxed{R(\pi + 2)}.$$



Step 2: Moments The moment about the x -axis is

$$\begin{aligned} M_x &= \int_C y \delta ds = R^2 \int_0^\pi \sin \theta (1 + \sin \theta) d\theta \\ &= R^2 \left(2 + \frac{\pi}{2} \right). \end{aligned}$$

For the y -axis,

$$M_y = R^2 \int_0^\pi \cos \theta (1 + \sin \theta) d\theta = 0.$$

Step 3: Center of mass

$$\bar{x} = \frac{M_y}{M} = 0, \quad \bar{y} = \frac{M_x}{M} = \boxed{\frac{R(\pi + 4)}{2(\pi + 2)}}.$$

Thus

$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{R(\pi + 4)}{2(\pi + 2)} \right)}.$$

Verification Left-right symmetry forces $\bar{x} = 0$. Also $0 < (\pi + 4)/[2(\pi + 2)] < 1$, so $0 < \bar{y} < R$, placing the center of mass inside the semicircle's vertical range.

Question 1

Let C be the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$. Evaluate

$$\int_C y \, dx.$$

Tasks

1. Parametrize the curve using x as the parameter.
2. Replace dx correctly and evaluate the integral.
3. Explain why no arc-length factor appears.

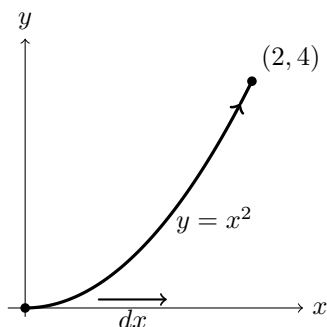
Question 1 – Solution

Strategy. For an integral with respect to x , use $dx = x'(t) dt$ rather than the arc-length element.

Step 1: Parametrize

$$\mathbf{r}(t) = \langle t, t^2 \rangle, \quad 0 \leq t \leq 2.$$

Thus $x = t$, $y = t^2$, and $dx = dt$.



Step 2: Evaluate

$$\int_C y \, dx = \int_0^2 t^2 \, dt = \left[\frac{t^3}{3} \right]_0^2 = \boxed{\frac{8}{3}}.$$

Step 3: Distinguish from ds The differential is $dx = x'(t) dt = dt$. The speed $\sqrt{1 + 4t^2}$ would be used only for an integral with respect to ds ; inserting it here would change the problem.

Verification Since x increases from 0 to 2 and $0 \leq y \leq 4$, the integral must lie between 0 and $4(2) = 8$. The value $8/3$ satisfies this bound.

Question 2

On the same parabola $C : y = x^2$ from $(0, 0)$ to $(2, 4)$, evaluate

$$\int_C x \, dy.$$

Tasks

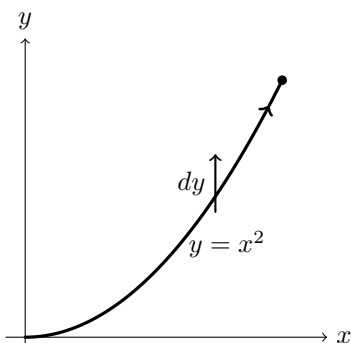
1. Express dy in terms of dx .
2. Evaluate the integral.
3. Compare the answer with $\int_C y \, dx = 8/3$ and explain why they differ.

Question 2 – Solution

Strategy. Use the graph relation to write $dy = y'(x) dx$.

Step 1: Convert the differential Along $y = x^2$,

$$dy = 2x dx, \quad 0 \leq x \leq 2.$$



Step 2: Evaluate

$$\int_C x dy = \int_0^2 x(2x) dx = 2 \left[\frac{x^3}{3} \right]_0^2 = \boxed{\frac{16}{3}}.$$

Step 3: Compare Although both integrals use the same curve, they measure change in different coordinates:

$$y dx = x^2 dx, \quad x dy = x(2x dx) = 2x^2 dx.$$

The second integrand is exactly twice the first, so its value is twice $8/3$.

Verification Here y increases from 0 to 4 and $0 \leq x \leq 2$, so the value must lie between 0 and $2(4) = 8$. The result $16/3$ does.

Question 3

Let C be the straight segment from $A = (0, 0)$ to $B = (2, 1)$. Evaluate

$$\int_C (x + y) dx + (x - y) dy.$$

Tasks

1. Parametrize the segment with the stated orientation.
2. Convert both differentials to the parameter.
3. Evaluate and check the endpoint orientation.

Question 3 – Solution

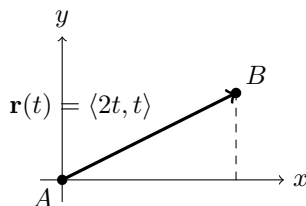
Strategy. A combined differential-form integral becomes one ordinary integral after substituting $dx = x'(t)dt$ and $dy = y'(t)dt$.

Step 1: Parameter data

$$\mathbf{r}(t) = \langle 2t, t \rangle, \quad 0 \leq t \leq 1,$$

so

$$x + y = 3t, \quad x - y = t, \quad dx = 2 dt, \quad dy = dt.$$



Step 2: Evaluate

$$\begin{aligned} \int_C (x + y) dx + (x - y) dy &= \int_0^1 [(3t)(2) + (t)(1)] dt \\ &= \int_0^1 7t dt = \boxed{\frac{7}{2}}. \end{aligned}$$

Verification At $t = 0$ the parametrization gives A , and at $t = 1$ it gives B , so the orientation is correct. Directly grouping the transformed integrand gives $[2(x + y) + (x - y)]dt = (3x + y)dt = 7t dt$, confirming the algebra.

Question 4

Let C_1 be the upper unit semicircle oriented from $(1, 0)$ to $(-1, 0)$, and let C_2 be the same arc with the reverse orientation. Compute

$$\int_C y \, dx$$

for both orientations.

Tasks

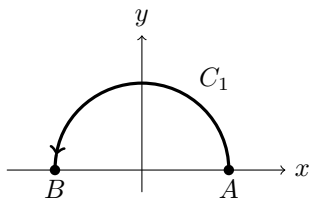
1. Parametrize C_1 and evaluate the integral.
2. Obtain the value on C_2 .
3. Explain why orientation matters for dx but not for ds .

Question 4 – Solution

Strategy. Compute once in the counterclockwise direction, then use the sign change caused by reversing dx .

Step 1: Forward orientation For C_1 , use

$$x = \cos \theta, \quad y = \sin \theta, \quad 0 \leq \theta \leq \pi, \quad dx = -\sin \theta d\theta.$$



reversal changes the sign

Therefore

$$\int_{C_1} y dx = - \int_0^\pi \sin^2 \theta d\theta = \boxed{-\frac{\pi}{2}}.$$

Step 2: Reverse orientation Reversing the same curve negates the coordinate differential, so

$$\boxed{\int_{C_2} y dx = \frac{\pi}{2}}.$$

Step 3: Why the rules differ Under reversal, $dx = x'(t)dt$ changes sign because the derivative reverses. By contrast, $ds = |\mathbf{r}'(t)|dt$ uses a magnitude and remains nonnegative.

Verification On C_1 , $y \geq 0$ while x decreases, so $dx \leq 0$ and the integral must be nonpositive. Its reverse must be nonnegative, matching the two answers.

Question 5

Let C be the boundary of the rectangle

$$0 \leq x \leq 3, \quad 0 \leq y \leq 2,$$

traversed counterclockwise. Evaluate

$$\oint_C y \, dx$$

by splitting the boundary into four segments.

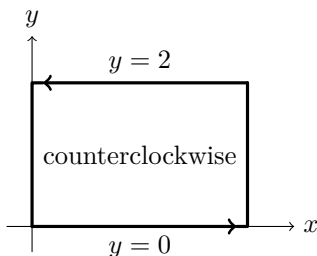
Tasks

1. Determine which edges contribute zero.
2. Evaluate the contribution from each horizontal edge.
3. Add the pieces and check the sign from the orientation.

Question 5 – Solution

Strategy. On vertical edges $dx = 0$, while y is constant on each horizontal edge.

Step 1: Traverse the edges



Along the bottom edge, $y = 0$, so its contribution is 0. On the right and left edges, x is constant, so $dx = 0$ and both contributions are also 0.

Step 2: Top edge Counterclockwise travel moves along the top from $(3, 2)$ to $(0, 2)$. Thus

$$\int_{\text{top}} y \, dx = \int_3^0 2 \, dx = 2[x]_3^0 = -6.$$

Step 3: Sum

$$\oint_C y \, dx = 0 + 0 - 6 + 0 = -6.$$

Verification The only nonzero edge has $y > 0$ and moves left, so $dx < 0$; a negative answer is necessary. Reversing the rectangle would change the answer to $+6$.

Question 6

Let C be the graph $y = x^3$ from $(0, 0)$ to $(1, 1)$. Evaluate

$$\int_C x^2 dy.$$

Tasks

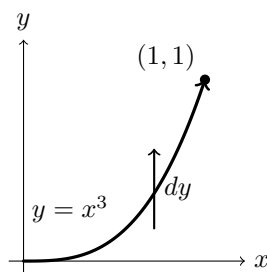
1. Convert dy using the graph equation.
2. Evaluate the resulting single-variable integral.
3. Verify the result by using y as the parameter instead.

Question 6 – Solution

Strategy. First calculate with x as parameter, then reparametrize by y to confirm invariance under a valid change of parameter.

Step 1: Use x

$$y = x^3, \quad dy = 3x^2 dx, \quad 0 \leq x \leq 1.$$



Therefore

$$\int_C x^2 dy = 3 \int_0^1 x^4 dx = 3 \left[\frac{x^5}{5} \right]_0^1 = \boxed{\frac{3}{5}}.$$

Step 2: Use y instead On this interval, $x = y^{1/3}$, so $x^2 = y^{2/3}$. Since the differential is already dy ,

$$\int_C x^2 dy = \int_0^1 y^{2/3} dy = \left[\frac{3}{5} y^{5/3} \right]_0^1 = \frac{3}{5}.$$

Verification Both parametrizations preserve the motion from $(0,0)$ to $(1,1)$ and give the same value, confirming the differential conversion.

Question 7

Let C be the vertical segment $x = 2$ from $(2, 0)$ to $(2, 3)$. Evaluate both

$$\int_C (x + y) \, dx \quad \text{and} \quad \int_C (x + y) \, dy.$$

Tasks

1. Parametrize the vertical segment.
2. Evaluate both integrals.
3. Explain why treating y as a function of x is inappropriate here.

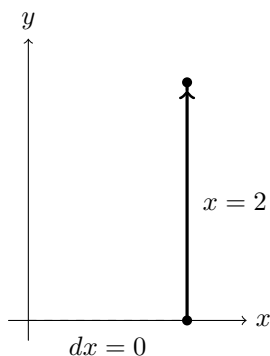
Question 7 – Solution

Strategy. Parametrize by the changing coordinate y ; the constant coordinate has zero differential.

Step 1: Parametrize

$$\mathbf{r}(t) = \langle 2, t \rangle, \quad 0 \leq t \leq 3.$$

Thus $dx = 0$, $dy = dt$, and $x + y = 2 + t$.

**Step 2: Evaluate**

$$\boxed{\int_C (x + y) dx = 0}$$

because $dx = 0$ everywhere. Meanwhile,

$$\int_C (x + y) dy = \int_0^3 (2 + t) dt = \left[2t + \frac{t^2}{2} \right]_0^3 = \boxed{\frac{21}{2}}.$$

Step 3: Representation issue A vertical segment assigns many y -values to the single value $x = 2$, so it cannot be written as a single-valued graph $y = g(x)$. Parametrization avoids this limitation.

Verification The second integrand rises linearly from 2 to 5; its average $7/2$ times the y -interval length 3 gives $21/2$.

Question 8

Let

$$C : \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Evaluate

$$\int_C z \, dx + x \, dy + y \, dz.$$

Tasks

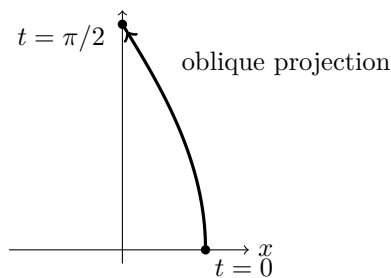
1. Compute dx , dy , and dz .
2. Convert the line integral to one parameter and evaluate it.
3. Verify the cancellation among the three contributions.

Question 8 – Solution

Strategy. Treat each coordinate differential separately, then combine all three ordinary integrals.

Step 1: Differential data

$$\begin{aligned}x &= \cos t, & y &= \sin t, & z &= t, \\dx &= -\sin t \, dt, & dy &= \cos t \, dt, & dz &= dt.\end{aligned}$$



Step 2: Evaluate

$$\int_C z \, dx + x \, dy + y \, dz = \int_0^{\pi/2} (-t \sin t + \cos^2 t + \sin t) \, dt.$$

The three contributions are

$$\int_0^{\pi/2} -t \sin t \, dt = -1, \quad \int_0^{\pi/2} \cos^2 t \, dt = \frac{\pi}{4}, \quad \int_0^{\pi/2} \sin t \, dt = 1.$$

Hence

$$\boxed{\int_C z \, dx + x \, dy + y \, dz = \frac{\pi}{4}}.$$

Verification The first and third contributions cancel exactly, leaving only $\int_0^{\pi/2} \cos^2 t \, dt = \pi/4$.

Question 9

On the parabola $C : y = x^2$ from $(0, 0)$ to $(1, 1)$, a constant a satisfies

$$\int_C ax \, dx + y \, dy = 3.$$

Determine a .

Tasks

1. Convert the entire integral to the parameter x .
2. Solve the resulting equation for a .
3. Substitute the result to verify the specified value.

Question 9 – Solution

Strategy. Use $dy = 2x \, dx$ to make the unknown coefficient appear in a simple algebraic equation.

Step 1: Convert Since $y = x^2$ and $dy = 2x \, dx$,

$$\int_C ax \, dx + y \, dy = \int_0^1 (ax + x^2(2x)) \, dx.$$

Therefore

$$\int_0^1 (ax + 2x^3) \, dx = \frac{a}{2} + \frac{1}{2}.$$

Step 2: Solve

$$\frac{a}{2} + \frac{1}{2} = 3, \quad \frac{a}{2} = \frac{5}{2}, \quad \boxed{a = 5}.$$

Step 3: Verify With $a = 5$,

$$\int_0^1 (5x + 2x^3) \, dx = \left[\frac{5x^2}{2} + \frac{x^4}{2} \right]_0^1 = \frac{5}{2} + \frac{1}{2} = 3.$$

Verification The computed value satisfies the original integral exactly. Because the coefficient of a is $\int_0^1 x \, dx = 1/2 \neq 0$, the solution is unique.

Question 10

Two curves connect $A = (0, 0)$ to $B = (1, 1)$:

$$C_1 : y = x, \quad C_2 : y = x^2, \quad 0 \leq x \leq 1.$$

Compare

$$\int_C y \, dx$$

along the two paths.

Tasks

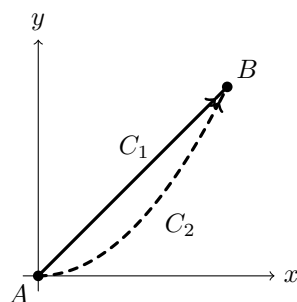
1. Evaluate the integral on C_1 .
2. Evaluate it on C_2 .
3. Explain what the unequal values demonstrate about path dependence.

Question 10 – Solution

Strategy. Use the shared coordinate parameter x on both paths, then compare the scalar functions traced between the same endpoints.

Step 1: Straight path On C_1 , $y = x$, so

$$\int_{C_1} y \, dx = \int_0^1 x \, dx = \boxed{\frac{1}{2}}.$$



Step 2: Parabolic path On C_2 , $y = x^2$, so

$$\int_{C_2} y \, dx = \int_0^1 x^2 \, dx = \boxed{\frac{1}{3}}.$$

Step 3: Interpretation Both curves have the same initial point, terminal point, and orientation, yet

$$\frac{1}{2} \neq \frac{1}{3}.$$

Therefore the value of $\int_C y \, dx$ is not determined by endpoints alone; it depends on the path between them.

Verification For $0 < x < 1$, the line $y = x$ lies above the parabola $y = x^2$. With $dx > 0$, its integral must be larger, consistent with $1/2 > 1/3$.

Question 1

Let

$$\mathbf{F}(x, y) = \langle y, x^2 \rangle, \quad \mathbf{r}(t) = \langle t, t^2 \rangle, \quad 0 \leq t \leq 1.$$

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

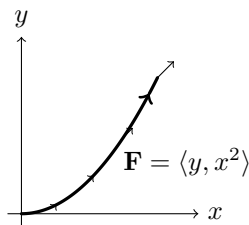
1. Evaluate $\mathbf{F}$ along the curve and compute $\mathbf{r}'(t)$.
2. Form the dot product and evaluate the integral.
3. State how the sign of the integrand agrees with the geometry.

Question 1 – Solution

Strategy. Use $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$.

Step 1: Restrict the field to the curve Since $x = t$ and $y = t^2$,

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^2, t^2 \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t \rangle.$$



Step 2: Form the scalar integrand

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t^2 + 2t^3.$$

Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^3) dt = \left[\frac{t^3}{3} + \frac{t^4}{2} \right]_0^1 = \boxed{\frac{5}{6}}.$$

Verification For $0 \leq t \leq 1$, both components of the field and both components of the velocity are nonnegative. Their dot product must be nonnegative, consistent with the positive answer.

Question 2

A particle follows the helix

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \leq t \leq 2\pi,$$

through the constant force field $\mathbf{F} = \langle 2, -1, 3 \rangle$.

Tasks

1. Compute the work done by the force.
2. Identify the contributions from the horizontal and vertical motion.
3. Check the result using the net displacement.

Question 2 – Solution

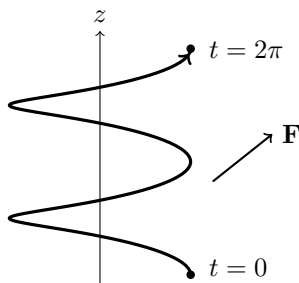
Strategy. Dot the constant force with the velocity, then use the displacement as an independent check.

Step 1: Velocity and dot product

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle,$$

so

$$\mathbf{F} \cdot \mathbf{r}'(t) = -2 \sin t - \cos t + 3.$$

**Step 2: Work**

$$\begin{aligned} W &= \int_0^{2\pi} (-2 \sin t - \cos t + 3) dt \\ &= 0 + 0 + 6\pi = \boxed{6\pi}. \end{aligned}$$

The horizontal coordinates complete a full turn, so their signed contributions cancel. The vertical rise is 2π , producing work $3(2\pi)$.

Verification The net displacement is $\mathbf{r}(2\pi) - \mathbf{r}(0) = \langle 0, 0, 2\pi \rangle$. For a constant force, its dot product with this displacement is $\langle 2, -1, 3 \rangle \cdot \langle 0, 0, 2\pi \rangle = 6\pi$.

Question 3

Let C travel from $A = (0, 0)$ to $B = (2, 0)$ and then from B to $C = (2, 1)$. For

$$\mathbf{F}(x, y) = \langle -y, x \rangle,$$

evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

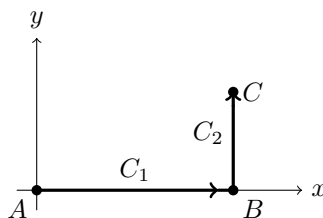
1. Parametrize each segment with the stated orientation.
2. Compute the contribution from each segment.
3. Explain geometrically why the first contribution vanishes.

Question 3 – Solution

Strategy. Split the piecewise path and add the two oriented line integrals.

Step 1: Horizontal segment Use $\mathbf{r}_1(t) = \langle t, 0 \rangle$, $0 \leq t \leq 2$. Then

$$\mathbf{F}(\mathbf{r}_1(t)) = \langle 0, t \rangle, \quad \mathbf{r}'_1(t) = \langle 1, 0 \rangle, \quad \mathbf{F} \cdot \mathbf{r}'_1 = 0.$$



The field is vertical along this segment while the motion is horizontal, so the vectors are perpendicular.

Step 2: Vertical segment Use $\mathbf{r}_2(s) = \langle 2, s \rangle$, $0 \leq s \leq 1$. Thus

$$\mathbf{F}(\mathbf{r}_2(s)) = \langle -s, 2 \rangle, \quad \mathbf{r}'_2(s) = \langle 0, 1 \rangle,$$

and

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2 \, ds = 2.$$

Step 3: Add Hence

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 2 = 2}.$$

Verification Only the upward component on the vertical segment contributes, and that component is the constant 2 over a distance of 1.

Question 4

Let C be the unit circle and let

$$\mathbf{F}(x, y) = \langle -y, x \rangle.$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ when C is traversed (a) counterclockwise and (b) clockwise.

Tasks

1. Parametrize the counterclockwise circle and evaluate the integral.
2. Use orientation reversal to obtain the clockwise value.
3. Interpret the signs using the direction of the field.

Question 4 – Solution

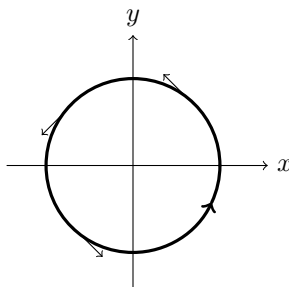
Strategy. On the unit circle this field is the counterclockwise unit tangent, making the dot product especially simple.

Step 1: Counterclockwise traversal Take

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq 2\pi.$$

Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle = \mathbf{r}'(t).$$



field and traversal are counterclockwise

Consequently,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{r}'(t) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} 1 dt = \boxed{2\pi}.$$

Step 2: Reverse the orientation Reversing a vector-field line integral changes its sign, so the clockwise value is

$$\boxed{-2\pi}.$$

Verification The field assists counterclockwise motion at every point and opposes clockwise motion at every point, so the respective signs must be positive and negative.

Question 5

Let

$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle, \quad \mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \quad 0 \leq t \leq 1.$$

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Evaluate the field along the space curve.
2. Compute and integrate the dot product.
3. Verify the calculation by expanding $z \, dx + x \, dy + y \, dz$.

Question 5 – Solution

Strategy. Substitute all three coordinates before taking the dot product with the three-dimensional velocity.

Step 1: Field and velocity

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^3, t, t^2 \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle.$$

Step 2: Dot and integrate

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= t^3 + 2t^2 + 3t^4, \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \left[\frac{t^4}{4} + \frac{2t^3}{3} + \frac{3t^5}{5} \right]_0^1 \\ &= \frac{1}{4} + \frac{2}{3} + \frac{3}{5} = \boxed{\frac{91}{60}}. \end{aligned}$$

Step 3: Differential-form check Along the curve,

$$z \, dx = t^3 \, dt, \quad x \, dy = t(2t \, dt) = 2t^2 \, dt, \quad y \, dz = t^2(3t^2 \, dt) = 3t^4 \, dt.$$

These are exactly the three terms above.

Verification Every term is nonnegative on $[0, 1]$, and the common-denominator sum $(15 + 40 + 36)/60 = 91/60$ confirms the arithmetic.

Question 6

For a constant a , let

$$\mathbf{F}_a(x, y) = \langle ay, x \rangle.$$

Along the parabola $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$, the work done is 2. Determine a .

Tasks

1. Express the work in terms of a .
2. Solve for a .
3. Substitute the result and verify the specified work.

Question 6 – Solution

Strategy. Restrict the one-parameter family of fields to the given curve; the work condition becomes a linear equation in a .

Step 1: Build the integrand

$$\mathbf{F}_a(\mathbf{r}(t)) = \langle at^2, t \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t \rangle.$$

Thus

$$\mathbf{F}_a(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = at^2 + 2t^2 = (a + 2)t^2.$$

Step 2: Impose the work condition

$$2 = \int_0^1 (a + 2)t^2 dt = \frac{a + 2}{3}.$$

Therefore $a + 2 = 6$, and

$$\boxed{a = 4}.$$

Step 3: Verify With $a = 4$,

$$\int_0^1 (4 + 2)t^2 dt = 6 \left[\frac{t^3}{3} \right]_0^1 = 2,$$

which is the required work.

Uniqueness check The coefficient of a in the work is $\int_0^1 t^2 dt = 1/3 \neq 0$, so no second value of a can satisfy the condition.

Question 7

The field $\mathbf{F}(x, y) = \langle -y, x \rangle$ acts between $A = (0, 0)$ and $B = (1, 1)$. Compare the work along

$$C_1 : \mathbf{r}(t) = \langle t, t \rangle, \quad 0 \leq t \leq 1,$$

and the broken path C_2 that goes from $(0, 0)$ to $(1, 0)$ to $(1, 1)$.

Tasks

1. Compute the work along C_1 .
2. Compute the work along both pieces of C_2 .
3. Explain what the comparison demonstrates.

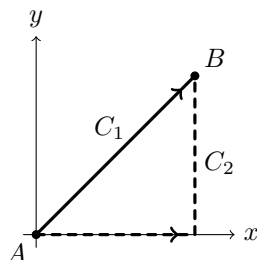
Question 7 – Solution

Strategy. Evaluate the same vector-field line integral on two explicitly parametrized paths with common endpoints.

Step 1: Diagonal path On C_1 ,

$$\mathbf{F}(\mathbf{r}(t)) = \langle -t, t \rangle, \quad \mathbf{r}'(t) = \langle 1, 1 \rangle.$$

The dot product is $-t + t = 0$, so $W_1 = \boxed{0}$.



Step 2: Broken path On the horizontal piece $\langle t, 0 \rangle$, the dot product $\langle 0, t \rangle \cdot \langle 1, 0 \rangle$ is zero. On the vertical piece $\langle 1, s \rangle$,

$$\langle -s, 1 \rangle \cdot \langle 0, 1 \rangle = 1, \quad 0 \leq s \leq 1.$$

Thus

$$W_2 = 0 + \int_0^1 1 \, ds = \boxed{1}.$$

Step 3: Compare Since $W_1 \neq W_2$ despite identical endpoints, the work for this field depends on the path taken.

Verification Along the diagonal the field is perpendicular to the motion. Along the final vertical piece of C_2 , its upward component is 1, explaining the extra unit of work.

Question 8

Let C be the counterclockwise boundary of the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$. For

$$\mathbf{F}(x, y) = \langle y^2, x \rangle,$$

compute the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly from the four edges.

Tasks

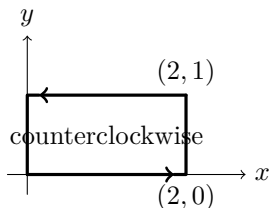
1. Compute each oriented edge contribution.
2. Add the four contributions.
3. Check which signs are forced by the traversal direction.

Question 8 – Solution

Strategy. Write $\mathbf{F} \cdot d\mathbf{r} = y^2 dx + x dy$ and exploit the constant coordinate on each edge.

Step 1: Bottom and right Along the bottom, $y = 0$ and $dy = 0$, so the contribution is 0. Along the right edge, $x = 2$ and $y : 0 \rightarrow 1$, giving

$$\int_0^1 2 dy = 2.$$



Step 2: Top and left Along the top, $y = 1$ and counterclockwise travel has $x : 2 \rightarrow 0$:

$$\int_2^0 1 dx = -2.$$

On the left, $x = 0$, and $dx = 0$, so the contribution is 0.

Step 3: Sum

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 + 2 - 2 + 0 = 0.$$

Verification The right-edge contribution is positive because both the field's vertical component and dy are positive. The top contribution is negative because $y^2 > 0$ while $dx < 0$; their equal magnitudes cancel.

Question 9

Let C be the line segment from $(0,0)$ to $(1,2)$ in the field

$$\mathbf{F}(x,y) = \langle x-y, x+y \rangle.$$

Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ using both

$$\mathbf{r}_1(t) = \langle t, 2t \rangle, \ 0 \leq t \leq 1, \quad \text{and} \quad \mathbf{r}_2(u) = \langle u^2, 2u^2 \rangle, \ 0 \leq u \leq 1.$$

Tasks

1. Evaluate the integral with each parametrization.
2. Explain why the zero velocity of $\mathbf{r}_2$ at $u = 0$ causes no problem.
3. State the invariance illustrated by the equal answers.

Question 9 – Solution

Strategy. Carry the parameter derivative through both computations; the slower start must be included in dr .

Step 1: First parametrization Here

$$\mathbf{F}(\mathbf{r}_1(t)) = \langle -t, 3t \rangle, \quad \mathbf{r}'_1(t) = \langle 1, 2 \rangle.$$

Thus

$$\int_0^1 (-t + 6t) dt = \int_0^1 5t dt = \boxed{\frac{5}{2}}.$$

Step 2: Nonuniform parametrization Now

$$\mathbf{F}(\mathbf{r}_2(u)) = \langle -u^2, 3u^2 \rangle, \quad \mathbf{r}'_2(u) = \langle 2u, 4u \rangle.$$

Therefore

$$\int_0^1 (-2u^3 + 12u^3) du = \int_0^1 10u^3 du = \boxed{\frac{5}{2}}.$$

Step 3: Interpret Although $\mathbf{r}'_2(0) = \mathbf{0}$, the curve pauses only at one endpoint and then traces the segment once with the same orientation. The derivative factor compensates for its varying speed.

Verification Both maps begin at $(0, 0)$, end at $(1, 2)$, and move monotonically along the same segment, so parameter independence requires the matching values obtained above.

Question 10

On the counterclockwise circle $C : x^2 + y^2 = 9$, consider

$$\mathbf{F}(x, y) = \langle x - 2y, 2x + y \rangle.$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by separating $\mathbf{F}$ into radial and tangential parts.

Tasks

1. Decompose the field into radial and tangential vectors.
2. Determine the work contributed by each part.
3. Verify the answer by direct parametrization.

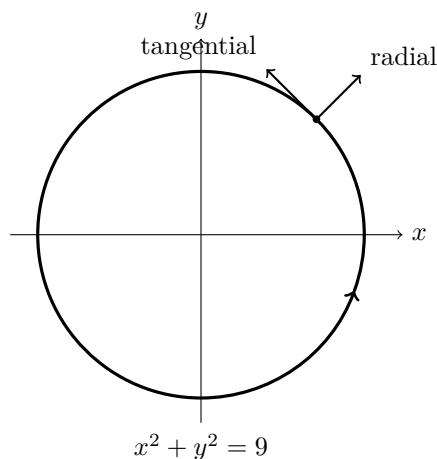
Question 10 – Solution

Strategy. Write the field as $\langle x, y \rangle + 2\langle -y, x \rangle$; on a circle the first part is normal and the second is tangent.

Step 1: Decompose

$$\mathbf{F} = \underbrace{\langle x, y \rangle}_{\text{radial}} + \underbrace{2\langle -y, x \rangle}_{\text{counterclockwise tangential}}.$$

The radial part is perpendicular to the circle's tangent, so it contributes zero work.



On the radius-3 circle, $\langle -y, x \rangle$ has magnitude 3 in the positive tangent direction. Hence the doubled part has tangential magnitude 6. Multiplying by the circumference 6π gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 + 6(6\pi) = \boxed{36\pi}.$$

Step 2: Direct verification Set $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t \rangle$, $0 \leq t \leq 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 18,$$

because the radial terms cancel and the tangential terms sum to $18(\sin^2 t + \cos^2 t)$. Therefore

$$\int_0^{2\pi} 18 \, dt = 36\pi,$$

confirming both the magnitude and the positive orientation.

Question 1

Let

$$f(x, y, z) = x^2y + e^z,$$

and let C be any smooth curve from $A = (1, 0, 0)$ to $B = (2, 1, \ln 2)$. Evaluate

$$\int_C \nabla f \cdot d\mathbf{r}.$$

Tasks

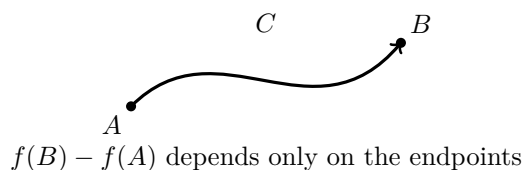
1. State the Fundamental Theorem for Line Integrals.
2. Evaluate the integral without parametrizing C .
3. Explain why the unspecified shape of C is irrelevant.

Question 1 – Solution

Strategy. Since the field is explicitly ∇f , apply the endpoint formula $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$.

Step 1: Evaluate the endpoint values

$$f(B) = 2^2(1) + e^{\ln 2} = 4 + 2 = 6, \quad f(A) = 1^2(0) + e^0 = 1.$$



Step 2: Apply the theorem

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 6 - 1 = \boxed{5}.$$

Step 3: Explain path independence Along any smooth parametrization $\mathbf{r}(t)$, $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ is the derivative of $f(\mathbf{r}(t))$. Its integral therefore records only the net change in f .

Verification The logarithmic coordinate was chosen so that $e^{\ln 2} = 2$; using $\ln 2$ itself in $f(B)$ would be an endpoint-substitution error.

Question 2

Let $f(x, y) = x^2 + y^2$. A smooth curve C travels from $A = (3, 4)$ to $B = (0, 5)$.

Tasks

1. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$.
2. Interpret the result using level curves of f .
3. Decide whether a different smooth path from A to B could change the value.

Question 2 – Solution

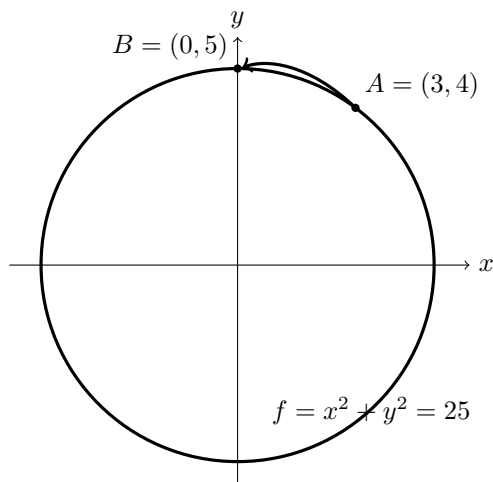
Strategy. Compare the potential values at the endpoints before doing any curve calculation.

Step 1: Endpoint values

$$f(A) = 3^2 + 4^2 = 25, \quad f(B) = 0^2 + 5^2 = 25.$$

Hence

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 0.}$$



Step 2: Interpret Both endpoints lie on the level circle $x^2 + y^2 = 25$. The curve between them need not stay on that circle: f may rise and fall along the way, but its total change is still zero.

Step 3: Compare paths Any other smooth path with the same orientation from A to B gives the same endpoint difference, so it cannot change the integral.

Verification The result is not claiming that ∇f vanishes. Indeed $\nabla f = \langle 2x, 2y \rangle$ is nonzero at both endpoints; only the net change in f is zero.

Question 3

Let

$$f(x, y, z) = xy + z^2,$$

and let C be any piecewise smooth closed curve contained in the domain of f . Evaluate

$$\oint_C \nabla f \cdot d\mathbf{r}.$$

Tasks

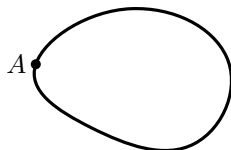
1. Apply the endpoint form of the theorem to the closed curve.
2. Explain the role of the curve's initial and terminal points.
3. State whether self-intersections affect the result.

Question 3 – Solution

Strategy. Regard a closed curve as an oriented path whose terminal point equals its initial point.

Step 1: Name the common endpoint Let A be the point at which the parametrization starts and ends. The theorem gives

$$\oint_C \nabla f \cdot d\mathbf{r} = f(A) - f(A) = \boxed{0}.$$



initial point = terminal point

Step 2: Interpret The integral measures the accumulated change of f during one traversal. Returning to the starting point returns to the same value of f , so all increases and decreases cancel.

Step 3: Self-intersections Self-intersections do not alter the endpoint calculation. Each piece contributes the change in f across that piece, and the intermediate values telescope when the pieces are added.

Verification Reversing the closed curve would negate its integral, but the negative of zero is still zero, consistent with the answer.

Question 4

Let $f(x, y) = xe^y$. A curve C is oriented from $A = (0, 0)$ to $B = (2, \ln 3)$, and $-C$ denotes the same curve with the reverse orientation.

Tasks

1. Compute $\int_C \nabla f \cdot d\mathbf{r}$.
2. Compute $\int_{-C} \nabla f \cdot d\mathbf{r}$.
3. Explain the sign change using both endpoints and orientation reversal.

Question 4 – Solution

Strategy. Evaluate the same endpoint difference in the two opposite orders.

Step 1: Forward orientation

$$f(A) = 0e^0 = 0, \quad f(B) = 2e^{\ln 3} = 6.$$

Thus

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = \boxed{6}.$$

Step 2: Reverse orientation The initial point of $-C$ is B and its terminal point is A , so

$$\int_{-C} \nabla f \cdot d\mathbf{r} = f(A) - f(B) = \boxed{-6}.$$

Step 3: Connect the rules Swapping the endpoints reverses the subtraction. This is exactly the general orientation rule

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Verification Adding the forward and reverse traversals makes a closed trip, and $6 + (-6) = 0$, as the theorem requires for a gradient field.

Question 5

Let $f(x, y) = x^2y$ and let C be the quarter unit circle

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ in two ways.

Tasks

1. Use the Fundamental Theorem for Line Integrals.
2. Evaluate directly from $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$.
3. Identify the chain-rule antiderivative in the direct calculation.

Question 5 – Solution

Strategy. First use the endpoints, then verify that the parameter integrand is the derivative of $f(\mathbf{r}(t))$.

Step 1: Endpoint method The curve begins at $(1, 0)$ and ends at $(0, 1)$. Both endpoint values are zero:

$$f(1, 0) = 0, \quad f(0, 1) = 0.$$

Therefore the theorem gives $\boxed{0}$.

Step 2: Direct method Since $\nabla f = \langle 2xy, x^2 \rangle$,

$$\begin{aligned}\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \langle 2 \cos t \sin t, \cos^2 t \rangle \cdot \langle -\sin t, \cos t \rangle \\ &= -2 \cos t \sin^2 t + \cos^3 t.\end{aligned}$$

This is

$$\frac{d}{dt}(\cos^2 t \sin t) = \frac{d}{dt}f(\mathbf{r}(t)).$$

Hence

$$\int_0^{\pi/2} \frac{d}{dt}(\cos^2 t \sin t) dt = [\cos^2 t \sin t]_0^{\pi/2} = \boxed{0}.$$

Verification The two methods agree, and the displayed antiderivative explicitly exhibits the chain rule behind the theorem.

Question 6

Let $f(x, y) = x^2 + y^2$. A curve begins at $A = (1, 2)$ and ends at a first-quadrant point B on the line $y = x + 1$. Suppose

$$\int_C \nabla f \cdot d\mathbf{r} = 20.$$

Determine B .

Tasks

1. Convert the work condition into a level-curve condition on B .
2. Solve the line-circle intersection equation.
3. Use the quadrant restriction to select and verify the endpoint.

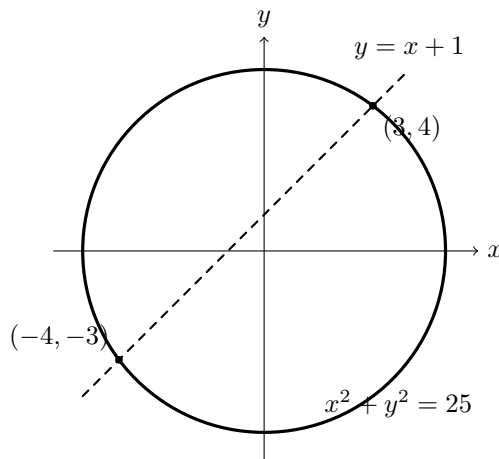
Question 6 – Solution

Strategy. Use the theorem in reverse: the known integral determines the terminal potential value.

Step 1: Determine the level Since $f(A) = 1^2 + 2^2 = 5$,

$$20 = f(B) - 5, \quad f(B) = 25.$$

Thus B lies on $x^2 + y^2 = 25$ as well as $y = x + 1$.



Step 2: Solve the intersection Substitute $y = x + 1$:

$$x^2 + (x + 1)^2 = 25 \implies 2x^2 + 2x - 24 = 0 \implies (x - 3)(x + 4) = 0.$$

The candidates are $(3, 4)$ and $(-4, -3)$.

Step 3: Apply the restriction Only $(3, 4)$ lies in the first quadrant, so

$$\boxed{B = (3, 4)}.$$

Verification The line condition gives $4 = 3 + 1$, and $f(B) - f(A) = 25 - 5 = 20$, exactly the specified integral.

Question 7

Let

$$f(x, y) = x^3 - 3xy + y^2.$$

A piecewise smooth path C travels from $A = (0, 0)$ to $(1, 0)$, then to $(1, 1)$, and finally to $B = (2, 1)$.

Tasks

1. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ using the theorem.
2. Show how the endpoint changes telescope across the three pieces.
3. Explain why the corners do not invalidate the theorem.

Question 7 – Solution

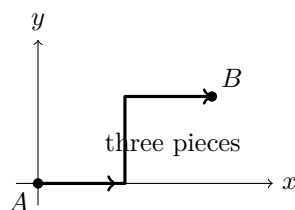
Strategy. Apply the theorem to the whole piecewise smooth path, then expose the same cancellation piece by piece.

Step 1: Whole-path calculation

$$f(A) = 0, \quad f(B) = 2^3 - 3(2)(1) + 1^2 = 8 - 6 + 1 = 3.$$

Therefore

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = 3}.$$



Step 2: Telescope Let $P = (1, 0)$ and $Q = (1, 1)$. Then

$$\begin{aligned} \int_C \nabla f \cdot d\mathbf{r} &= [f(P) - f(A)] + [f(Q) - f(P)] + [f(B) - f(Q)] \\ &= f(B) - f(A) = 3. \end{aligned}$$

Every intermediate potential value appears once positively and once negatively.

Step 3: Corners Smoothness is required on each segment, not at the finitely many joining corners. Adding the valid integral on each piece yields the integral on the piecewise smooth path.

Verification Direct values $f(P) = 1$ and $f(Q) = -1$ give contributions 1, -2 , 4, whose sum is 3.

Question 8

Let

$$f(x, y, z) = xyz + \frac{1}{2}(x^2 + y^2 + z^2).$$

Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ for any smooth space curve from $A = (1, -1, 0)$ to $B = (2, 1, 3)$.

Tasks

1. Compute $f(A)$ and $f(B)$ carefully.
2. Apply the Fundamental Theorem for Line Integrals.
3. Verify that the associated vector field is three-dimensional by computing ∇f .

Question 8 – Solution

Strategy. The spatial geometry of C is unnecessary because the integrand is the gradient of the supplied scalar function.

Step 1: Initial value

$$f(1, -1, 0) = (1)(-1)(0) + \frac{1}{2}(1 + 1 + 0) = 1.$$

Step 2: Terminal value

$$f(2, 1, 3) = (2)(1)(3) + \frac{1}{2}(4 + 1 + 9) = 6 + 7 = 13.$$

Thus

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 13 - 1 = 12}.$$

Step 3: Display the field Differentiating componentwise gives

$$\nabla f = \langle yz + x, xz + y, xy + z \rangle.$$

This confirms that the line integral uses all three coordinate components, even though no parametrization is needed.

Verification The quadratic contribution changes from 1 to 7, a gain of 6, and the product contribution changes from 0 to 6, another gain of 6. The total change is 12.

Question 9

A force is derived from the potential energy

$$U(x, y) = x^2 + 2y^2$$

by $\mathbf{F} = -\nabla U$. A particle moves along any smooth curve from $A = (2, 1)$ to $B = (0, 3)$.

Tasks

1. Compute the work $W = \int_C \mathbf{F} \cdot d\mathbf{r}$.
2. Relate the work to the change in potential energy.
3. Explain the sign of the answer.

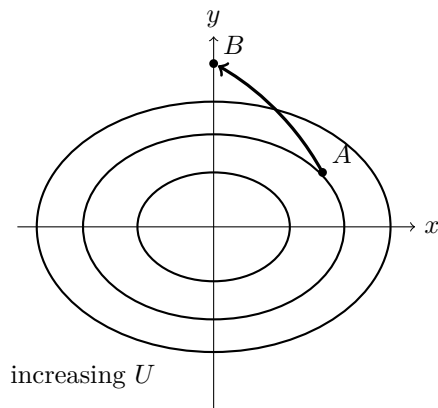
Question 9 – Solution

Strategy. Apply the theorem to $-U$: work by $-\nabla U$ is the negative of the change in U .

Step 1: Potential-energy change

$$U(A) = 2^2 + 2(1^2) = 6, \quad U(B) = 0^2 + 2(3^2) = 18.$$

Therefore $\Delta U = U(B) - U(A) = 12$.



Step 2: Work Since $\mathbf{F} = \nabla(-U)$,

$$\begin{aligned} W &= (-U)(B) - (-U)(A) \\ &= U(A) - U(B) = 6 - 18 = \boxed{-12}. \end{aligned}$$

Equivalently, $W = -\Delta U = -12$.

Step 3: Interpret the sign The particle ends at a point with greater potential energy. The force derived from $-\nabla U$ therefore does negative net work during the motion.

Verification The identity $W + \Delta U = -12 + 12 = 0$ checks the sign. Reversing the trip would produce work $+12$.

Question 10

Let f have continuous first partial derivatives on an open region containing a smooth curve $\mathbf{r}(t)$, $a \leq t \leq b$. Prove

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Tasks

1. Rewrite the line integral using the parametrization.
2. Apply the multivariable chain rule.
3. Finish with the one-variable Fundamental Theorem of Calculus and identify the hypotheses used.

Question 10 – Solution

Strategy. Show that the vector-field integrand is exactly the ordinary derivative of the composite function $f \circ \mathbf{r}$.

Step 1: Parameter form Write $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. By definition,

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Expanding the dot product gives

$$\int_a^b (f_x x'(t) + f_y y'(t) + f_z z'(t)) dt,$$

where the partial derivatives are evaluated at $\mathbf{r}(t)$.

Step 2: Apply the chain rule The multivariable chain rule states

$$\frac{d}{dt} f(\mathbf{r}(t)) = f_x x'(t) + f_y y'(t) + f_z z'(t).$$

Thus the line integral is $\int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt$.

Step 3: Apply the one-variable theorem

$$\int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt = \boxed{f(\mathbf{r}(b)) - f(\mathbf{r}(a))}.$$

The smoothness of $\mathbf{r}$ and continuity of the first partial derivatives make the chain-rule derivative continuous and integrable. The same proof applies piece by piece to a piecewise smooth curve, with intermediate endpoint values canceling.

Verification The right side reverses sign when a and b are interchanged, matching orientation reversal of the line integral.

Question 1

Determine whether

$$\mathbf{F}(x, y) = \langle 2xy + 3, x^2 + 4y \rangle$$

is conservative on $\mathbb{R}^2$. If it is, find a potential function f .

Tasks

1. Compare the relevant mixed partial derivatives.
2. Construct f and determine the one-variable correction term.
3. Differentiate the result to verify the field.

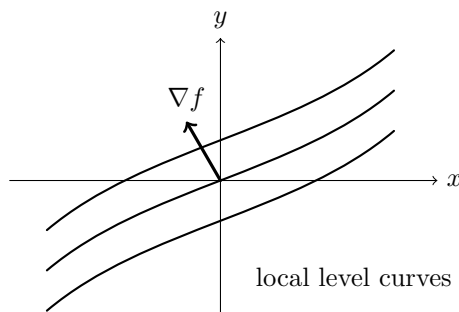
Question 1 – Solution

Strategy. Test $P_y = Q_x$ on the simply connected domain, then integrate one component and recover the missing function.

Step 1: Test the field With $P = 2xy + 3$ and $Q = x^2 + 4y$,

$$P_y = 2x, \quad Q_x = 2x.$$

The components have continuous first partial derivatives on all of $\mathbb{R}^2$, which is simply connected. Thus the equality guarantees that $\mathbf{F}$ is conservative.



Step 2: Construct a potential Integrating $f_x = P$ with respect to x gives

$$f = x^2y + 3x + g(y).$$

Then $f_y = x^2 + g'(y) = Q = x^2 + 4y$, so $g'(y) = 4y$ and $g(y) = 2y^2 + C$.

$$\boxed{f(x, y) = x^2y + 3x + 2y^2 + C}.$$

Verification Differentiation gives $\nabla f = \langle 2xy + 3, x^2 + 4y \rangle = \mathbf{F}$. The arbitrary constant has zero gradient.

Question 2

Consider

$$\mathbf{F}(x, y) = \langle y^2, 2xy + x \rangle$$

on $\mathbb{R}^2$.

Tasks

1. Test whether $\mathbf{F}$ is conservative.
2. Identify the exact obstruction.
3. Explain why no potential function can exist even locally near any point.

Question 2 – Solution

Strategy. A continuously differentiable gradient field must satisfy equality of the cross partials.

Step 1: Compute the cross partials Let $P = y^2$ and $Q = 2xy + x$. Then

$$P_y = 2y, \quad Q_x = 2y + 1.$$

Since $P_y \neq Q_x$ at every point, the necessary condition fails.

$\mathbf{F}$ is not conservative on $\mathbb{R}^2$.

Step 2: Identify the contradiction If $\mathbf{F} = \nabla f$ for a twice continuously differentiable f , then

$$f_x = P, \quad f_y = Q,$$

and hence $f_{xy} = P_y$ and $f_{yx} = Q_x$. Equality of mixed partials would require $2y = 2y + 1$, which is impossible.

Step 3: Local conclusion The mismatch is the constant 1 everywhere, not merely at an isolated point. Every open neighborhood contains points where the necessary equality fails, so no local potential exists on any such neighborhood.

Verification Integrating P gives $f = xy^2 + g(y)$, whose y -derivative is $2xy + g'(y)$. Matching Q would require $g'(y) = x$, impossible because g' cannot depend on x .

Question 3

For a real constant a , let

$$\mathbf{F}_a(x, y) = \langle ay + 2xy, 3x + x^2 \rangle.$$

Determine the value of a for which $\mathbf{F}_a$ is conservative on $\mathbb{R}^2$, and find a potential for that value.

Tasks

1. Use the mixed-partial condition to determine a .
2. Construct a potential function.
3. Verify both components and uniqueness of the parameter.

Question 3 – Solution

Strategy. Make the cross-partial identity hold for every point, not just at a selected point.

Step 1: Determine the parameter Write $P = ay + 2xy$ and $Q = 3x + x^2$. Then

$$P_y = a + 2x, \quad Q_x = 3 + 2x.$$

Equality for all x requires

$$\boxed{a = 3}.$$

No other constant can make the two affine expressions identical.

Step 2: Construct the potential For $a = 3$,

$$f_x = 3y + 2xy.$$

Integrating with respect to x gives

$$f = 3xy + x^2y + g(y).$$

Now $f_y = 3x + x^2 + g'(y)$ must equal $Q = 3x + x^2$, so $g'(y) = 0$.

$$\boxed{f(x, y) = 3xy + x^2y + C}.$$

Verification The gradient is $\langle 3y + 2xy, 3x + x^2 \rangle$, exactly $\mathbf{F}_3$. The coefficient comparison also proves the value $a = 3$ is unique.

Question 4

On the punctured plane $D = \mathbb{R}^2 \setminus \{(0, 0)\}$, consider

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

Investigate whether $\mathbf{F}$ is conservative on D .

Tasks

1. Verify that $P_y = Q_x$ on D .
2. Evaluate the line integral around the counterclockwise unit circle.
3. Reconcile the two results using the geometry of the domain.

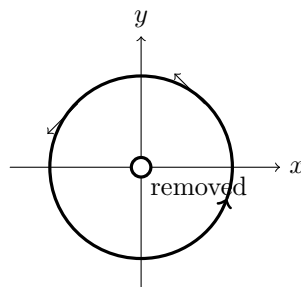
Question 4 – Solution

Strategy. The cross-partial test needs a suitable domain; use a closed-loop integral to detect the hole.

Step 1: Cross partials For $(x, y) \neq (0, 0)$,

$$P_y = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad Q_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Thus the local equality holds throughout D .



$\mathbb{R}^2 \setminus \{(0, 0)\}$ has a hole

Step 2: Test a closed circle Set $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle = \mathbf{r}'(t),$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 \, dt = \boxed{2\pi}.$$

A conservative field must have zero integral around every closed curve. Therefore

$$\boxed{\mathbf{F} \text{ is not conservative on } D}.$$

Step 3: Reconcile The punctured plane is not simply connected: the unit circle encloses the missing origin and cannot contract to a point within D . Consequently $P_y = Q_x$ is not sufficient here.

Question 5

Restrict the field

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

to the right half-plane $H = \{(x, y) : x > 0\}$.

Tasks

1. Find a potential function on H .
2. Verify its gradient.
3. Evaluate the line integral from $A = (1, 1)$ to $B = (\sqrt{3}, 1)$ along any path in H .

Question 5 – Solution

Strategy. The right half-plane permits a single-valued angle function with no branch crossing.

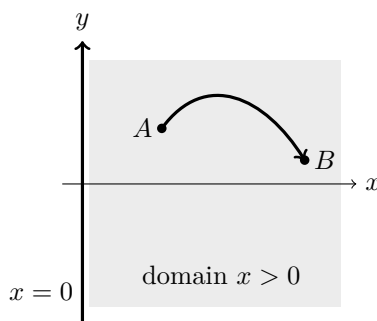
Step 1: Choose the angle potential On $x > 0$, define

$$f(x, y) = \arctan\left(\frac{y}{x}\right).$$

Using the chain rule,

$$f_x = \frac{-y}{x^2 + y^2}, \quad f_y = \frac{x}{x^2 + y^2}.$$

Hence $\nabla f = \mathbf{F}$ on H .



Step 2: Evaluate by endpoints The endpoint angles in the branch $(-\pi/2, \pi/2)$ are

$$f(A) = \arctan(1) = \frac{\pi}{4}, \quad f(B) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Therefore

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) = -\frac{\pi}{12}}.$$

Verification Both endpoints and the entire allowed path lie in $x > 0$, where the selected arctangent branch is continuous. The negative sign matches the decrease in polar angle from $\pi/4$ to $\pi/6$.

Question 6

Determine whether the three-dimensional field

$$\mathbf{F}(x, y, z) = \langle 2xy + z, x^2 + 2y, x + 2z \rangle$$

is conservative on $\mathbb{R}^3$. If so, find a potential and use it to evaluate the integral from $(0, 0, 0)$ to $(1, 2, 3)$.

Tasks

1. Check the three pairwise cross-partial relations.
2. Construct and verify a potential.
3. Evaluate the requested line integral by endpoints.

Question 6 – Solution

Strategy. Verify compatibility of all three components, then integrate successively while retaining correction functions.

Step 1: Compatibility Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$P_y = 2x = Q_x, \quad P_z = 1 = R_x, \quad Q_z = 0 = R_y.$$

On the simply connected domain $\mathbb{R}^3$, these identities show the field is conservative.

Step 2: Construct f Integrating $f_x = P$ gives

$$f = x^2y + xz + g(y, z).$$

From $f_y = x^2 + g_y = Q$, we get $g_y = 2y$, so $g = y^2 + h(z)$. Finally,

$$f_z = x + h'(z) = R = x + 2z,$$

so $h = z^2 + C$. Thus

$$\boxed{f = x^2y + xz + y^2 + z^2 + C}.$$

Step 3: Endpoint evaluation

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 3) - f(0, 0, 0) = 2 + 3 + 4 + 9 = \boxed{18}.$$

Verification Differentiating the boxed potential returns all three original components; the additive constant cancels from the endpoint difference.

Question 7

Suppose f and g are potential functions for the same continuous vector field $\mathbf{F}$ on a connected open region D .

Tasks

1. Prove that $f - g$ is constant on D .
2. Explain why connectedness is needed.
3. State the resulting uniqueness principle for potential functions.

Question 7 – Solution

Strategy. Subtract the two gradient equations and use paths inside the connected open region.

Step 1: Subtract gradients Since $\nabla f = \mathbf{F} = \nabla g$,

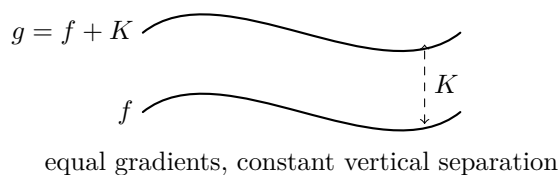
$$\nabla(f - g) = \mathbf{0}.$$

Let $A, B \in D$. An open connected region in $\mathbb{R}^n$ is path connected, so choose a piecewise smooth curve C in D from A to B . Then

$$(f - g)(B) - (f - g)(A) = \int_C \nabla(f - g) \cdot d\mathbf{r} = 0.$$

Thus $(f - g)(B) = (f - g)(A)$ for every pair, and

$$\boxed{f - g = K \text{ on } D}.$$



Step 2: Why connectedness matters On separate components, the same zero gradient permits a different constant on each component because no path in D joins them.

Step 3: Uniqueness principle A potential on a connected region is unique up to one additive constant. Adding a constant changes function values but leaves every partial derivative unchanged.

Verification If $g = f + K$, then $\nabla g = \nabla f + \nabla K = \nabla f$, confirming that every additive constant really does produce another potential.

Question 8

Let $\mathbf{F}$ be continuous on a region where the indicated curves lie. Prove that the following statements are equivalent:

1. line integrals of $\mathbf{F}$ are path independent;
2. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C .

Tasks

1. Prove path independence implies the closed-loop condition.
2. Prove the converse using two paths with common endpoints.
3. Track orientation carefully in the concatenated curve.

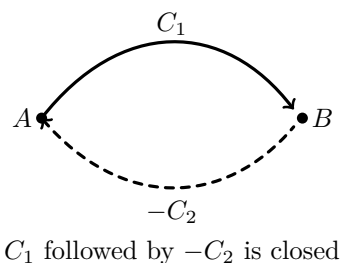
Question 8 – Solution

Strategy. A closed curve can be compared with a constant path, while two endpoint-sharing paths form a closed curve when one is reversed.

Step 1: Path independence $\Rightarrow$ zero loops Let a closed curve start and end at A . Compare it with the constant path at A , whose line integral is zero. Path independence gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Step 2: Zero loops $\Rightarrow$ path independence Let C_1 and C_2 both run from A to B . Follow C_1 from A to B , then follow $-C_2$ back to A . This concatenation is closed.



By the assumed closed-loop condition,

$$0 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Therefore

$$\boxed{\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}},$$

which is path independence.

Verification The reversal is essential: following C_2 in its original direction would begin at A , not at the terminal point B of C_1 , and would not close the loop.

Question 9

A field on $\mathbb{R}^2$ has the form

$$\mathbf{F}(x, y) = \langle 2xy + y, Q(x, y) \rangle.$$

It is conservative and satisfies $Q(0, y) = e^y$.

Tasks

1. Determine the most general Q allowed by the mixed-partial condition.
2. Use the boundary data to determine Q uniquely.
3. Find and verify a potential function.

Question 9 – Solution

Strategy. Integrate the compatibility equation for the unknown component, then use the supplied trace data to fix its arbitrary function.

Step 1: Compatibility equation With $P = 2xy + y$,

$$P_y = 2x + 1.$$

Conservativeness requires $Q_x = P_y$, so integrating with respect to x gives

$$Q(x, y) = x^2 + x + h(y).$$

Step 2: Apply the boundary data At $x = 0$,

$$Q(0, y) = h(y) = e^y.$$

Thus

$$\boxed{Q(x, y) = x^2 + x + e^y}.$$

Step 3: Find a potential Integrating $f_x = P$ yields

$$f = x^2y + xy + g(y).$$

Then

$$f_y = x^2 + x + g'(y) = Q$$

requires $g'(y) = e^y$, so

$$\boxed{f(x, y) = x^2y + xy + e^y + C}.$$

Verification Its gradient is $\langle 2xy + y, x^2 + x + e^y \rangle$, and setting $x = 0$ in the second component gives exactly e^y .

Question 10

Classify the linear vector fields

$$\mathbf{F}(x, y) = \langle ax + by, cx + dy \rangle,$$

where a, b, c, d are real constants.

Tasks

1. Find the necessary and sufficient condition for $\mathbf{F}$ to be conservative on $\mathbb{R}^2$.
2. Construct the general potential when the condition holds.
3. Under that condition, compute the work from $(0, 0)$ to (p, q) .

Question 10 – Solution

Strategy. Compare the constant cross partials, then integrate symbolically without assigning numerical coefficients.

Step 1: Classify Let $P = ax + by$ and $Q = cx + dy$. Then

$$P_y = b, \quad Q_x = c.$$

Because $\mathbb{R}^2$ is simply connected, equality is both necessary and sufficient. Hence

$$\boxed{\mathbf{F} \text{ is conservative exactly when } b = c}.$$

Step 2: Construct the potential Assume $b = c$. Integrating $f_x = ax + by$ gives

$$f = \frac{a}{2}x^2 + bxy + g(y).$$

Since $f_y = bx + g'(y) = bx + dy$, we have $g'(y) = dy$, so

$$\boxed{f(x, y) = \frac{a}{2}x^2 + bxy + \frac{d}{2}y^2 + C}.$$

Step 3: Compute the work The Fundamental Theorem for Line Integrals gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(p, q) - f(0, 0) = \boxed{\frac{a}{2}p^2 + bpq + \frac{d}{2}q^2}.$$

Verification Differentiating the potential gives $\langle ax + by, bx + dy \rangle$, which equals the original field precisely because $c = b$. The additive constant cancels from the work.

Question 1

Let C be the positively oriented boundary of the rectangle

$$D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 3\}.$$

Evaluate

$$\oint_C (x^2 - y) dx + (x + y^2) dy$$

using Green's Theorem.

Tasks

1. Identify P , Q , and the required derivative difference.
2. Convert the line integral to a double integral.
3. Evaluate and check the positive orientation.

Question 1 – Solution

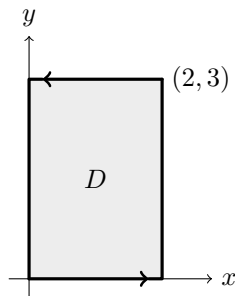
Strategy. Apply the circulation form $\oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA$.

Step 1: Differentiate Here

$$P = x^2 - y, \quad Q = x + y^2,$$

so

$$Q_x - P_y = 1 - (-1) = 2.$$



Step 2: Integrate over the region

$$\oint_C P dx + Q dy = \iint_D 2 dA = 2 \text{Area}(D) = 2(2)(3) = \boxed{12}.$$

Step 3: Orientation Positive orientation means counterclockwise traversal, with the region on the left. That is the orientation for which Green's Theorem gives the displayed positive sign.

Verification The derivative difference is the constant 2 and the rectangle has area 6, so the value must be 12; reversing C would give -12 .

Question 2

Let C be the counterclockwise ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Evaluate

$$\frac{1}{2} \oint_C (-y \, dx + x \, dy)$$

and explain its geometric meaning.

Tasks

1. Apply Green's Theorem to the given integral.
2. Compute the enclosed area.
3. Explain the effect of reversing the ellipse's orientation.

Question 2 – Solution

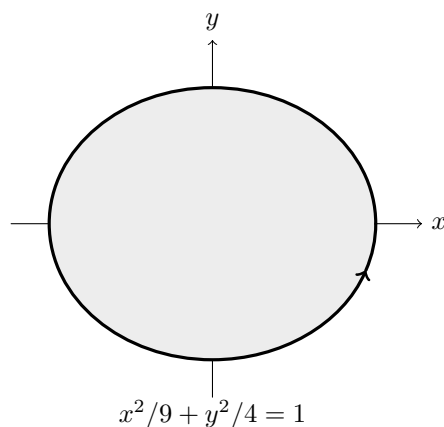
Strategy. The chosen coefficients make $Q_x - P_y = 1$, so the line integral measures area.

Step 1: Set P and Q Write

$$P = -\frac{y}{2}, \quad Q = \frac{x}{2}.$$

Then

$$Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1.$$



Step 2: Apply Green's Theorem

$$\frac{1}{2} \oint_C (-y \, dx + x \, dy) = \iint_D 1 \, dA = \text{Area}(D).$$

The ellipse has semiaxes 3 and 2, so

$$\boxed{\frac{1}{2} \oint_C (-y \, dx + x \, dy) = \pi(3)(2) = 6\pi}.$$

Step 3: Reverse orientation Clockwise traversal negates the line integral, producing -6π . Geometric area itself remains 6π ; the sign records orientation.

Verification The area formula reduces to πab , and substituting $a = 3$, $b = 2$ confirms 6π .

Question 3

The planar field

$$\mathbf{F}(x, y) = \langle x^2, xy \rangle$$

acts on the disk $D : (x - 1)^2 + y^2 \leq 1$. Let $C = \partial D$ be counterclockwise. Find the outward flux of $\mathbf{F}$ across C .

Tasks

1. State the flux form of Green's Theorem.
2. Compute the divergence and its integral over D .
3. Use symmetry or the centroid of the disk to finish.

Question 3 – Solution

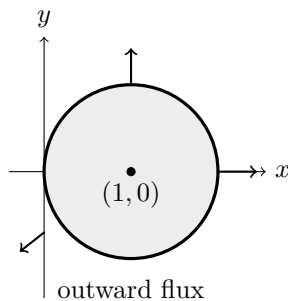
Strategy. Convert outward flux to the double integral of the planar divergence.

Step 1: Flux form For $\mathbf{F} = \langle M, N \rangle$,

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C M \, dy - N \, dx = \iint_D (M_x + N_y) \, dA.$$

Here

$$M_x + N_y = 2x + x = 3x.$$



Step 2: Integrate The disk has area π and centroid $(1, 0)$. Therefore its average x -coordinate is 1, so

$$\iint_D x \, dA = 1 \cdot \pi = \pi.$$

Consequently,

$$\boxed{\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D 3x \, dA = 3\pi.}$$

Verification The disk lies in $0 \leq x \leq 2$, so the divergence $3x$ is nonnegative. A positive outward flux is therefore consistent with the sign of the integrand.

Question 4

Let D be the annulus $1 \leq x^2 + y^2 \leq 4$ and

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

The boundary consists of the outer circle and the inner circle.

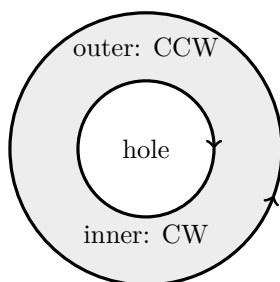
Tasks

1. Specify the positive orientation of both boundary components.
2. Apply Green's Theorem on the annulus.
3. Compute each circle's contribution and verify the cancellation.

Question 4 – Solution

Strategy. On a region with a hole, positive orientation is counterclockwise outside and clockwise around the hole.

Step 1: Orientation Traverse the radius-2 circle counterclockwise and the radius-1 circle clockwise. In both cases the annular region remains on the left.



Step 2: Apply Green's Theorem On D , the field is continuously differentiable and

$$Q_x - P_y = 0.$$

Therefore

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D 0 \, dA = \boxed{0}.$$

Step 3: Check each component On any counterclockwise circle centered at the origin, the integral is 2π , independent of radius. Hence

$$\int_{C_{\text{outer}}} \mathbf{F} \cdot d\mathbf{r} = 2\pi, \quad \int_{C_{\text{inner}}} \mathbf{F} \cdot d\mathbf{r} = -2\pi$$

because the inner circle is clockwise. Their sum is zero.

Verification If both circles were traversed counterclockwise, the total would be 4π ; that is not the positively oriented boundary of the annulus.

Question 5

Let C be the counterclockwise boundary of the region between

$$y = x^2 \quad \text{and} \quad y = x, \quad 0 \leq x \leq 1.$$

Evaluate

$$\oint_C -y \, dx + x^2 \, dy.$$

Tasks

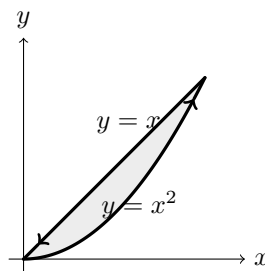
1. Describe the positively oriented boundary.
2. Set up the Green's Theorem double integral with correct bounds.
3. Evaluate and verify the sign.

Question 5 – Solution

Strategy. Replace the two curved boundary integrals with one Type I double integral over the enclosed region.

Step 1: Derivative difference Here

$$P = -y, \quad Q = x^2, \quad Q_x - P_y = 2x + 1.$$



Positive orientation travels from $(0, 0)$ to $(1, 1)$ along the lower parabola and returns along the upper line.

Step 2: Set up and evaluate

$$\begin{aligned} \oint_C P dx + Q dy &= \int_0^1 \int_{x^2}^x (2x + 1) dy dx \\ &= \int_0^1 (2x + 1)(x - x^2) dx \\ &= \int_0^1 (x + x^2 - 2x^3) dx \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = \boxed{\frac{1}{3}}. \end{aligned}$$

Verification On $0 \leq x \leq 1$, both $2x + 1$ and the vertical thickness $x - x^2$ are nonnegative, so the positive result is required.

Question 6

Use a Green's Theorem area formula to find the area enclosed by the cardioid

$$r = 1 + \cos \theta, \quad 0 \leq \theta \leq 2\pi,$$

traversed counterclockwise.

Tasks

1. Begin with $A = \frac{1}{2} \oint_C (x \, dy - y \, dx)$.
2. Show that $x \, dy - y \, dx = r^2 \, d\theta$.
3. Evaluate the resulting trigonometric integral.

Question 6 – Solution

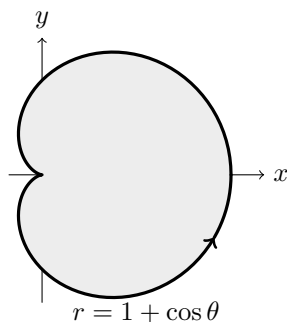
Strategy. Combine Green's area formula with polar boundary coordinates.

Step 1: Convert the differential With $x = r \cos \theta$ and $y = r \sin \theta$,

$$dx = (r' \cos \theta - r \sin \theta) d\theta, \quad dy = (r' \sin \theta + r \cos \theta) d\theta.$$

The rr' terms cancel, leaving

$$x dy - y dx = r^2(\cos^2 \theta + \sin^2 \theta) d\theta = r^2 d\theta.$$



Step 2: Evaluate the area

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{1}{2} (2\pi + 0 + \pi) = \boxed{\frac{3\pi}{2}}. \end{aligned}$$

Verification The standard polar area formula is $\frac{1}{2} \int r^2 d\theta$, exactly what Green's Theorem produced. The nonnegative integrand confirms the positive area.

Question 7

Let C be the counterclockwise triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 1)$. For

$$P(x, y) = -y^2, \quad Q(x, y) = x^2,$$

verify Green's Theorem by computing $\oint_C P dx + Q dy$ both directly and as a double integral.

Tasks

1. Evaluate all three boundary segments directly.
2. Evaluate $\iint_D (Q_x - P_y) dA$.
3. Compare the two results.

Question 7 – Solution

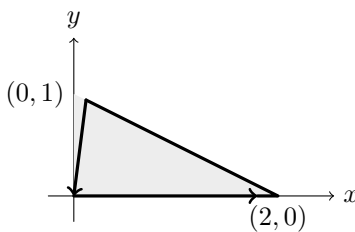
Strategy. Only the slanted edge contributes directly; then use $0 \leq y \leq 1 - x/2$ for the interior.

Step 1: Boundary calculation The horizontal and vertical coordinate-axis edges contribute zero. Parametrize the slanted edge from $(2, 0)$ to $(0, 1)$ by

$$\mathbf{r}(t) = \langle 2 - 2t, t \rangle, \quad 0 \leq t \leq 1.$$

Then $dx = -2dt$, $dy = dt$, and

$$\int_{\text{slant}} P dx + Q dy = \int_0^1 [2t^2 + 4(1 - t)^2] dt = \boxed{2}.$$



Step 2: Interior calculation Since $Q_x - P_y = 2x + 2y$,

$$\begin{aligned} \iint_D (2x + 2y) dA &= \int_0^2 \int_0^{1-x/2} (2x + 2y) dy dx \\ &= \int_0^2 \left(1 + x - \frac{3x^2}{4} \right) dx = \boxed{2}. \end{aligned}$$

Verification Both methods give 2. The slanted parametrization runs from $(2, 0)$ to $(0, 1)$, which is the required counterclockwise direction after traversing the bottom edge.

Question 8

For a constant a , let

$$\mathbf{F}_a(x, y) = \langle ay^2, x^2 \rangle.$$

The counterclockwise circulation around the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ is -2 . Determine a .

Tasks

1. Express the circulation as a double integral.
2. Solve for a .
3. Verify uniqueness and the stated circulation.

Question 8 – Solution

Strategy. Green's Theorem turns the circulation condition into a linear equation for a .

Step 1: Form the integrand With $P = ay^2$ and $Q = x^2$,

$$Q_x - P_y = 2x - 2ay.$$

Thus

$$\begin{aligned}\oint_C \mathbf{F}_a \cdot d\mathbf{r} &= \int_0^1 \int_0^1 (2x - 2ay) dy dx \\ &= \int_0^1 (2x - a) dx = 1 - a.\end{aligned}$$

Step 2: Impose the condition

$$1 - a = -2 \quad \implies \quad \boxed{a = 3}.$$

Step 3: Verify For $a = 3$,

$$\int_0^1 \int_0^1 (2x - 6y) dy dx = 1 - 3 = -2,$$

exactly the specified counterclockwise circulation.

Uniqueness check The circulation $1 - a$ has nonzero coefficient -1 on a , so the equation has exactly one solution.

Question 9

Let C be a positively oriented, simple, piecewise smooth closed curve enclosing a region D . Derive the three area formulas

$$A = \oint_C x \, dy, \quad A = - \oint_C y \, dx, \quad A = \frac{1}{2} \oint_C (x \, dy - y \, dx).$$

Tasks

1. Choose P, Q for each of the first two formulas.
2. Derive the symmetric formula from Green's Theorem or by averaging.
3. Explain the orientation requirement.

Question 9 – Solution

Strategy. Make the Green's Theorem integrand $Q_x - P_y$ equal to 1 in three different ways.

Step 1: Formula using $x \, dy$ Choose $P = 0$, $Q = x$. Then $Q_x - P_y = 1$, so

$$\text{Area}(D) = \iint_D 1 \, dA = \boxed{\oint_C x \, dy}.$$

Step 2: Formula using $-y \, dx$ Choose $P = -y$, $Q = 0$. Again $Q_x - P_y = 1$, giving

$$\text{Area}(D) = \boxed{-\oint_C y \, dx}.$$

Step 3: Symmetric formula Choose $P = -y/2$ and $Q = x/2$. Then

$$Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1,$$

and therefore

$$\boxed{\text{Area}(D) = \frac{1}{2} \oint_C (x \, dy - y \, dx)}.$$

This is also the average of the first two formulas.

Verification Green's Theorem uses positive, counterclockwise orientation. Clockwise traversal negates all three line integrals, so a minus sign would be needed to recover positive geometric area.

Question 10

Let D be the ellipse bounded counterclockwise by

$$x = 2 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq 2\pi.$$

Evaluate

$$\iint_D (3x^2 - 2y) \, dA$$

by converting it to a boundary integral.

Tasks

1. Choose P, Q so that $Q_x - P_y = 3x^2 - 2y$.
2. Parametrize the resulting boundary integral.
3. Evaluate using symmetry and a standard trigonometric integral.

Question 10 – Solution

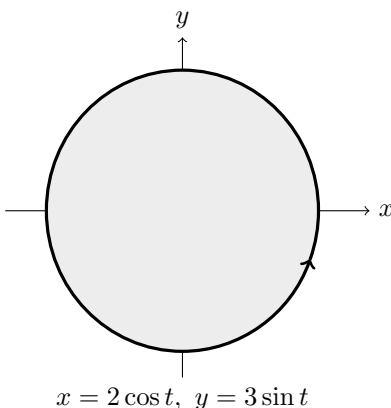
Strategy. Reverse Green's Theorem with $P = y^2$ and $Q = x^3$, then use the supplied ellipse parametrization.

Step 1: Choose a boundary field Set

$$P = y^2, \quad Q = x^3.$$

Then $Q_x - P_y = 3x^2 - 2y$, so

$$\iint_D (3x^2 - 2y) dA = \oint_C y^2 dx + x^3 dy.$$



Step 2: Substitute the boundary Since $dx = -2 \sin t dt$ and $dy = 3 \cos t dt$,

$$\oint_C y^2 dx + x^3 dy = \int_0^{2\pi} [-18 \sin^3 t + 24 \cos^4 t] dt.$$

The sine-cubed term integrates to zero over a full period, while

$$\int_0^{2\pi} \cos^4 t dt = \frac{3\pi}{4}.$$

Therefore

$$\boxed{\iint_D (3x^2 - 2y) dA = 24 \left(\frac{3\pi}{4} \right) = 18\pi}.$$

Verification The $-2y$ term is odd across the x -axis and contributes zero. Also $3x^2 \geq 0$, so the positive result has the required sign.