

Question 1

Let

$$R = [0, 2] \times [0, 1], \quad f(x, y) = 6 - x^2 - y.$$

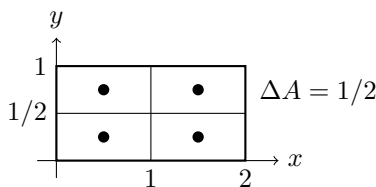
Partition R into two equal columns and two equal rows, and use the midpoint of each subrectangle as its sample point.

Tasks

1. List the four sample points and the common subrectangle area.
2. Compute the midpoint Riemann sum.
3. Interpret the sum geometrically and check it against bounds from the minimum and maximum of f on R .

Question 1 – Solution

Strategy. Translate the partition into four base areas, evaluate the height at each midpoint, and add the resulting prism volumes.



Step 1: Partition data Here $\Delta x = 1$, $\Delta y = 1/2$, and

$$\Delta A = \Delta x \Delta y = \frac{1}{2}.$$

The midpoints are

$$\left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{3}{2}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{3}{2}, \frac{3}{4}\right).$$

Step 2: Heights and sum In the same order, the four function values are

$$\frac{11}{2}, \quad \frac{7}{2}, \quad 5, \quad 3.$$

Therefore

$$S = \frac{1}{2} \left(\frac{11}{2} + \frac{7}{2} + 5 + 3 \right) = \boxed{\frac{17}{2}}.$$

Step 3: Interpretation and bounds The sum is the total volume of four midpoint prisms. On R , f decreases as either x or y increases, so

$$1 = f(2, 1) \leq f(x, y) \leq f(0, 0) = 6.$$

Since $\text{area}(R) = 2$, every Riemann-sum estimate lies between 2 and 12; indeed $17/2$ does. This also verifies that every sampled height was positive and on the correct scale.

Question 2

On $R = [0, 1] \times [0, 2]$, let $f(x, y) = x + 2y$. Divide the x -interval into m equal parts and the y -interval into n equal parts. Use the upper-right corner of each subrectangle as its sample point.

Tasks

1. Write the resulting double Riemann sum $S_{m,n}$.
2. Simplify the finite sum exactly using summation formulas.
3. Take the limit as $m, n \rightarrow \infty$ to evaluate the double integral from its definition.

Question 2 – Solution

Strategy. Express every sample point and area in terms of its indices, then separate the finite sums algebraically before taking the two-parameter limit.

Step 1: Grid and samples We have

$$\Delta x = \frac{1}{m}, \quad \Delta y = \frac{2}{n}, \quad \Delta A = \frac{2}{mn},$$

and the upper-right sample in cell (i, j) is $(i/m, 2j/n)$. Thus

$$S_{m,n} = \sum_{i=1}^m \sum_{j=1}^n \left(\frac{i}{m} + \frac{4j}{n} \right) \frac{2}{mn}.$$

Step 2: Exact finite sum Using $\sum_{k=1}^N k = N(N+1)/2$,

$$\begin{aligned} S_{m,n} &= \frac{2}{m^2 n} \sum_{i=1}^m \sum_{j=1}^n i + \frac{8}{mn^2} \sum_{i=1}^m \sum_{j=1}^n j \\ &= \frac{2}{m^2 n} \left(n \frac{m(m+1)}{2} \right) + \frac{8}{mn^2} \left(m \frac{n(n+1)}{2} \right) \\ &= \frac{m+1}{m} + 4 \frac{n+1}{n}. \end{aligned}$$

Step 3: Definition limit Both correction terms vanish independently, so

$$\boxed{\iint_R (x + 2y) dA = \lim_{m,n \rightarrow \infty} S_{m,n} = 1 + 4 = 5}.$$

The answer lies between $0 \cdot 2$ and $5 \cdot 2 = 10$, the bounds obtained from the extreme values of f on the area-2 rectangle.

Question 3

Let $f(x, y) = xy$ on the unit square $R = [0, 1] \times [0, 1]$. Partition R into an $n \times n$ uniform grid.

Tasks

1. Compute the lower sum L_n and upper sum U_n .
2. Use them to determine $\iint_R xy \, dA$ without iterated integration.
3. Find the least integer n for which the gap $U_n - L_n$ is less than 0.01.

Question 3 – Solution

Strategy. Because xy increases in each coordinate on the unit square, lower-left and upper-right corners produce the lower and upper sums.

Step 1: Lower sum Every cell has area $1/n^2$. Using indices $i, j = 0, \dots, n-1$ at lower-left corners,

$$\begin{aligned} L_n &= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{i}{n} \frac{j}{n} = \frac{1}{n^4} \left(\sum_{i=0}^{n-1} i \right)^2 \\ &= \frac{1}{n^4} \left(\frac{n(n-1)}{2} \right)^2 = \frac{(n-1)^2}{4n^2}. \end{aligned}$$

Step 2: Upper sum and squeeze Upper-right corners give

$$U_n = \frac{1}{n^4} \left(\sum_{i=1}^n i \right)^2 = \frac{(n+1)^2}{4n^2}.$$

Both sums approach $1/4$. Since $L_n \leq \iint_R xy \, dA \leq U_n$,

$$\boxed{\iint_R xy \, dA = \frac{1}{4}}.$$

Step 3: Resolution The uncertainty is

$$U_n - L_n = \frac{(n+1)^2 - (n-1)^2}{4n^2} = \frac{1}{n}.$$

Thus $1/n < 0.01$ requires $n > 100$, and the least integer is n=101. The strict inequality rules out $n = 100$.

Question 4

On $R = [-2, 2] \times [0, 3]$, define

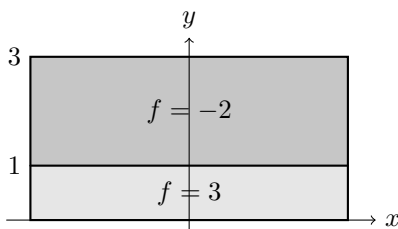
$$f(x, y) = \begin{cases} 3, & 0 \leq y \leq 1, \\ -2, & 1 < y \leq 3. \end{cases}$$

Tasks

1. Evaluate $\iint_R f \, dA$ by signed volume.
2. Evaluate $\iint_R |f| \, dA$ and explain the difference.
3. Find the average value of f on R .

Question 4 – Solution

Strategy. Split the base into the two constant-height bands and distinguish algebraic volume from total unsigned volume.



Step 1: Signed volume The lower band has area $4(1) = 4$ and height 3, so it contributes 12. The upper band has area $4(2) = 8$ and height -2 , so it contributes -16 . Hence

$$\iint_R f \, dA = 12 - 16 = -4.$$

Step 2: Unsigned volume Absolute value changes the second height from -2 to 2. Therefore

$$\iint_R |f| \, dA = 12 + 16 = 28.$$

The first integral records volume below the xy -plane negatively; the second records all geometric volume positively.

Step 3: Average value Since $\text{area}(R) = 4 \cdot 3 = 12$,

$$f_{\text{avg}} = \frac{1}{12} \iint_R f \, dA = -\frac{1}{3}.$$

As a check, $-1/3$ lies between the only two function values, -2 and 3 , and the wider negative band explains its negative sign.

Question 5

Let $R = [-1, 2] \times [0, 2]$. A continuous function f satisfies

$$2 \leq f(x, y) \leq 5, \quad \iint_R f \, dA = 19.$$

A second integrable function g satisfies $|g(x, y) - f(x, y)| \leq 0.1$ throughout R .

Tasks

1. Check that the stated integral of f is consistent with its pointwise bounds.
2. Evaluate $\iint_R (3f - 4) \, dA$.
3. Give the sharp bounds guaranteed by the data for $\iint_R g \, dA$.

Question 5 – Solution

Strategy. Convert pointwise inequalities to integral inequalities using the area of the rectangle, and use linearity for the transformed function.

Step 1: Comparison bounds The rectangle has area

$$A = (2 - (-1))(2 - 0) = 6.$$

Integrating $2 \leq f \leq 5$ over R gives

$$12 \leq \iint_R f \, dA \leq 30.$$

The given value 19 lies in this interval, so the data are consistent.

Step 2: Linearity Because the integral of the constant 4 is $4A = 24$,

$$\iint_R (3f - 4) \, dA = 3 \iint_R f \, dA - \iint_R 4 \, dA = 57 - 24 = \boxed{33}.$$

Step 3: Stability bound From $|g - f| \leq 0.1$,

$$-0.1 \leq g - f \leq 0.1.$$

Integration over an area-6 region yields

$$-0.6 \leq \iint_R (g - f) \, dA \leq 0.6.$$

Consequently,

$$\boxed{18.4 \leq \iint_R g \, dA \leq 19.6}.$$

These endpoints are attainable by $g = f - 0.1$ and $g = f + 0.1$, so no tighter bounds follow from the supplied information.

Question 6

On the centrally symmetric rectangle

$$R = [-2, 2] \times [-1, 1],$$

consider $f(x, y) = 5 + x^3 + xy^2 + 2y$.

Tasks

1. Show that the nonconstant part cancels under central reflection.
2. Evaluate $\iint_R f \, dA$ and find the average value of f .
3. Explain precisely which symmetry of the region makes the argument valid.

Question 6 – Solution

Strategy. Pair each point (x, y) with its central reflection $(-x, -y)$ and average the two function values.

Step 1: Pairing Let $h(x, y) = x^3 + xy^2 + 2y$. Then

$$h(-x, -y) = -x^3 - xy^2 - 2y = -h(x, y).$$

Thus the contributions of h at every reflected pair cancel. Equivalently,

$$f(x, y) + f(-x, -y) = 10.$$

Step 2: Integral and average Central reflection maps R onto itself without changing area. Therefore the paired average of f is 5 everywhere, and

$$\iint_R f \, dA = 5 \operatorname{area}(R) = 5(4)(2) = \boxed{40}.$$

Dividing by the area 8 gives

$$\boxed{f_{\text{avg}} = 5}.$$

Step 3: Symmetry check The required property is

$$(x, y) \in R \iff (-x, -y) \in R.$$

Symmetry in only one coordinate would not automatically cancel the entire h : xy^2 is odd in x , while $2y$ is odd in y . Central symmetry reverses both coordinates at once and cancels all three terms.

Question 7

The plane $z = 6 + x - 2y$ lies above the rectangle

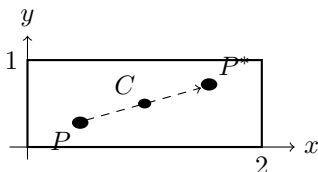
$$R = [0, 2] \times [0, 1].$$

Tasks

1. Pair each point of R with its reflection through the center $(1, 1/2)$ and compare their heights.
2. Use the pairing to find the volume under the plane without iterated integration.
3. Verify the result using the four corner heights.

Question 7 – Solution

Strategy. An affine function has complementary values at points reflected through a rectangle's center, so every reflected pair has the same average height.



Step 1: Reflected heights The reflection of $P = (x, y)$ through $C = (1, 1/2)$ is

$$P^* = (2 - x, 1 - y).$$

For $f(x, y) = 6 + x - 2y$,

$$\begin{aligned} f(P) + f(P^*) &= (6 + x - 2y) + (6 + (2 - x) - 2(1 - y)) \\ &= 12. \end{aligned}$$

Thus every pair has average height $6 = f(C)$.

Step 2: Volume The base area is 2, and the plane's minimum corner height is $4 > 0$, so the double integral is geometric volume:

$$V = \iint_R (6 + x - 2y) dA = 6(2) = 12.$$

Step 3: Corner check The corner heights are

$$f(0, 0) = 6, \quad f(2, 0) = 8, \quad f(0, 1) = 4, \quad f(2, 1) = 6.$$

Their average is $(6 + 8 + 4 + 6)/4 = 6$, agreeing with the center and pairing calculation. Multiplication by base area again gives 12.

Question 8

An affine function $f(x, y) = a + bx + cy$ is defined on the unit square. Its average value is 4 on the whole square, 5 on the right half $[1/2, 1] \times [0, 1]$, and 3 on the top half $[0, 1] \times [1/2, 1]$.

Tasks

1. Use center values of affine functions to obtain three equations for a, b, c .
2. Determine f uniquely.
3. Predict the integral of f over the lower-left quarter $[0, 1/2] \times [0, 1/2]$ and verify consistency.

Question 8 – Solution

Strategy. The average of an affine function over a rectangle equals its value at that rectangle's center; use the three given centers to reconstruct the coefficients.

Step 1: Average equations The relevant centers are $(1/2, 1/2)$, $(3/4, 1/2)$, and $(1/2, 3/4)$. Hence

$$\begin{aligned}a + \frac{b}{2} + \frac{c}{2} &= 4, \\a + \frac{3b}{4} + \frac{c}{2} &= 5, \\a + \frac{b}{2} + \frac{3c}{4} &= 3.\end{aligned}$$

The center-value rule follows from the same reflection pairing used for any affine graph over a rectangle.

Step 2: Reconstruct Subtracting the first equation from the second gives $b/4 = 1$, so $b = 4$. Subtracting the first from the third gives $c/4 = -1$, so $c = -4$. The first equation then gives $a = 4$. Therefore

$$\boxed{f(x, y) = 4 + 4x - 4y}.$$

Step 3: New prediction The lower-left quarter has center $(1/4, 1/4)$ and area $1/4$. Its average is

$$f\left(\frac{1}{4}, \frac{1}{4}\right) = 4 + 1 - 1 = 4,$$

so

$$\boxed{\iint_{[0, 1/2]^2} f \, dA = 4 \left(\frac{1}{4}\right) = 1}.$$

The equal x - and y -center coordinates cancel the opposite slopes, which checks the predicted average.

Question 9

Define the bounded function on the unit square by

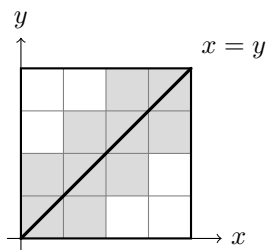
$$f(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

Tasks

1. Explain why every lower sum is zero.
2. Use a uniform $n \times n$ grid to bound an upper sum by $3/n$.
3. Prove that f is Riemann integrable and find its double integral, despite its discontinuities.

Question 9 – Solution

Strategy. The diagonal has no area: trap every possible nonzero upper-sum contribution in a shrinking collection of grid squares.



Step 1: Lower sums Every subrectangle with positive area contains a point off the diagonal, where $f = 0$. Its infimum is therefore 0, so every lower sum equals 0.

Step 2: Upper-sum cover In an $n \times n$ grid, each square has area $1/n^2$. The n diagonal squares meet $x = y$. At each of the $n - 1$ interior diagonal grid vertices, two additional off-diagonal squares have closures meeting the diagonal. Thus at most

$$n + 2(n - 1) = 3n - 2$$

closed grid squares have supremum 1; all others have supremum 0. Consequently,

$$0 \leq U_n \leq \frac{3n - 2}{n^2} < \frac{3}{n}.$$

Step 3: Integrability Given $\varepsilon > 0$, choose $n > 3/\varepsilon$. Then

$$0 \leq U_n - L_n < \frac{3}{n} < \varepsilon.$$

The Darboux criterion proves integrability, and the integral is squeezed between zero lower sums and upper sums tending to zero:

$$\boxed{\iint_{[0,1]^2} f \, dA = 0}.$$

The function is discontinuous at every point of the diagonal, but that one-dimensional set contributes no area.

Question 10

A function f on the unit square satisfies the Lipschitz estimate

$$|f(P) - f(Q)| \leq 2\|P - Q\|$$

for all points P, Q . Let S_n be the midpoint Riemann sum on an $n \times n$ uniform grid.

Tasks

1. Prove the error bound $|\iint_R f \, dA - S_n| \leq \sqrt{2}/n$.
2. Find the least n certified by this bound to give error less than 0.01.
3. If $S_{142} = 2.731$, give a guaranteed interval containing the true integral.

Question 10 – Solution

Strategy. Control the variation within each cell by its maximum distance from the midpoint, then add the cellwise error bounds.

Step 1: One-cell error The Lipschitz estimate makes f continuous, hence integrable on the closed square. A grid square has side $1/n$, area $1/n^2$, and maximum distance from its midpoint to a corner

$$d_n = \frac{\sqrt{2}}{2n}.$$

If M_{ij} is its midpoint, the Lipschitz condition gives

$$|f(P) - f(M_{ij})| \leq 2d_n = \frac{\sqrt{2}}{n}$$

throughout that cell. Its integral differs from midpoint height times area by at most $\sqrt{2}/n^3$.

Step 2: Global error There are n^2 cells, so the triangle inequality yields

$$\left| \iint_R f \, dA - S_n \right| \leq n^2 \frac{\sqrt{2}}{n^3} = \frac{\sqrt{2}}{n}.$$

To guarantee strict error below 0.01, require $n > 100\sqrt{2} \approx 141.421$. The least integer is therefore n=142.

Step 3: Numerical enclosure For $n = 142$,

$$\frac{\sqrt{2}}{142} \approx 0.009959.$$

Thus

$$2.731 - \frac{\sqrt{2}}{142} \leq \iint_R f \, dA \leq 2.731 + \frac{\sqrt{2}}{142},$$

or approximately

$$2.72104 \leq \iint_R f \, dA \leq 2.74096.$$

The half-diagonal, rather than the full cell diagonal, is essential to the stated constant.

Question 1

Evaluate the double integral of

$$f(x, y) = 3x^2y - 2x + 4$$

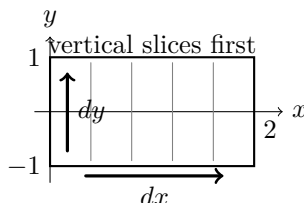
over $R = [0, 2] \times [-1, 1]$ by iterated integration.

Tasks

1. Evaluate by integrating with respect to y first.
2. Evaluate again by integrating with respect to x first.
3. Explain why the two answers must agree.

Question 1 – Solution

Strategy. Write both constant-bound orders explicitly and simplify the inner integral before performing the outer one.



Step 1: Integrate in y first

$$\begin{aligned} \int_0^2 \int_{-1}^1 (3x^2y - 2x + 4) dy dx &= \int_0^2 \left[\frac{3}{2}x^2y^2 - 2xy + 4y \right]_{-1}^1 dx \\ &= \int_0^2 (-4x + 8) dx \\ &= [-2x^2 + 8x]_0^2 = 8. \end{aligned}$$

The $3x^2y$ term vanishes because it is odd in y over a symmetric interval.

Step 2: Integrate in x first

$$\begin{aligned} \int_{-1}^1 \int_0^2 (3x^2y - 2x + 4) dx dy &= \int_{-1}^1 [x^3y - x^2 + 4x]_0^2 dy \\ &= \int_{-1}^1 (8y + 4) dy = 8. \end{aligned}$$

Thus

$$\boxed{\iint_R f dA = 8}.$$

Verification The polynomial is continuous on the closed rectangle, so Fubini's theorem permits either order. In the second order the odd term $8y$ cancels, leaving constant height 4 across a y -interval of length 2, again giving 8.

Question 2

Evaluate

$$\iint_R e^{2x} \cos y \, dA, \quad R = [0, \ln 3] \times \left[0, \frac{\pi}{3}\right].$$

Tasks

1. Recognize and use the product structure of the integrand.
2. Evaluate the integral in each order.
3. Verify the exact result by multiplying the two one-variable integrals.

Question 2 – Solution

Strategy. The integrand is a product of an x -only factor and a y -only factor, so constant rectangular bounds make the two integrations separate.

Step 1: Integrate in y first

$$\begin{aligned}\int_0^{\ln 3} \int_0^{\pi/3} e^{2x} \cos y \, dy \, dx &= \int_0^{\ln 3} e^{2x} [\sin y]_0^{\pi/3} \, dx \\ &= \frac{\sqrt{3}}{2} \left[\frac{1}{2} e^{2x} \right]_0^{\ln 3} \\ &= \frac{\sqrt{3}}{4} (9 - 1) = 2\sqrt{3}.\end{aligned}$$

Step 2: Reverse the order

$$\begin{aligned}\int_0^{\pi/3} \int_0^{\ln 3} e^{2x} \cos y \, dx \, dy &= \int_0^{\pi/3} \cos y \left[\frac{1}{2} e^{2x} \right]_0^{\ln 3} \, dy \\ &= 4[\sin y]_0^{\pi/3} = 2\sqrt{3}.\end{aligned}$$

Therefore

$$\boxed{\iint_R e^{2x} \cos y \, dA = 2\sqrt{3}}.$$

Verification Direct factorization gives

$$\left(\int_0^{\ln 3} e^{2x} \, dx \right) \left(\int_0^{\pi/3} \cos y \, dy \right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3},$$

confirming both orders and the endpoint $e^{2\ln 3} = 9$.

Question 3

Consider

$$I = \int_0^1 \int_0^2 x e^{xy} dy dx.$$

Tasks

1. Evaluate I using the displayed order.
2. Write the reversed iterated integral and its inner antiderivative for $y \neq 0$.
3. Explain why the displayed order is substantially more efficient and check the behavior at $x = 0$.

Question 3 – Solution

Strategy. Treat x as constant in the inner integral; the factor x is exactly the derivative needed for e^{xy} with respect to y .

Step 1: Efficient order For $x \neq 0$,

$$\int_0^2 x e^{xy} dy = [e^{xy}]_0^2 = e^{2x} - 1.$$

At $x = 0$ the original integrand is identically zero and the same expression gives $e^0 - 1 = 0$, so the formula extends continuously. Hence

$$\begin{aligned} I &= \int_0^1 (e^{2x} - 1) dx = \left[\frac{1}{2} e^{2x} - x \right]_0^1 \\ &= \boxed{\frac{e^2 - 3}{2}}. \end{aligned}$$

Step 2: Reversed order The rectangle permits

$$I = \int_0^2 \int_0^1 x e^{xy} dx dy.$$

For $y \neq 0$, integration by parts gives

$$\int x e^{xy} dx = \frac{e^{xy}(xy - 1)}{y^2} + C,$$

so the inner result is

$$\frac{e^y(y - 1) + 1}{y^2}.$$

Step 3: Efficiency and verification The reversed expression has a removable singular appearance at $y = 0$ and requires another nonobvious antiderivative. The original order collapses immediately by substitution. Continuity of $x e^{xy}$ on the rectangle guarantees by Fubini's theorem that either completed order has the same value.

Question 4

A student is given

$$I = \int_{-1}^1 \int_0^2 (x - 2y) \, dx \, dy$$

and claims the inner integral is $[x^2/2 - 2xy]_{-1}^1$.

Tasks

1. Diagnose the student's error using the differential and bounds.
2. Evaluate the integral correctly as written.
3. Reverse the order on the same rectangle and verify the result.

Question 4 – Solution

Strategy. Read an iterated integral from the inside out: the inner differential determines both the variable of integration and which pair of bounds belongs to it.

Step 1: Diagnose The inner differential is dx , so x runs from 0 to 2 while y is held constant. The student incorrectly used the outer y -bounds, -1 and 1 , as x -bounds.

Step 2: Evaluate as written

$$\begin{aligned} I &= \int_{-1}^1 \left[\frac{x^2}{2} - 2yx \right]_0^2 dy = \int_{-1}^1 (2 - 4y) dy \\ &= [2y - 2y^2]_{-1}^1 = \boxed{4}. \end{aligned}$$

The odd contribution $-4y$ cancels over $[-1, 1]$.

Step 3: Reverse correctly Constant rectangular bounds reverse to

$$I = \int_0^2 \int_{-1}^1 (x - 2y) dy dx.$$

Then

$$I = \int_0^2 [xy - y^2]_{-1}^1 dx = \int_0^2 2x dx = [x^2]_0^2 = 4.$$

The agreement verifies both the order reversal and the association of each bound pair with its variable.

Question 5

Evaluate

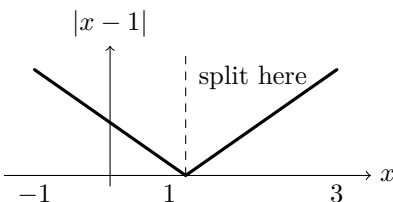
$$\iint_R |x - 1|(1 + y) \, dA, \quad R = [-1, 3] \times [0, 2].$$

Tasks

1. Choose an efficient order of integration.
2. Split at the correct location to remove the absolute value.
3. Evaluate the integral and its average value over R .

Question 5 – Solution

Strategy. Integrate the positive y -factor first, then split the remaining x -integral where $x - 1$ changes sign.



Step 1: Inner integral Using dy first,

$$\begin{aligned}\int_{-1}^3 \int_0^2 |x - 1|(1 + y) dy dx &= \int_{-1}^3 |x - 1| \left[y + \frac{y^2}{2} \right]_0^2 dx \\ &= 4 \int_{-1}^3 |x - 1| dx.\end{aligned}$$

Step 2: Absolute-value split

$$\begin{aligned}\int_{-1}^3 |x - 1| dx &= \int_{-1}^1 (1 - x) dx + \int_1^3 (x - 1) dx \\ &= 2 + 2 = 4.\end{aligned}$$

Therefore

$$\boxed{\iint_R |x - 1|(1 + y) dA = 16}.$$

Step 3: Average and check The rectangle has area $4 \cdot 2 = 8$, so

$$\boxed{f_{\text{avg}} = \frac{16}{8} = 2}.$$

The two x -pieces are mirror images about $x = 1$, and the nonnegative integrand confirms the positive sign.

Question 6

For a real parameter a , define

$$I(a) = \int_0^2 \int_{-1}^1 (axy^2 + x^2 - 2y) \, dy \, dx.$$

Tasks

1. Evaluate $I(a)$ exactly.
2. Find the unique value of a for which the integrand's average value on the rectangle is 1.
3. Verify the resulting integral directly from the inner accumulation function.

Question 6 – Solution

Strategy. Use parity in the symmetric y -interval, retain the parameter symbolically, and convert the prescribed average into a total integral.

Step 1: Inner accumulation Since $-2y$ is odd,

$$\begin{aligned}\int_{-1}^1 (axy^2 + x^2 - 2y) dy &= ax \left[\frac{y^3}{3} \right]_{-1}^1 + x^2 [y]_{-1}^1 \\ &= \frac{2ax}{3} + 2x^2.\end{aligned}$$

Thus

$$I(a) = \int_0^2 \left(\frac{2ax}{3} + 2x^2 \right) dx = \frac{4a}{3} + \frac{16}{3}.$$

Step 2: Prescribed average The rectangle's area is $2 \cdot 2 = 4$. Average value 1 therefore requires $I(a) = 4$:

$$\frac{4a + 16}{3} = 4 \implies \boxed{a = -1}.$$

Step 3: Direct verification With $a = -1$, the inner accumulation is

$$-\frac{2x}{3} + 2x^2.$$

Consequently,

$$\int_0^2 \left(-\frac{2x}{3} + 2x^2 \right) dx = \left[-\frac{x^2}{3} + \frac{2x^3}{3} \right]_0^2 = -\frac{4}{3} + \frac{16}{3} = 4.$$

Dividing by area 4 returns the required average 1.

Question 7

A temperature field on a rectangular plate is

$$T(x, y) = 18 + 2x + y^2, \quad 0 \leq x \leq 3, \quad -1 \leq y \leq 1.$$

Tasks

1. Compute the average temperature using an iterated integral.
2. Compare it with the minimum and maximum temperatures on the plate.
3. Exhibit two distinct points where the temperature equals its average.

Question 7 – Solution

Strategy. Integrate the three terms separately, divide by plate area, and then solve the resulting level equation at convenient choices of y .

Step 1: Total and average The plate area is $A = 3 \cdot 2 = 6$. Its total temperature integral is

$$\begin{aligned}\int_0^3 \int_{-1}^1 (18 + 2x + y^2) dy dx &= 18(6) + \left(\int_0^3 2x dx \right) (2) + 3 \int_{-1}^1 y^2 dy \\ &= 108 + 18 + 2 = 128.\end{aligned}$$

Hence

$$T_{\text{avg}} = \frac{128}{6} = \frac{64}{3}.$$

Step 2: Range check The minimum is $T(0, 0) = 18$. The maximum 25 occurs at $(3, 1)$ and $(3, -1)$. Since

$$18 < \frac{64}{3} < 25,$$

the average lies strictly inside the temperature range, as required.

Step 3: Points at the average Solve

$$18 + 2x + y^2 = \frac{64}{3}.$$

Taking $y = 0$ gives $x = 5/3$, while taking $y = 1$ gives $x = 7/6$. Thus two examples are

$$\left(\frac{5}{3}, 0 \right) \quad \text{and} \quad \left(\frac{7}{6}, 1 \right).$$

Both lie on the plate, and substitution gives $64/3$ at each.

Question 8

Let p be continuous on $[a, b]$ and q continuous on $[c, d]$.

Tasks

1. Prove using iterated integrals that

$$\iint_{[a,b] \times [c,d]} p(x)q(y) \, dA = \left(\int_a^b p(x) \, dx \right) \left(\int_c^d q(y) \, dy \right).$$

2. Explain why the same proof works in either order.
3. Apply the result to $(1 + x^2) \cos y$ on $[-1, 1] \times [-\pi/2, \pi/2]$.

Question 8 – Solution

Strategy. In the inner integral, the factor depending only on the outer variable is constant and can be pulled outside.

Step 1: Factorization proof Integrating in y first gives

$$\begin{aligned}\int_a^b \int_c^d p(x)q(y) dy dx &= \int_a^b p(x) \left(\int_c^d q(y) dy \right) dx \\ &= \left(\int_a^b p(x) dx \right) \left(\int_c^d q(y) dy \right).\end{aligned}$$

The bracketed q -integral is a number, so it is constant in the outer x -integration.

Step 2: Other order Reversing the order first produces the constant $\int_a^b p(x) dx$ and then the q -integral. Continuity on the closed rectangle guarantees Fubini's theorem applies, so both forms represent the same double integral.

Step 3: Application Here

$$\int_{-1}^1 (1+x^2) dx = \left[x + \frac{x^3}{3} \right]_{-1}^1 = \frac{8}{3}$$

and

$$\int_{-\pi/2}^{\pi/2} \cos y dy = [\sin y]_{-\pi/2}^{\pi/2} = 2.$$

Therefore

$$\boxed{\iint_R (1+x^2) \cos y dA = \frac{16}{3}}.$$

Both factors are nonnegative on their intervals, verifying the sign.

Question 9

Evaluate

$$\iint_R (x^3 \cos y + x^2 \sin y + x^2) dA, \quad R = [-2, 2] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Tasks

1. Integrate with respect to y first and identify the term eliminated by y -symmetry.
2. Use x -symmetry in the outer integral.
3. Evaluate the surviving contribution and verify its sign.

Question 9 – Solution

Strategy. Use parity at each stage: the symmetric inner interval cancels the sine term, and the symmetric outer interval cancels the remaining odd power of x .

Step 1: Inner y -integration

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} (x^3 \cos y + x^2 \sin y + x^2) dy &= x^3 [\sin y]_{-\pi/2}^{\pi/2} - x^2 [\cos y]_{-\pi/2}^{\pi/2} + \pi x^2 \\ &= 2x^3 + \pi x^2.\end{aligned}$$

The $x^2 \sin y$ term vanishes because sine is odd in y .

Step 2: Outer x -integration Now $2x^3$ is odd on $[-2, 2]$, so it contributes zero. Therefore

$$\begin{aligned}\iint_R f dA &= \pi \int_{-2}^2 x^2 dx = \pi \left[\frac{x^3}{3} \right]_{-2}^2 \\ &= \boxed{\frac{16\pi}{3}}.\end{aligned}$$

Step 3: Verification The only surviving term is $x^2 \geq 0$, integrated over a y -interval of length π . Thus a positive result is necessary. Its one-variable integral is $16/3$, so multiplication by π confirms the scale and constant.

Question 10

Over the square $R = [-1, 1] \times [-1, 1]$, a solid lies between

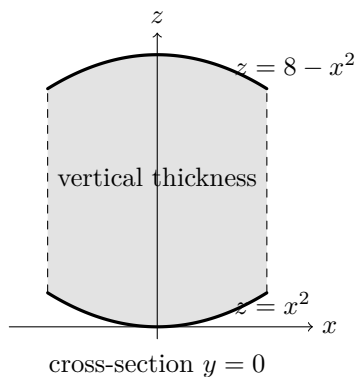
$$z_{\text{top}} = 8 - x^2 - y^2 \quad \text{and} \quad z_{\text{bottom}} = x^2 + y^2.$$

Tasks

1. Verify that the named top surface stays above the bottom surface on all of R .
2. Set up and evaluate an iterated integral for the volume.
3. Check the answer using symmetry and separate one-variable integrals.

Question 10 – Solution

Strategy. Integrate vertical thickness over the rectangular base, after confirming that the thickness never changes sign.



Step 1: Thickness The vertical thickness is

$$h(x, y) = 8 - 2x^2 - 2y^2.$$

On R , $x^2 + y^2 \leq 2$, so $h \geq 8 - 4 = 4 > 0$. Thus no top-bottom reversal or absolute value is needed.

Step 2: Iterated integral

$$\begin{aligned} V &= \int_{-1}^1 \int_{-1}^1 (8 - 2x^2 - 2y^2) dy dx \\ &= \int_{-1}^1 \left(16 - 4x^2 - \frac{4}{3} \right) dx \\ &= \left[\frac{44}{3}x - \frac{4}{3}x^3 \right]_{-1}^1 = \boxed{\frac{80}{3}}. \end{aligned}$$

Step 3: Independent check The constant term contributes $8 \text{ area}(R) = 32$. Also,

$$\iint_R x^2 dA = \left(\int_{-1}^1 x^2 dx \right) (2) = \frac{4}{3},$$

and identically $\iint_R y^2 dA = 4/3$. Hence

$$V = 32 - 2 \left(\frac{4}{3} \right) - 2 \left(\frac{4}{3} \right) = 32 - \frac{16}{3} = \frac{80}{3},$$

confirming the iterated calculation.

Question 1

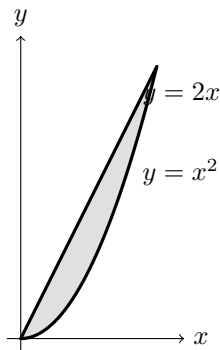
Let D be bounded by $y = x^2$ and $y = 2x$.

Tasks

1. Find the intersection points and describe D as a Type I region.
2. Set up and evaluate $\iint_D (x + y) dA$.
3. Verify that the bounds keep the lower curve below the upper curve.

Question 1 – Solution

Strategy. Solve the boundary intersection first, then use vertical slices from the parabola to the line.



Step 1: Region The curves meet where $x^2 = 2x$, so $x = 0, 2$. For $0 \leq x \leq 2$, $x^2 \leq 2x$. Thus

$$D = \{(x, y) : 0 \leq x \leq 2, x^2 \leq y \leq 2x\}.$$

Step 2: Integral

$$\begin{aligned} \iint_D (x + y) dA &= \int_0^2 \int_{x^2}^{2x} (x + y) dy dx \\ &= \int_0^2 \left[xy + \frac{y^2}{2} \right]_{x^2}^{2x} dx \\ &= \int_0^2 \left(4x^2 - x^3 - \frac{x^4}{2} \right) dx \\ &= \frac{32}{3} - 4 - \frac{16}{5} = \boxed{\frac{52}{15}}. \end{aligned}$$

Verification The slice height is $2x - x^2 = x(2 - x) \geq 0$ on $[0, 2]$. Also $x + y \geq 0$ throughout D , consistent with the positive answer.

Question 2

Let D be bounded by $x = y^2$ and $x = 2y$.

Tasks

1. Describe D as a Type II region.
2. Evaluate $\iint_D x \, dA$ using horizontal slices.
3. Check the sign and the intersection endpoints.

Question 2 – Solution

Strategy. Treat x as the inner variable and determine which curve is leftmost between the intersection values of y .

Step 1: Region From $y^2 = 2y$, the curves meet at $y = 0, 2$. On this interval, $y^2 \leq 2y$, so

$$D = \{(x, y) : 0 \leq y \leq 2, y^2 \leq x \leq 2y\}.$$

Step 2: Integral

$$\begin{aligned}\iint_D x \, dA &= \int_0^2 \int_{y^2}^{2y} x \, dx \, dy = \frac{1}{2} \int_0^2 (4y^2 - y^4) \, dy \\ &= \left[\frac{2y^3}{3} - \frac{y^5}{5} \right]_0^2 = \frac{16}{3} - \frac{16}{5} = \boxed{\frac{32}{15}}.\end{aligned}$$

Verification Both boundaries have $x \geq 0$ on $0 \leq y \leq 2$, so the integral of x must be nonnegative. The width $2y - y^2 = y(2 - y)$ vanishes at both intersection endpoints as the geometry requires.

Question 3

The region D lies between $y = x$ and $y = \sqrt{x}$ in the first quadrant.

Tasks

1. Describe D using vertical slices and using horizontal slices.
2. Write $\iint_D 1 \, dA$ in both orders.
3. Evaluate the area and verify the two descriptions agree.

Question 3 – Solution

Strategy. Solve each boundary for the inner variable and use the common endpoints $(0, 0)$ and $(1, 1)$.

Step 1: Two descriptions Since $x \leq \sqrt{x}$ on $[0, 1]$,

$$D = \{0 \leq x \leq 1, x \leq y \leq \sqrt{x}\}.$$

Solving for x , the parabola is $x = y^2$ and the line is $x = y$. For $0 \leq y \leq 1$, $y^2 \leq y$, so

$$D = \{0 \leq y \leq 1, y^2 \leq x \leq y\}.$$

Step 2: Both orders

$$\iint_D 1 \, dA = \int_0^1 \int_x^{\sqrt{x}} 1 \, dy \, dx = \int_0^1 \int_{y^2}^y 1 \, dx \, dy.$$

Step 3: Area The horizontal description gives

$$\text{area}(D) = \int_0^1 (y - y^2) \, dy = \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \boxed{\frac{1}{6}}.$$

Directly, $\int_0^1 (\sqrt{x} - x) \, dx = 2/3 - 1/2 = 1/6$, verifying the reversal.

Question 4

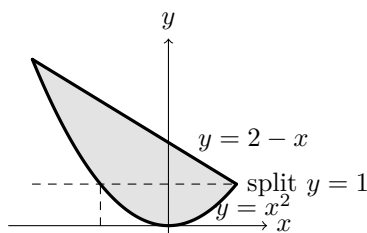
Let D be the region between $y = x^2$ and $y = 2 - x$.

Tasks

1. Describe D with vertical slices.
2. Reverse the order, explaining why horizontal slices require two integrals.
3. Compute the area in the reversed order and check it in the original order.

Question 4 – Solution

Strategy. Locate the intersection points and the height where the right boundary changes from the parabola to the line.



Step 1: Vertical description Solving $x^2 = 2 - x$ gives $x = -2, 1$. Hence

$$D = \{(x, y) : -2 \leq x \leq 1, x^2 \leq y \leq 2 - x\}.$$

Step 2: Reverse and split Horizontal slices begin at $x = -\sqrt{y}$. For $0 \leq y \leq 1$, they end at $x = \sqrt{y}$; for $1 \leq y \leq 4$, they end on the line $x = 2 - y$. Thus

$$\iint_D 1 \, dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} 1 \, dx \, dy + \int_1^4 \int_{-\sqrt{y}}^{2-y} 1 \, dx \, dy.$$

Step 3: Area

$$\begin{aligned} A &= \int_0^1 2\sqrt{y} \, dy + \int_1^4 (2 - y + \sqrt{y}) \, dy \\ &= \frac{4}{3} + \left[2y - \frac{y^2}{2} + \frac{2}{3}y^{3/2} \right]_1^4 = \frac{4}{3} + \frac{19}{6} = \boxed{\frac{9}{2}}. \end{aligned}$$

The original order gives $\int_{-2}^1 (2 - x - x^2) dx = 9/2$, confirming the split.

Question 5

Let

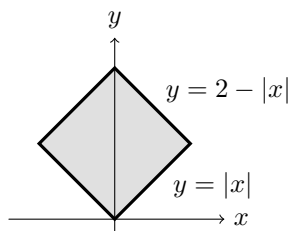
$$D = \{(x, y) : -1 \leq x \leq 1, |x| \leq y \leq 2 - |x|\}.$$

Tasks

1. Sketch or describe the region and identify its symmetry.
2. Evaluate $\iint_D y \, dA$ by splitting at $x = 0$ or using $|x|$ directly.
3. Find the area of D and hence the average value of y .

Question 5 – Solution

Strategy. Each vertical slice has endpoints symmetric about the horizontal line $y = 1$, making its average height immediate.



Step 1: Geometry The vertices are $(0, 0), (1, 1), (0, 2), (-1, 1)$. The diamond is symmetric about both $x = 0$ and $y = 1$.

Step 2: Integral of y

$$\begin{aligned}\iint_D y \, dA &= \int_{-1}^1 \int_{|x|}^{2-|x|} y \, dy \, dx \\ &= \frac{1}{2} \int_{-1}^1 ((2 - |x|)^2 - |x|^2) \, dx \\ &= \int_{-1}^1 (2 - 2|x|) \, dx = \boxed{2}.\end{aligned}$$

Step 3: Area and average The slice width is $2 - 2|x|$, so

$$A = \int_{-1}^1 (2 - 2|x|) \, dx = 2.$$

Therefore $y_{\text{avg}} = 2/2 = \boxed{1}$, exactly the symmetry line. This independently checks the integral: total equals average times area.

Question 6

Consider

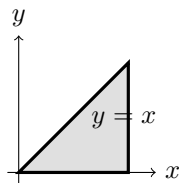
$$I = \int_0^1 \int_y^1 e^{x^2} dx dy.$$

Tasks

1. Describe the triangular region represented by the bounds.
2. Explain why the displayed order is not convenient for elementary integration.
3. Reverse the order and evaluate I exactly.

Question 6 – Solution

Strategy. Draw the triangle $0 \leq y \leq x \leq 1$; reversing the order creates an inner interval whose length supplies the factor needed for substitution.



Step 1: Region The displayed bounds say $0 \leq y \leq 1$ and $y \leq x \leq 1$. Equivalently,

$$D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}.$$

Step 2: Why reverse In the given order, the inner antiderivative $\int e^{x^2} dx$ is not elementary. With dy first, e^{x^2} is constant in the inner variable.

Step 3: Reversed evaluation

$$\begin{aligned} I &= \int_0^1 \int_0^x e^{x^2} dy dx = \int_0^1 x e^{x^2} dx \\ &= \frac{1}{2} \int_0^1 e^u du = \left[\frac{e-1}{2} \right], \quad u = x^2. \end{aligned}$$

The integrand and region are positive, so the positive result is consistent. Differentiating $e^{x^2}/2$ returns $x e^{x^2}$.

Question 7

A solid lies above the triangle

$$D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$$

and below the plane $z = 1 - x - y$.

Tasks

1. Write Cartesian bounds for D .
2. Evaluate the volume using a double integral.
3. Verify the answer using the geometry of a tetrahedron.

Question 7 – Solution

Strategy. Use vertical slices of the triangular base, then compare the integral with the intercept-form tetrahedron.

Step 1: Bounds For $0 \leq x \leq 1$, the variable y runs from 0 to $1 - x$. Therefore

$$V = \int_0^1 \int_0^{1-x} (1 - x - y) dy dx.$$

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx \\ &= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[x - x^2 + \frac{x^3}{3} \right]_0^1 \\ &= \boxed{\frac{1}{6}}. \end{aligned}$$

Step 3: Geometric check The coordinate planes and $x + y + z = 1$ enclose a tetrahedron with mutually perpendicular intercept lengths 1, 1, 1. Its volume is

$$\frac{1}{3}(\text{base area})(\text{height}) = \frac{1}{3} \left(\frac{1}{2} \right) (1) = \frac{1}{6}.$$

This matches the integral and verifies the top surface remains nonnegative over D .

Question 8

For $k > 0$, let D_k be the region enclosed by $y = x^2$ and $y = kx$.

Tasks

1. Find the intersections and write bounds for D_k .
2. Derive a formula for the area as a function of k .
3. Determine the unique $k > 0$ for which the area is $9/2$.

Question 8 – Solution

Strategy. Keep the line slope symbolic; the nonzero intersection becomes the outer endpoint of a vertical-slice area integral.

Step 1: Intersections and bounds From $x^2 = kx$,

$$x(x - k) = 0,$$

so the intersections occur at $x = 0, k$. Since $k > 0$, $x^2 \leq kx$ on $[0, k]$, and

$$D_k = \{(x, y) : 0 \leq x \leq k, x^2 \leq y \leq kx\}.$$

Step 2: Area formula

$$\begin{aligned} A(k) &= \int_0^k \int_{x^2}^{kx} 1 \, dy \, dx = \int_0^k (kx - x^2) \, dx \\ &= \left[\frac{kx^2}{2} - \frac{x^3}{3} \right]_0^k = \boxed{\frac{k^3}{6}}. \end{aligned}$$

Step 3: Inverse condition Set $k^3/6 = 9/2$. Then $k^3 = 27$, and the positive restriction gives

$$\boxed{k = 3}.$$

Substitution yields $A(3) = 27/6 = 9/2$. Since $A(k) = k^3/6$ is strictly increasing for $k > 0$, this positive solution is unique.

Question 9

Let

$$D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq \min(x, 2 - x)\}.$$

Tasks

1. Explain why the vertical-slice description must split at $x = 1$.
2. Give a single Type II description of D .
3. Evaluate $\iint_D (x + 2y) \, dA$ using the more efficient description.

Question 9 – Solution

Strategy. Recognize the triangle with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$; horizontal slices use one left and one right boundary.

Step 1: Vertical split The smaller upper boundary is x on $[0, 1]$ and $2 - x$ on $[1, 2]$. Thus vertical integration would require

$$\int_0^1 \int_0^x (\cdots) dy dx + \int_1^2 \int_0^{2-x} (\cdots) dy dx.$$

Step 2: Single horizontal description For $0 \leq y \leq 1$, the line $y = x$ gives $x = y$ and $y = 2 - x$ gives $x = 2 - y$. Hence

$$D = \{(x, y) : 0 \leq y \leq 1, y \leq x \leq 2 - y\}.$$

Step 3: Evaluate

$$\begin{aligned} \iint_D (x + 2y) dA &= \int_0^1 \int_y^{2-y} (x + 2y) dx dy \\ &= \int_0^1 \left[\frac{x^2}{2} + 2yx \right]_y^{2-y} dy \\ &= \int_0^1 (2 + 2y - 4y^2) dy = \left[2y + y^2 - \frac{4}{3}y^3 \right]_0^1 \\ &= \boxed{\frac{5}{3}}. \end{aligned}$$

The integrand is nonnegative on D , agreeing with the sign.

Question 10

Let

$$D = \{(x, y) : 0 \leq y \leq 1, -y \leq x \leq y\}.$$

Tasks

1. Describe the region in the reverse order, including any required split.
2. Evaluate $\iint_D (x^2 + y) dA$ in the given order.
3. Use symmetry and the reversed description to check the structure of the answer.

Question 10 – Solution

Strategy. Use horizontal symmetry about the y -axis for evaluation, then express the lower boundary as $y = |x|$ when reversing.

Step 1: Reverse description The triangle has vertices $(0, 0)$, $(-1, 1)$, $(1, 1)$. In one compact form,

$$-1 \leq x \leq 1, \quad |x| \leq y \leq 1.$$

Without absolute values it splits into $-1 \leq x \leq 0$, $-x \leq y \leq 1$, and $0 \leq x \leq 1$, $x \leq y \leq 1$.

Step 2: Given order

$$\begin{aligned} \iint_D (x^2 + y) dA &= \int_0^1 \int_{-y}^y (x^2 + y) dx dy \\ &= \int_0^1 \left(\frac{2y^3}{3} + 2y^2 \right) dy \\ &= \frac{1}{6} + \frac{2}{3} = \boxed{\frac{5}{6}}. \end{aligned}$$

Step 3: Structural check In the reversed order, the two x -halves contribute equally because $x^2 + y$ is even in x . Thus

$$\iint_D (x^2 + y) dA = 2 \int_0^1 \int_x^1 (x^2 + y) dy dx.$$

Evaluating gives $2(5/12) = 5/6$. This verifies both the factor of two and the lower boundary $y = |x|$.

Question 1

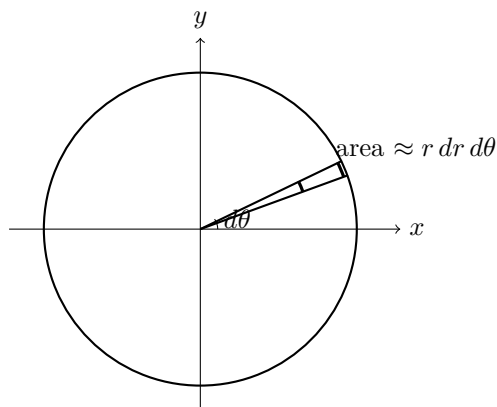
Let D be the disk $x^2 + y^2 \leq 4$.

Tasks

1. Convert $x^2 + y^2$ and dA to polar form.
2. Set up and evaluate $\iint_D (x^2 + y^2) dA$.
3. Explain the Jacobian factor geometrically.

Question 1 – Solution

Strategy. Radial symmetry turns the integrand into r^2 and the area element into $r \, dr \, d\theta$.



Step 1: Conversion The disk is $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, and $x^2 + y^2 = r^2$. Hence

$$\iint_D (x^2 + y^2) dA = \int_0^{2\pi} \int_0^2 r^2 (r \, dr \, d\theta).$$

Step 2: Evaluate

$$\int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\theta = \int_0^{2\pi} 4 \, d\theta = \boxed{8\pi}.$$

Verification A thin polar cell has radial side dr and arc-length side approximately $r \, d\theta$, so its area is $r \, dr \, d\theta$. Omitting r would give the wrong units and value.

Question 2

Let D be the annular sector

$$1 \leq x^2 + y^2 \leq 9, \quad \frac{\pi}{6} \leq \arg(x + iy) \leq \frac{\pi}{2}.$$

Tasks

1. Write polar bounds.
2. Evaluate $\iint_D 1/\sqrt{x^2 + y^2} dA$.
3. Check finiteness and units.

Question 2 – Solution

Strategy. The factor $1/r$ cancels the polar Jacobian, leaving a rectangular integral in (r, θ) .

Step 1: Bounds The radial inequality gives $1 \leq r \leq 3$, while $\pi/6 \leq \theta \leq \pi/2$.

Step 2: Evaluate

$$\begin{aligned}\iint_D \frac{1}{\sqrt{x^2 + y^2}} dA &= \int_{\pi/6}^{\pi/2} \int_1^3 \frac{1}{r} (r \, dr \, d\theta) \\ &= \int_{\pi/6}^{\pi/2} 2 \, d\theta = \boxed{\frac{2\pi}{3}}.\end{aligned}$$

Verification The region stays at least one unit from the origin, so the integrand is bounded and no singularity occurs. After cancellation the radial length is 2 and the angular width is $\pi/3$.

Question 3

Find the area enclosed by the cardioid $r = 1 + \cos \theta$.

Tasks

1. Choose a complete nonduplicating angular interval.
2. Derive the polar area integral from a double integral.
3. Evaluate it exactly.

Question 3 – Solution

Strategy. Integrate 1 with radial bounds from the pole to the cardioid over one full turn.

Step 1: Bounds Since $1 + \cos \theta \geq 0$, one traversal is $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1 + \cos \theta$.

Step 2: Area integral

$$A = \int_0^{2\pi} \int_0^{1+\cos \theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 \, d\theta.$$

Step 3: Evaluate Over a full period, $\int \cos \theta = 0$ and $\int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$. Thus

$$A = \frac{1}{2}(2\pi + 0 + \pi) = \boxed{\frac{3\pi}{2}}.$$

The radial bound vanishes only at $\theta = \pi$, so the interval neither omits nor duplicates positive-area portions.

Question 4

Find the area inside $r = 2 \cos \theta$ but outside the unit circle $r = 1$.

Tasks

1. Determine the angles where the radial bounds are ordered.
2. Set up and evaluate the area integral.
3. Interpret the boundary intersections.

Question 4 – Solution

Strategy. Solve $2 \cos \theta = 1$ to locate the two intersection rays, then integrate from the unit circle to the offset circle.

Step 1: Angles The curves meet when $\cos \theta = 1/2$, so $\theta = \pm\pi/3$. Between these angles, $2 \cos \theta \geq 1$.

Step 2: Area

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \int_1^{2 \cos \theta} r \, dr \, d\theta \\ &= \frac{1}{2} \int_{-\pi/3}^{\pi/3} (4 \cos^2 \theta - 1) d\theta \\ &= \frac{1}{2} \left(4 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) - \frac{2\pi}{3} \right) = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}. \end{aligned}$$

Verification At both endpoint rays the inner and outer radii equal 1, so slice width vanishes. At $\theta = 0$ the slice runs from 1 to 2, confirming the order.

Question 5

For $a > 0$, evaluate

$$\iint_{x^2+y^2 \leq a^2} e^{-(x^2+y^2)} dA.$$

Tasks

1. Convert to polar coordinates.
2. Evaluate as a function of a .
3. Find the limit as $a \rightarrow \infty$ and verify monotonicity.

Question 5 – Solution

Strategy. The exponent and disk are radial; use $u = r^2$ after including the Jacobian.

Step 1: Set up

$$I(a) = \int_0^{2\pi} \int_0^a e^{-r^2} r \, dr \, d\theta.$$

Step 2: Evaluate With $u = r^2$, $du = 2r \, dr$,

$$I(a) = 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^a = \boxed{\pi(1 - e^{-a^2})}.$$

Step 3: Checks Since $I'(a) = 2\pi a e^{-a^2} > 0$, enlarging the disk increases the integral. Also

$$\boxed{\lim_{a \rightarrow \infty} I(a) = \pi}.$$

The result approaches a finite limit because the exponential decay dominates the growing circumference.

Question 6

Let D be the first-quadrant triangle $x \geq 0$, $y \geq 0$, $x + y \leq 2$.

Tasks

1. Convert the line boundary to a radial bound.
2. Use polar coordinates to compute the area.
3. Verify with elementary triangle geometry.

Question 6 – Solution

Strategy. A ray at angle θ meets $x + y = 2$ when $r(\cos \theta + \sin \theta) = 2$.

Step 1: Polar bounds First-quadrant rays give $0 \leq \theta \leq \pi/2$, and

$$0 \leq r \leq \frac{2}{\cos \theta + \sin \theta}.$$

Step 2: Area

$$A = \frac{1}{2} \int_0^{\pi/2} \frac{4}{(\cos \theta + \sin \theta)^2} d\theta.$$

Let $u = \tan \theta$. Since $(\cos \theta + \sin \theta)^2 = (1 + u)^2 / (1 + u^2)$ and $d\theta = du / (1 + u^2)$,

$$A = 2 \int_0^\infty \frac{du}{(1 + u)^2} = \boxed{2}.$$

Verification The Cartesian triangle has legs of length 2, so its area is $\frac{1}{2}(2)(2) = 2$. The nonconstant radial bound is essential; a constant bound would describe a sector, not the triangle.

Question 7

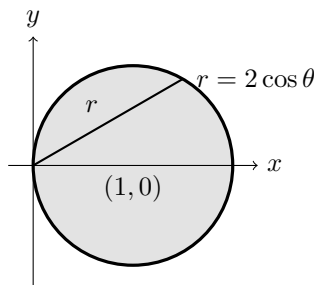
Let D be the disk $(x - 1)^2 + y^2 \leq 1$. Evaluate $\iint_D x \, dA$ in polar coordinates.

Tasks

1. Convert the offset circle to polar bounds.
2. Evaluate the integral.
3. Verify the result using area and symmetry.

Question 7 – Solution

Strategy. Rewrite the circle as $r = 2 \cos \theta$ and remember that $x = r \cos \theta$ contributes another factor of r with dA .



Step 1: Bounds The boundary equation becomes $r^2 - 2r \cos \theta = 0$. Thus

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2 \cos \theta.$$

Step 2: Evaluate

$$\begin{aligned} \iint_D x \, dA &= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^2 \cos \theta \, dr \, d\theta \\ &= \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^4 \theta \, d\theta = \frac{8}{3} \cdot \frac{3\pi}{8} = \boxed{\pi}. \end{aligned}$$

Verification The disk has area π and is symmetric about the vertical line $x = 1$, so its average x -coordinate is 1. Hence its first moment is $1 \cdot \pi = \pi$.

Question 8

Find the area of one petal of the rose $r = \cos(2\theta)$ centered on the positive x -axis.

Tasks

1. Determine the angular interval for exactly that petal.
2. Set up and evaluate its area.
3. Explain why negative r values are excluded.

Question 8 – Solution

Strategy. The desired petal begins and ends where $\cos(2\theta) = 0$ around $\theta = 0$.

Step 1: Bounds The adjacent zeros are $\theta = \pm\pi/4$, and $\cos(2\theta) \geq 0$ between them. Thus $0 \leq r \leq \cos(2\theta)$.

Step 2: Area

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r \, dr \, d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2(2\theta) d\theta \\ &= \frac{1}{4} \left[\theta + \frac{\sin 4\theta}{4} \right]_{-\pi/4}^{\pi/4} = \boxed{\frac{\pi}{8}}. \end{aligned}$$

Verification The sine endpoint terms vanish. Extending beyond these zeros introduces negative radial values, which trace different petals rather than enlarging the positive- x petal.

Question 9

An unknown annulus $a \leq r \leq b$, with $0 < a < b$, has area 3π and satisfies

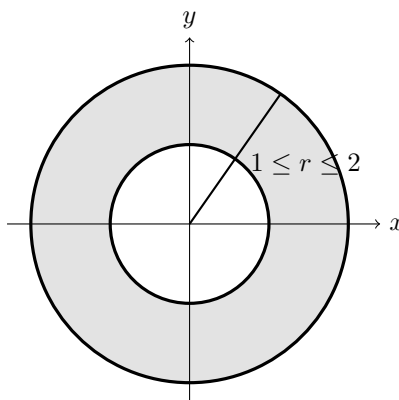
$$\iint_D (x^2 + y^2) dA = \frac{15\pi}{2}.$$

Tasks

1. Translate both conditions into equations for a^2, b^2 .
2. Recover the radii.
3. Verify both measurements.

Question 9 – Solution

Strategy. Set $A = a^2$ and $B = b^2$; area gives their difference, while the radial moment gives a difference of squares.



Step 1: Equations Area gives

$$\pi(b^2 - a^2) = 3\pi, \quad \text{so} \quad B - A = 3.$$

Also

$$\int_0^{2\pi} \int_a^b r^3 dr d\theta = \frac{\pi}{2}(b^4 - a^4) = \frac{15\pi}{2},$$

so $(B - A)(B + A) = 15$.

Step 2: Solve Since $B - A = 3$, we get $B + A = 5$. Therefore $B = 4$, $A = 1$, and positivity gives

$$\boxed{a = 1, \quad b = 2}.$$

Verification The area is $\pi(4 - 1) = 3\pi$, and the moment is $\frac{\pi}{2}(16 - 1) = 15\pi/2$. Both conditions hold exactly.

Question 10

Let D be the upper half-disk $x^2 + y^2 \leq 9$, $y \geq 0$. Evaluate

$$\iint_D (x^2 + y^2 + xy) dA.$$

Tasks

1. Convert the region and integrand to polar coordinates.
2. Use angular symmetry to eliminate the mixed term.
3. Evaluate the remaining integral and verify its sign.

Question 10 – Solution

Strategy. In polar form the radial term is r^2 , while $xy = r^2 \cos \theta \sin \theta$ cancels over $0 \leq \theta \leq \pi$.

Step 1: Set up

$$0 \leq \theta \leq \pi, \quad 0 \leq r \leq 3,$$

so

$$I = \int_0^\pi \int_0^3 r^3 (1 + \cos \theta \sin \theta) dr d\theta.$$

Step 2: Symmetry Since

$$\int_0^\pi \cos \theta \sin \theta d\theta = 0,$$

the xy contribution vanishes, matching reflection symmetry across the y -axis.

Step 3: Evaluate

$$I = \int_0^\pi \left[\frac{r^4}{4} \right]_0^3 d\theta = \int_0^\pi \frac{81}{4} d\theta = \boxed{\frac{81\pi}{4}}.$$

The surviving integrand $x^2 + y^2$ is nonnegative, so the positive result is necessary. Its scale is radius squared times half-disk area, as expected.

Question 1

Evaluate $\iiint_B (x + 2y + 3z) \, dV$ on $B = [0, 1] \times [0, 2] \times [-1, 1]$.

Tasks

1. Write a Cartesian iterated integral.
2. Evaluate it.
3. Verify by average values.

Question 1 – Solution

Strategy. Integrate termwise over the box.

Step 1: Set up $I = \int_0^1 \int_0^2 \int_{-1}^1 (x + 2y + 3z) dz dy dx$.

Step 2: Evaluate The z term is odd; the other terms give $2x + 4y$. Thus $I = \int_0^1 \int_0^2 (2x + 4y) dy dx = \int_0^1 (4x + 8) dx = \boxed{10}$.

Verification The box volume is 4 and its coordinate averages are $(1/2, 1, 0)$, so the integrand average is $5/2$ and $4(5/2) = 10$.

Question 2

Let $E = \{0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 2 - x - y\}$.

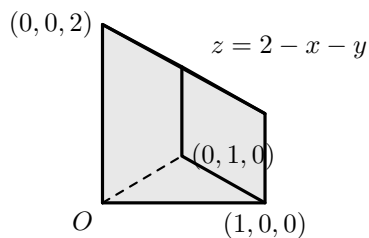
Tasks

1. Interpret the bounds.
2. Evaluate $\iiint_E 1 \, dV$.
3. Check positivity of every bound.

Question 2 – Solution

Strategy. Integrate vertical height over the triangular projection.

Step 1: Geometry The projection is the triangle $x \geq 0$, $y \geq 0$, $x + y \leq 1$. Above it, the solid runs from $z = 0$ to the sloping plane $z = 2 - x - y$.



Step 2: Evaluate $V = \int_0^1 \int_0^{1-x} \int_0^{2-x-y} 1 \, dz \, dy \, dx = \int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2} \right) dx = \boxed{\frac{2}{3}}.$

Verification On the projection, $2 - x - y \geq 1$, so every vertical bound is positive.

Question 3

Find the volume of the first-octant tetrahedron $x/2 + y/3 + z/6 \leq 1$.

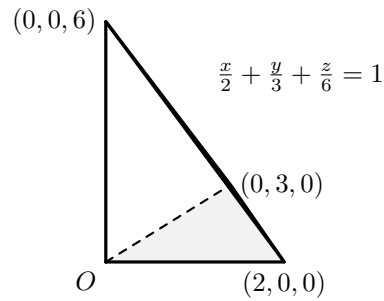
Tasks

1. Write bounds in the order $dz\,dy\,dx$.
2. Evaluate.
3. Verify geometrically.

Question 3 – Solution

Strategy. Solve the plane successively for z and for the projected y -boundary.

Step 1: Geometry The coordinate-plane faces meet the sloping face at the three axis intercepts shown.



Step 2: Bounds and integral $0 \leq x \leq 2$, $0 \leq y \leq 3 - 3x/2$, and $0 \leq z \leq 6 - 3x - 2y$. Hence $V = \int_0^2 \int_0^{3-3x/2} (6 - 3x - 2y) dy dx = \int_0^2 \frac{9}{4} (2 - x)^2 dx = \boxed{6}$.

Verification The intercept formula gives $V = (1/6)(2)(3)(6) = 6$.

Question 4

A box $0 \leq x \leq 2$, $0 \leq y \leq 1$, $0 \leq z \leq 3$ has density $\rho = 1 + x + z$.

Tasks

1. Find its mass.
2. Find the average density.
3. Check against the density range.

Question 4 – Solution

Strategy. Integrate density and divide by volume.

Step 1: Mass $M = \int_0^2 \int_0^1 \int_0^3 (1 + x + z) dz dy dx = \int_0^2 (15/2 + 3x) dx = \boxed{21}$.

Step 2: Average The volume is 6, so $\boxed{\rho_{\text{avg}} = 21/6 = 7/2}$.

Verification Here $1 \leq \rho \leq 6$; $7/2$ lies in this interval. Also the coordinate-average rule gives $1 + 1 + 3/2 = 7/2$.

Question 5

Find the average of $f = x^2 + y^2 + z^2$ on $[-1, 1]^3$.

Tasks

1. Use symmetry and separability.
2. Compute the average.
3. Compare with the range.

Question 5 – Solution

Strategy. The three squared-coordinate contributions are identical.

Step 1: Integral $\iiint_B x^2 dV = (\int_{-1}^1 x^2 dx)(2)(2) = 8/3$. Thus $\iiint_B f dV = 3(8/3) = 8$.

Step 2: Average Since $\text{vol}(B) = 8$, $\boxed{f_{\text{avg}} = 1}$.

Verification $0 \leq f \leq 3$, and each coordinate square has average $1/3$, so their sum has average 1 .

Question 6

Evaluate $\iiint_B (x^3 y z^2 + 4) \, dV$ on $B = [-2, 2] \times [-1, 1] \times [0, 3]$.

Tasks

1. Identify a cancellation symmetry.
2. Evaluate the integral.
3. State why the region matters.

Question 6 – Solution

Strategy. Pair x with $-x$; the nonconstant term is odd in x .

Step 1: Cancellation x^3yz^2 integrates to zero over the symmetric x -interval.

Step 2: Constant term The box volume is $(4)(2)(3) = 24$, so $\boxed{\iiint_B (x^3yz^2 + 4)dV = 4(24) = 96}$.

Verification Reflection $(x, y, z) \mapsto (-x, y, z)$ preserves B and reverses only the nonconstant term. Without that regional symmetry the cancellation would not follow.

Question 7

For $E = \{x, y, z \geq 0 : x + y + z \leq 1\}$, rewrite $\iiint_E z \, dV$ in the orders $dz \, dy \, dx$ and $dx \, dz \, dy$, then evaluate.

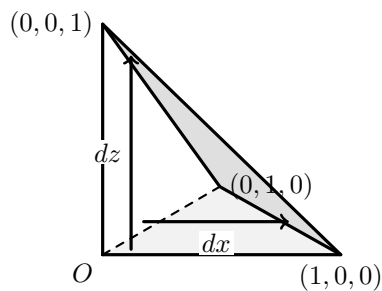
Tasks

1. Give both sets of bounds.
2. Evaluate one order.
3. Verify by symmetry.

Question 7 – Solution

Strategy. Each order peels coordinates from the simplex inequality.

Step 1: Geometry The simplex is bounded by the coordinate planes and $x + y + z = 1$. The two marked fibers show which coordinate is peeled off first.



Step 2: Orders $I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \int_0^1 \int_0^{1-y} \int_0^{1-y-z} z \, dx \, dz \, dy$. The first form gives $I = \frac{1}{6} \int_0^1 (1-x)^3 dx = \boxed{\frac{1}{24}}$.

Verification The simplex volume is $1/6$ and its centroid has $\bar{z} = 1/4$, so the moment is $(1/6)(1/4) = 1/24$.

Question 8

Let $I(a) = \int_0^1 \int_0^2 \int_0^3 (a + x - y + z) \, dz \, dy \, dx$. Find a so the average integrand is 5.

Tasks

1. Evaluate $I(a)$.
2. Solve the average condition.
3. Verify using coordinate averages.

Question 8 – Solution

Strategy. Convert average 5 into total 5 times box volume.

Step 1: Integral The volume is 6 and the coordinate averages are $(1/2, 1, 3/2)$, hence $I(a) = 6(a + 1)$.

Step 2: Solve $I(a)/6 = 5$ gives $a + 1 = 5$, so $\boxed{a = 4}$.

Verification At $a = 4$, the average is $4 + 1/2 - 1 + 3/2 = 5$, and the total integral is 30.

Question 9

For continuous f, g, h on their respective intervals, prove the product rule for $f(x)g(y)h(z)$ on $[a, b] \times [c, d] \times [e, k]$, then apply it to $(1+x)y^2e^z$ on $[0, 1] \times [-1, 1] \times [0, \ln 2]$.

Tasks

1. Prove factorization.
2. Evaluate the example.
3. Check the sign.

Question 9 – Solution

Strategy. Pull factors independent of the current integration variable outside each inner integral.

Step 1: Proof Successive integration gives $\iiint fgh \, dV = (\int_a^b f)(\int_c^d g)(\int_e^k h)$. Continuity guarantees the orders are valid.

Step 2: Example $\int_0^1 (1+x)dx = 3/2$, $\int_{-1}^1 y^2 dy = 2/3$, and $\int_0^{\ln 2} e^z dz = 1$. Their product is $\boxed{1}$.

Verification All factors are nonnegative on the box, matching the positive result.

Question 10

A solid E has volume 12, and $2 \leq f \leq 7$ throughout E . Another integrable function g satisfies $|g - f| \leq 0.05$. Given $\iiint_E f \, dV = 50$, bound $\iiint_E g \, dV$.

Tasks

1. Check consistency of the data.
2. Derive sharp bounds for the new integral.
3. Explain sharpness.

Question 10 – Solution

Strategy. Integrate pointwise inequalities and scale uniform error by volume.

Step 1: Consistency $24 \leq \iiint_E f \, dV \leq 84$, so 50 is possible.

Step 2: Perturbation From $-0.05 \leq g - f \leq 0.05$ and $\text{vol}(E) = 12$,

$$-0.6 \leq \iiint_E (g - f) \, dV \leq 0.6.$$

Therefore $\boxed{49.4 \leq \iiint_E g \, dV \leq 50.6}.$

Verification The endpoints occur for $g = f - 0.05$ and $g = f + 0.05$, so the supplied data guarantee no tighter interval.

Question 1

Let

$$E = \{(x, y, z) : x^2 + y^2 \leq 4, x \geq 0, y \geq 0, -1 \leq z \leq 2\}.$$

Evaluate $\iiint_E (x^2 + y^2) dV$ using cylindrical coordinates.

Tasks

1. Convert the region, integrand, and volume element.
2. Evaluate the cylindrical integral.
3. Verify the result using an average value.

Question 1 – Solution

Strategy. Treat the solid as a quarter-cylinder; radial symmetry makes the integrand r^2 .

Step 1: Convert Cylindrical coordinates give

$$x = r \cos \theta, \quad y = r \sin \theta, \quad x^2 + y^2 = r^2, \quad dV = r \, dz \, dr \, d\theta.$$

The first-quadrant quarter-cylinder has bounds

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2, \quad -1 \leq z \leq 2.$$

Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^2 \int_{-1}^2 r^2 (r \, dz \, dr \, d\theta) \\ &= \left(\frac{\pi}{2}\right) (3) \left[\frac{r^4}{4}\right]_0^2 = \boxed{6\pi}. \end{aligned}$$

Verification The volume is $(\pi 2^2/4)(3) = 3\pi$. On a radius-2 disk, the average of r^2 is $2^2/2 = 2$; restricting to one quadrant does not change that radial average. Thus $I = (3\pi)(2) = 6\pi$.

Question 2

Find the volume inside the paraboloid

$$z = 9 - x^2 - y^2$$

and above the xy -plane.

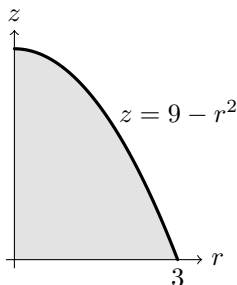
Tasks

1. Determine the cylindrical bounds from the surface intersection.
2. Evaluate the volume.
3. Check the answer against a containing cylinder.

Question 2 – Solution

Strategy. In cylindrical coordinates the paraboloid becomes $z = 9 - r^2$; its intersection with $z = 0$ determines the radial endpoint.

Step 1: Geometry Setting $z = 0$ gives $r = 3$. A meridian half-plane shows the height above each radius.



Thus $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 3$, and $0 \leq z \leq 9 - r^2$.

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r \, dz \, dr \, d\theta \\ &= 2\pi \int_0^3 (9r - r^3) \, dr = 2\pi \left[\frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_0^3 = \boxed{\frac{81\pi}{2}}. \end{aligned}$$

Verification The solid lies inside the cylinder of radius 3 and height 9, whose volume is 81π . The computed volume is positive and exactly half that containing-cylinder volume.

Question 3

Let E lie above the cone $z = \sqrt{x^2 + y^2}$ and below the paraboloid $z = 2 - x^2 - y^2$.

Tasks

1. Find the curve where the two surfaces meet.
2. Set up and evaluate $\iiint_E 1 \, dV$ in cylindrical coordinates.
3. Verify that the vertical bounds are correctly ordered.

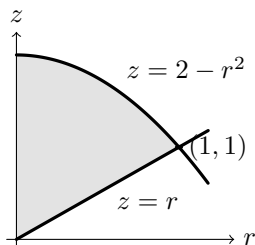
Question 3 – Solution

Strategy. Convert both surfaces to functions of r , solve for their positive intersection radius, and integrate top minus bottom.

Step 1: Intersection The surfaces are $z = r$ and $z = 2 - r^2$. Their intersection satisfies

$$r = 2 - r^2 \iff (r - 1)(r + 2) = 0.$$

Since $r \geq 0$, the boundary circle is $r = 1, z = 1$.



Step 2: Integral

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r \, dz \, dr \, d\theta \\ &= 2\pi \int_0^1 (2 - r^2 - r)r \, dr = 2\pi \left(1 - \frac{1}{4} - \frac{1}{3} \right) = \boxed{\frac{5\pi}{6}}. \end{aligned}$$

Verification For $0 \leq r \leq 1$, $2 - r^2 - r = (1 - r)(r + 2) \geq 0$, so the upper and lower z -bounds never reverse.

Question 4

Let

$$E = \{(x, y, z) : x^2 + y^2 \leq 4x, 0 \leq z \leq \sqrt{x^2 + y^2}\}.$$

Find the volume of E using cylindrical coordinates.

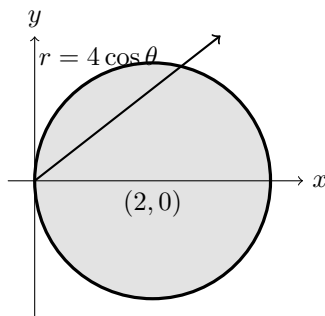
Tasks

1. Convert the offset cylindrical boundary and determine a nonduplicating angular interval.
2. Evaluate the volume.
3. Check the radial bounds at the endpoint and central rays.

Question 4 – Solution

Strategy. Convert the footprint to $r \leq 4 \cos \theta$ and use the conical height $z = r$.

Step 1: Footprint From $r^2 \leq 4r \cos \theta$, nonnegative radii satisfy $0 \leq r \leq 4 \cos \theta$. This requires $-\pi/2 \leq \theta \leq \pi/2$ and describes the disk centered at $(2, 0)$ with radius 2.



Step 2: Evaluate

$$\begin{aligned} V &= \int_{-\pi/2}^{\pi/2} \int_0^{4 \cos \theta} \int_0^r r \, dz \, dr \, d\theta \\ &= \frac{64}{3} \int_{-\pi/2}^{\pi/2} \cos^3 \theta \, d\theta = \frac{64}{3} \left(\frac{4}{3} \right) = \boxed{\frac{256}{9}}. \end{aligned}$$

Verification At $\theta = \pm\pi/2$ the radial interval collapses to $r = 0$; at $\theta = 0$ it reaches $r = 4$. These are exactly the leftmost and rightmost points of the footprint, so no rays are omitted or duplicated.

Question 5

A solid annular cylinder is described by

$$1 \leq x^2 + y^2 \leq 4, \quad 0 \leq z \leq 3,$$

and has density $\rho(x, y, z) = \sqrt{x^2 + y^2} + z$.

Tasks

1. Find its mass using cylindrical coordinates.
2. Find its average density.
3. Compare the average with the density range.

Question 5 – Solution

Strategy. The annular cylinder becomes a rectangular box in (r, θ, z) , while the density becomes $r + z$.

Step 1: Mass The bounds are $0 \leq \theta \leq 2\pi$, $1 \leq r \leq 2$, and $0 \leq z \leq 3$. Therefore

$$\begin{aligned} M &= \int_0^{2\pi} \int_1^2 \int_0^3 (r + z)r \, dz \, dr \, d\theta \\ &= 2\pi \int_1^2 \left(3r^2 + \frac{9}{2}r \right) dr = 2\pi \left[r^3 + \frac{9}{4}r^2 \right]_1^2 = \boxed{\frac{55\pi}{2}}. \end{aligned}$$

Step 2: Average density The volume is $\pi(2^2 - 1^2)(3) = 9\pi$. Hence

$$\boxed{\rho_{\text{avg}} = \frac{M}{V} = \frac{55}{18}}.$$

Verification On the solid, $1 \leq r + z \leq 5$, and $55/18 \approx 3.06$ lies strictly in that range. The units of M/V are density units, as required.

Question 6

The cone

$$0 \leq z \leq 3 - \sqrt{x^2 + y^2}$$

has constant density $\rho = 2$.

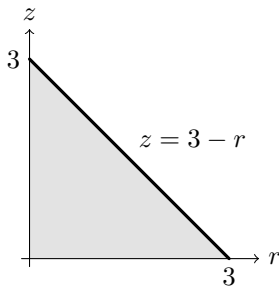
Tasks

1. Find its mass.
2. Find its moment of inertia I_z about the z -axis.
3. Find its radius of gyration $k_z = \sqrt{I_z/M}$.

Question 6 – Solution

Strategy. Use $x^2 + y^2 = r^2$ for the squared distance to the axis and $dV = r \, dz \, dr \, d\theta$.

Step 1: Region The meridian triangle gives $0 \leq r \leq 3$ and $0 \leq z \leq 3 - r$.



Step 2: Mass

$$M = \int_0^{2\pi} \int_0^3 \int_0^{3-r} 2r \, dz \, dr \, d\theta = 4\pi \int_0^3 r(3-r) \, dr = \boxed{18\pi}.$$

Step 3: Moment and radius of gyration

$$I_z = \iiint_E (x^2 + y^2) \rho \, dV = 4\pi \int_0^3 r^3(3-r) \, dr = \boxed{\frac{243\pi}{5}},$$
$$k_z = \sqrt{\frac{I_z}{M}} = \boxed{\sqrt{\frac{27}{10}}}.$$

Verification $k_z \approx 1.64$ lies between the minimum and maximum distances 0 and 3 from the axis. Also $M = 2 \cdot \frac{1}{3}\pi(3^2)(3) = 18\pi$ agrees with the cone-volume formula.

Question 7

Let E be the ball $x^2 + y^2 + z^2 \leq 4$. Without using spherical coordinates, evaluate

$$\iiint_E (x^2 + y^2) \, dV.$$

Tasks

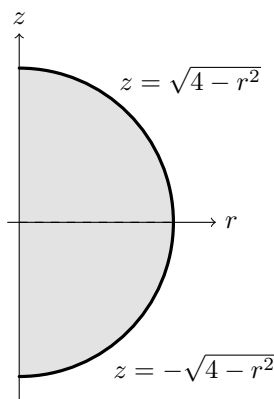
1. Express the ball with cylindrical bounds.
2. Evaluate the integral exactly.
3. Check positivity and magnitude.

Question 7 – Solution

Strategy. At fixed cylindrical radius r , the sphere supplies symmetric upper and lower z -bounds.

Step 1: Bounds Since $r^2 + z^2 \leq 4$,

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}.$$



Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2 (r \, dz \, dr \, d\theta) \\ &= 4\pi \int_0^2 r^3 \sqrt{4-r^2} \, dr. \end{aligned}$$

With $u = 4 - r^2$, $r^3 dr = -(4 - u)du/2$, so

$$\int_0^2 r^3 \sqrt{4-r^2} \, dr = \frac{1}{2} \int_0^4 (4-u)u^{1/2} du = \frac{64}{15}.$$

Thus $\boxed{I = 256\pi/15}$.

Verification The integrand is nonnegative and at most 4. Since the ball volume is $32\pi/3$, the required bound $0 < I < 128\pi/3$ holds.

Question 8

Let E be the solid between the paraboloid $z = x^2 + y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.

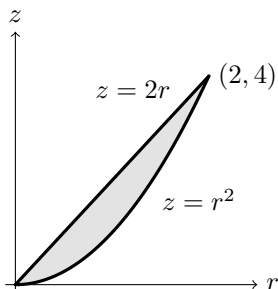
Tasks

1. Determine the cylindrical bounds.
2. Compute $\iiint_E z \, dV$.
3. Find the average value of z on E .

Question 8 – Solution

Strategy. Find the nonzero intersection radius, then integrate z between the radial graphs.

Step 1: Geometry The surfaces meet where $r^2 = 2r$, so $r = 0$ or $r = 2$. On $0 \leq r \leq 2$, the cone lies above the paraboloid.



Thus $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$, and $r^2 \leq z \leq 2r$.

Step 2: Integral of z

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^2 \int_{r^2}^{2r} zr \, dz \, dr \, d\theta \\ &= 2\pi \int_0^2 \left(2r^3 - \frac{1}{2}r^5 \right) dr = \boxed{\frac{16\pi}{3}}. \end{aligned}$$

Step 3: Average The volume is

$$V = 2\pi \int_0^2 (2r - r^2)r \, dr = \frac{8\pi}{3}.$$

Therefore $\boxed{z_{\text{avg}} = I/V = 2}$. This lies between the solid's minimum and maximum heights, 0 and 4.

Question 9

Explain why cylindrical coordinates use

$$dV = r \, dz \, dr \, d\theta,$$

then apply the formula to evaluate $\iiint_E (x^2 + y^2) \, dV$, where

$$1 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{3}, \quad 0 \leq z \leq 4.$$

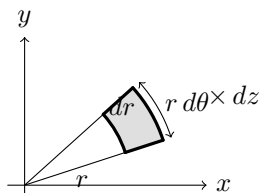
Tasks

1. Derive the scale factor r geometrically.
2. Evaluate the integral.
3. Verify the dimensional scaling.

Question 9 – Solution

Strategy. Approximate a small cylindrical cell by a rectangular box whose angular side is an arc of length $r d\theta$.

Step 1: Jacobian factor A small cell has radial length dr , vertical length dz , and circumferential length approximately $r d\theta$.



Consequently,

$$dV \approx (dr)(r d\theta)(dz) = r dz dr d\theta,$$

and the approximation becomes exact in the integral limit.

Step 2: Apply Since $x^2 + y^2 = r^2$,

$$\begin{aligned} I &= \int_0^{\pi/3} \int_1^2 \int_0^4 r^2 (r dz dr d\theta) \\ &= \left(\frac{\pi}{3}\right) (4) \left[\frac{r^4}{4}\right]_1^2 = \boxed{5\pi}. \end{aligned}$$

Verification The factors scale as angular width times height times radius to the fourth power. That produces length^5 , the correct units for integrating a squared distance over volume.

Question 10

For $a > 0$, let E_a be the solid

$$0 \leq z \leq a^2, \quad z \geq x^2 + y^2.$$

Its average z -coordinate is 6.

Tasks

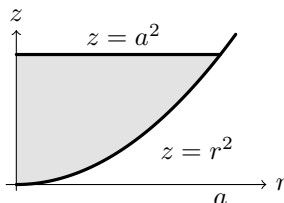
1. Express E_a in cylindrical coordinates.
2. Derive its volume and average z -coordinate as functions of a .
3. Recover a and the corresponding volume.

Question 10 – Solution

Strategy. The horizontal plane meets the paraboloid at $r = a$; compute both volume and first z -moment before imposing the average.

Step 1: Bounds The solid lies above $z = r^2$ and below $z = a^2$:

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq a, \quad r^2 \leq z \leq a^2.$$



Step 2: Volume and moment

$$V(a) = 2\pi \int_0^a (a^2 - r^2)r \, dr = \frac{\pi a^4}{2},$$
$$M_z(a) = 2\pi \int_0^a \int_{r^2}^{a^2} zr \, dz \, dr = \frac{\pi a^6}{3}.$$

Therefore

$$z_{\text{avg}} = \frac{M_z(a)}{V(a)} = \frac{2a^2}{3}.$$

Step 3: Recover the parameter The condition $2a^2/3 = 6$ gives $a^2 = 9$. Since $a > 0$,

$$\boxed{a = 3}, \quad \boxed{V(3) = \frac{81\pi}{2}}.$$

Verification For $a = 3$, the solid has $0 \leq z \leq 9$, and the average 6 lies inside that range. Substitution returns $2(3^2)/3 = 6$ exactly.

Question 1

Let E be the part of the spherical shell

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 3$$

in the first octant. Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV$$

using spherical coordinates.

Tasks

1. Convert the region, integrand, and volume element.
2. Evaluate the integral.
3. Check the result using the integrand range and region volume.

Question 1 – Solution

Strategy. A first-octant spherical shell has constant bounds in all three spherical variables, and the radial integrand cancels one power of the Jacobian.

Step 1: Convert Using

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,$$

the bounds are

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 1 \leq \rho \leq 3,$$

and $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.

Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^3 \frac{1}{\rho} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left(\frac{\pi}{2}\right) (1) \left[\frac{\rho^2}{2}\right]_1^3 = \boxed{2\pi}. \end{aligned}$$

Verification The shell-octant volume is $\frac{1}{8} \frac{4\pi}{3} (3^3 - 1^3) = 13\pi/3$. Hence the average integrand is $I/V = 6/13$, which lies between its endpoint values $1/3$ and 1 .

Question 2

Find the volume inside the sphere $x^2 + y^2 + z^2 \leq 16$ and above the cone

$$z = \sqrt{3}\sqrt{x^2 + y^2}.$$

Tasks

1. Convert the cone to a constant ϕ -boundary.
2. Set up and evaluate the spherical integral.
3. Compare with the volume of the containing ball.

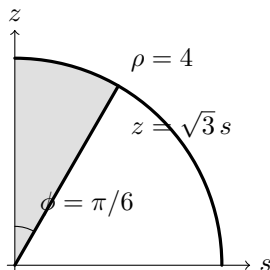
Question 2 – Solution

Strategy. In a meridian plane, the cone fixes the polar angle while the sphere fixes the radial endpoint.

Step 1: Angular boundary Since $z = \rho \cos \phi$ and $\sqrt{x^2 + y^2} = \rho \sin \phi$,

$$\rho \cos \phi = \sqrt{3} \rho \sin \phi \implies \tan \phi = \frac{1}{\sqrt{3}}.$$

The region above the cone therefore has $0 \leq \phi \leq \pi/6$.



Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/6} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 2\pi \left(1 - \frac{\sqrt{3}}{2}\right) \frac{64}{3} = \boxed{\frac{64\pi}{3}(2 - \sqrt{3})}. \end{aligned}$$

Verification The angular factor $1 - \sqrt{3}/2$ is between 0 and 1, so this positive sector is smaller than the upper half-ball and therefore smaller than the entire ball of volume $256\pi/3$.

Question 3

The upper half of the spherical shell $1 \leq \rho \leq 2$ has density $\delta = \rho$.

Tasks

1. Find its mass in spherical coordinates.
2. Find its average density.
3. Verify the average against the density range.

Question 3 – Solution

Strategy. The hemisphere contributes the full azimuthal interval but only $0 \leq \phi \leq \pi/2$; the radial density adds one power of ρ .

Step 1: Mass

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= (2\pi)(1) \left[\frac{\rho^4}{4} \right]_1^2 = \boxed{\frac{15\pi}{2}}. \end{aligned}$$

Step 2: Average density The region volume is half the difference of two ball volumes:

$$V = \frac{1}{2} \frac{4\pi}{3} (2^3 - 1^3) = \frac{14\pi}{3}.$$

Therefore

$$\boxed{\delta_{\text{avg}} = \frac{M}{V} = \frac{45}{28}}.$$

Verification Since $1 \leq \delta = \rho \leq 2$ throughout the shell, the value $45/28 \approx 1.61$ is admissible. The larger outer spherical layers correctly pull it above the midpoint $3/2$.

Question 4

Let E be the upper half of the ball $x^2 + y^2 + z^2 \leq 9$.

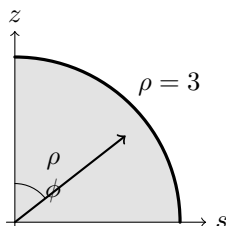
Tasks

1. Use spherical coordinates to compute $\iiint_E z \, dV$.
2. Find the average height z_{avg} .
3. Check its geometric range.

Question 4 – Solution

Strategy. Write $z = \rho \cos \phi$ and use the upper-hemisphere bounds $0 \leq \phi \leq \pi/2$.

Step 1: Geometry and integral



$$\begin{aligned}
 I &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= (2\pi) \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} \left[\frac{\rho^4}{4} \right]_0^3 = \boxed{\frac{81\pi}{4}}.
 \end{aligned}$$

Step 2: Average height The hemisphere volume is $V = \frac{1}{2} \frac{4\pi}{3} (3^3) = 18\pi$. Thus

$$z_{\text{avg}} = \frac{I}{V} = \frac{9}{8}.$$

Verification Every point has $0 \leq z \leq 3$, and $9/8$ lies in that interval. The value is below the mid-height $3/2$ because cross-sectional area is greatest near $z = 0$.

Question 5

Inside a ball of radius R , consider the cone sector

$$0 \leq \rho \leq R, \quad 0 \leq \phi \leq \alpha, \quad 0 \leq \theta \leq 2\pi,$$

where $0 < \alpha < \pi/2$. Its volume is one-third of the upper half-ball.

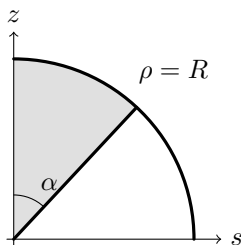
Tasks

1. Determine α .
2. Write the boundary cone in Cartesian coordinates.
3. Verify the stated volume ratio.

Question 5 – Solution

Strategy. The radial and azimuthal factors cancel from the ratio, leaving only the polar-angle factor $1 - \cos \alpha$.

Step 1: Sector volume



$$V_{\text{sector}} = \int_0^{2\pi} \int_0^\alpha \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{2\pi R^3}{3} (1 - \cos \alpha).$$

The upper half-ball has volume $2\pi R^3/3$. Hence

$$1 - \cos \alpha = \frac{1}{3}, \quad \boxed{\alpha = \arccos\left(\frac{2}{3}\right)}.$$

Step 2: Cartesian cone On $\phi = \alpha$, $z = s \cot \alpha$, where $s = \sqrt{x^2 + y^2}$. Since $\sin \alpha = \sqrt{5}/3$, we have $\cot \alpha = 2/\sqrt{5}$. Thus

$$\boxed{z = \frac{2}{\sqrt{5}} \sqrt{x^2 + y^2}}.$$

Verification Substituting $\cos \alpha = 2/3$ makes the angular factor $1 - \cos \alpha = 1/3$, so the sector volume is exactly one-third of the upper half-ball volume.

Question 6

Use spherical coordinates to find the volume inside

$$x^2 + y^2 + z^2 \leq 2z.$$

Tasks

1. Convert the boundary to a radial spherical bound.
2. Evaluate the volume.
3. Identify the Cartesian center and radius as a check.

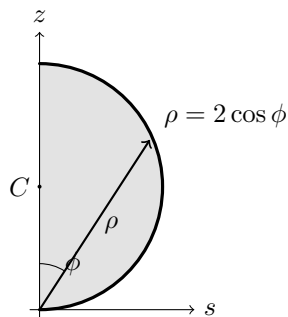
Question 6 – Solution

Strategy. The shifted sphere becomes $\rho \leq 2 \cos \phi$, so its nonconstant radial bound encodes the displacement from the origin.

Step 1: Bounds Substitution gives

$$\rho^2 \leq 2\rho \cos \phi.$$

For $\rho \geq 0$, the nontrivial radial interval is $0 \leq \rho \leq 2 \cos \phi$, which requires $0 \leq \phi \leq \pi/2$. Also $0 \leq \theta \leq 2\pi$.



Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \frac{16\pi}{3} \int_0^{\pi/2} \cos^3 \phi \sin \phi \, d\phi = \boxed{\frac{4\pi}{3}}. \end{aligned}$$

Verification Completing the square gives $x^2 + y^2 + (z - 1)^2 \leq 1$, a sphere of radius 1 centered at $(0, 0, 1)$. Its ordinary volume is $4\pi/3$, confirming the integral.

Question 7

Let

$$E = \{(x, y, z) : 2 \leq \sqrt{x^2 + y^2 + z^2} \leq 3, x \geq 0, y \geq x, z \leq 0\}.$$

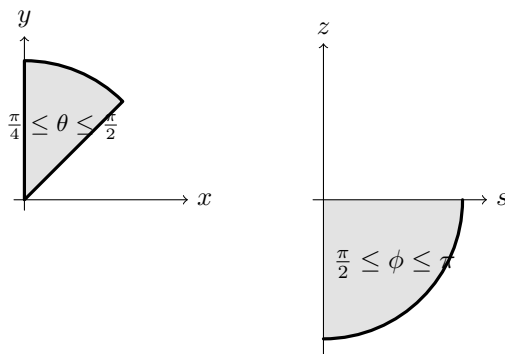
Tasks

1. Determine exact spherical bounds for E .
2. Find its volume.
3. Verify the angular fractions geometrically.

Question 7 – Solution

Strategy. Resolve the horizontal inequalities into a θ -wedge and the vertical inequality into a ϕ -hemisphere.

Step 1: Angular bounds The conditions $x \geq 0$ and $y \geq x$ select $\pi/4 \leq \theta \leq \pi/2$. The condition $z \leq 0$ selects $\pi/2 \leq \phi \leq \pi$, while the shell gives $2 \leq \rho \leq 3$.



Step 2: Volume

$$\begin{aligned}
 V &= \int_{\pi/4}^{\pi/2} \int_{\pi/2}^{\pi} \int_2^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \left(\frac{\pi}{4}\right) (1) \left(\frac{27-8}{3}\right) = \boxed{\frac{19\pi}{12}}.
 \end{aligned}$$

Verification The azimuthal wedge is $(\pi/4)/(2\pi) = 1/8$ of a full turn, and $z \leq 0$ supplies half the polar solid angle. Thus E occupies $1/16$ of the full shell; indeed $\frac{1}{16} \frac{4\pi}{3} (27-8) = 19\pi/12$.

Question 8

A full spherical shell $1 \leq \rho \leq 2$ has density $\delta = \rho$. Find its moment of inertia about the z -axis.

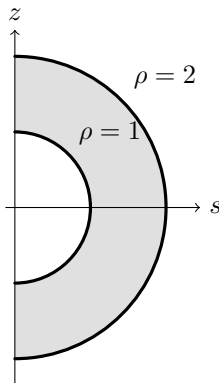
Tasks

1. Express the squared distance to the z -axis in spherical coordinates.
2. Compute the mass and I_z .
3. Find and check the radius of gyration $k_z = \sqrt{I_z/M}$.

Question 8 – Solution

Strategy. The distance to the z -axis is $\rho \sin \phi$; combine its square with the radial density and spherical Jacobian.

Step 1: Region and mass



$$M = \int_0^{2\pi} \int_0^\pi \int_1^2 \rho \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = (2\pi)(2) \frac{15}{4} = \boxed{15\pi}.$$

Step 2: Moment of inertia

$$\begin{aligned} I_z &= \iiint_E (\rho \sin \phi)^2 \delta \, dV \\ &= \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^5 \sin^3 \phi \, d\rho \, d\phi \, d\theta \\ &= (2\pi) \left(\frac{4}{3} \right) \left(\frac{2^6 - 1}{6} \right) = \boxed{28\pi}. \end{aligned}$$

Step 3: Radius of gyration

$$\boxed{k_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{28}{15}}}.$$

It lies between the minimum and maximum distances 0 and 2 from the axis, as required.

Question 9

Explain why spherical coordinates use

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta,$$

then evaluate $\iiint_E \rho^{-2} \, dV$ on the shell $1 \leq \rho \leq 3$.

Tasks

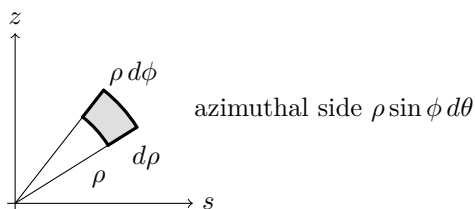
1. Derive the three local scale factors geometrically.
2. Evaluate the integral over the full shell.
3. Check the dimensional scaling and behavior near the origin.

Question 9 – Solution

Strategy. Approximate a small spherical cell by a box with one radial side and two angular arc-length sides.

Step 1: Volume element The local side lengths are

$$d\rho, \quad \rho d\phi, \quad \rho \sin \phi d\theta.$$



Their product is

$$(d\rho)(\rho d\phi)(\rho \sin \phi d\theta) = \rho^2 \sin \phi d\rho d\phi d\theta.$$

Step 2: Apply

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^\pi \int_1^3 \rho^{-2} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= (2\pi)(2)(3-1) = \boxed{8\pi}. \end{aligned}$$

Verification The shell excludes $\rho = 0$, so the integrand is bounded. Its units are length^{-2} ; multiplying by volume produces length, matching the radial factor $3 - 1$ in the result. Even if the inner radius tends to 0, the radial integral remains finite: $4\pi \int_0^3 d\rho = 12\pi$. Thus the singularity is integrable.

Question 10

For $a > 0$, a uniform spherical shell occupies

$$a \leq \rho \leq 2a.$$

Its average distance from the origin is $45/7$.

Tasks

1. Derive the shell volume as a function of a .
2. Derive the average value of ρ .
3. Recover a , the outer radius, and the shell volume.

Question 10 – Solution

Strategy. Divide the first radial moment by volume; the common angular factors cancel, leaving a linear function of a .

Step 1: Volume

$$\begin{aligned} V(a) &= \int_0^{2\pi} \int_0^\pi \int_a^{2a} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 4\pi \left[\frac{\rho^3}{3} \right]_a^{2a} = \frac{28\pi a^3}{3}. \end{aligned}$$

Step 2: Average radius The first radial moment is

$$M_\rho(a) = 4\pi \int_a^{2a} \rho^3 \, d\rho = 15\pi a^4.$$

Hence

$$\rho_{\text{avg}} = \frac{M_\rho(a)}{V(a)} = \frac{45a}{28}.$$

Step 3: Recover the shell Setting $45a/28 = 45/7$ gives $\boxed{a = 4}$. Therefore

$$\boxed{\text{outer radius} = 8}, \quad \boxed{V = \frac{1792\pi}{3}}.$$

Verification The average $45/7 \approx 6.43$ lies strictly between the recovered radii 4 and 8. Substituting $a = 4$ into $28\pi a^3/3$ gives the stated volume.

Question 1

Let

$$R = \{(x, y) : |x + y| \leq 2, |x - y| \leq 1\}.$$

Use $u = x + y$ and $v = x - y$ to evaluate

$$\iint_R (x + y)^2 dA.$$

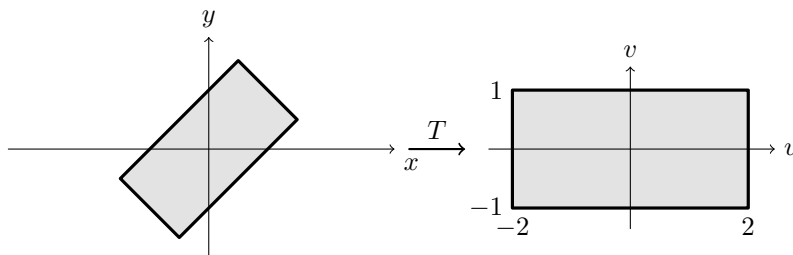
Tasks

1. Describe the image of R in the uv -plane.
2. Compute the inverse Jacobian and evaluate the integral.
3. Verify the sign and scale of the answer.

Question 1 – Solution

Strategy. The boundary expressions themselves become coordinates, turning the tilted rectangle into an axis-aligned rectangle.

Step 1: Transform the region



The inequalities become $-2 \leq u \leq 2$ and $-1 \leq v \leq 1$. Solving $u = x + y$, $v = x - y$ gives

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}.$$

Step 2: Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}, \quad dA = \frac{1}{2} du dv.$$

Step 3: Evaluate

$$\begin{aligned} I &= \int_{-2}^2 \int_{-1}^1 u^2 \left(\frac{1}{2} \right) dv du \\ &= \frac{1}{2}(2) \left[\frac{u^3}{3} \right]_{-2}^2 = \boxed{\frac{16}{3}}. \end{aligned}$$

Verification The integrand is nonnegative. The transformed rectangle has area 8, scaled by $1/2$, so $\text{area}(R) = 4$; the result corresponds to average value $4/3$, which is below the maximum $u^2 = 4$.

Question 2

In the first quadrant, let R be bounded by

$$xy = 1, \quad xy = 4, \quad \frac{x}{y} = 1, \quad \frac{x}{y} = 9.$$

Use $u = xy$ and $v = x/y$ to evaluate $\iint_R (x/y) \, dA$.

Tasks

1. Show that the transformation is one-to-one on R .
2. Find the inverse Jacobian.
3. Evaluate and check the result against the integrand range.

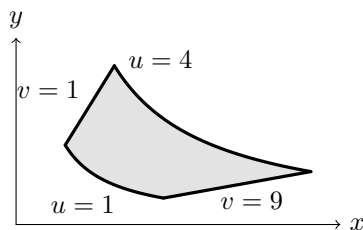
Question 2 – Solution

Strategy. Product and ratio coordinates straighten the hyperbolas and rays into a rectangle.

Step 1: Region and inverse In the first quadrant,

$$x = \sqrt{uv}, \quad y = \sqrt{\frac{u}{v}},$$

so positive (u, v) determine a unique (x, y) . The image is $1 \leq u \leq 4$, $1 \leq v \leq 9$.



Step 2: Jacobian Differentiating the inverse formulas gives

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{2v}.$$

Therefore the factor $x/y = v$ cancels the v in the denominator.

Step 3: Evaluate

$$I = \int_1^4 \int_1^9 v \left(\frac{1}{2v} \right) dv du = \frac{1}{2}(3)(8) = \boxed{12}.$$

Verification The region area is $\int_1^4 \int_1^9 (2v)^{-1} dv du = 3 \ln 3$. Since $1 \leq x/y \leq 9$, the bounds $3 \ln 3 \leq I \leq 27 \ln 3$ hold; $I = 12$ lies between them.

Question 3

Let

$$E = \left\{ (x, y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \right\}.$$

Use $x = 3u$, $y = 2v$ and then polar coordinates in the uv -plane to evaluate

$$\iint_E \left(\frac{x^2}{9} + \frac{y^2}{4} \right) dA.$$

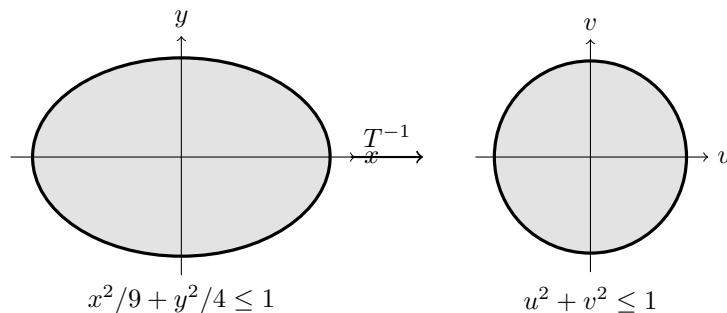
Tasks

1. Map the ellipse to a unit disk.
2. Track both Jacobian factors and evaluate.
3. Verify by comparing the integral with the ellipse area.

Question 3 – Solution

Strategy. First remove the unequal axis scales, then exploit radial symmetry on the resulting unit disk.

Step 1: Linear scaling



The map $x = 3u$, $y = 2v$ sends the unit disk D to E , with

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = 6, \quad \frac{x^2}{9} + \frac{y^2}{4} = u^2 + v^2.$$

Step 2: Polar coordinates in D Write $u = r \cos \theta$, $v = r \sin \theta$ and $du dv = r dr d\theta$. Then

$$\begin{aligned} I &= 6 \int_0^{2\pi} \int_0^1 r^2 (r dr d\theta) \\ &= 6(2\pi) \left[\frac{r^4}{4} \right]_0^1 = \boxed{3\pi}. \end{aligned}$$

Verification The ellipse area is 6π , and the transformed radial-square function has average $1/2$ on the unit disk. Thus the integral must be $(6\pi)(1/2) = 3\pi$.

Question 4

Let R be bounded by the four lines

$$x + 2y = 0, \quad x + 2y = 2, \quad 3x - y = 1, \quad 3x - y = 4.$$

Use $u = x + 2y$, $v = 3x - y$ to evaluate

$$\iint_R (x + 2y)(3x - y) \, dA.$$

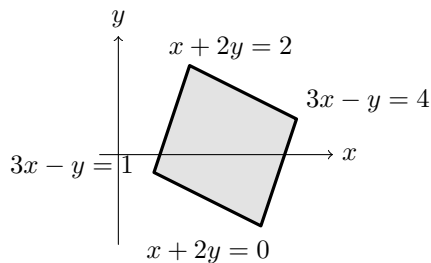
Tasks

1. Identify the transformed rectangle.
2. Compute the Jacobian and integral.
3. Verify that the transformation preserves one-to-one coverage.

Question 4 – Solution

Strategy. Each pair of parallel boundary lines becomes a pair of constant-coordinate sides.

Step 1: Geometry



The image is the rectangle $0 \leq u \leq 2$, $1 \leq v \leq 4$. The coefficient matrix has determinant

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7.$$

Since this determinant is nonzero, the linear map is globally one-to-one.

Step 2: Change variables We have $dA = \frac{1}{7} du dv$ and the integrand is uv . Hence

$$\begin{aligned} I &= \frac{1}{7} \int_0^2 \int_1^4 uv \, dv \, du \\ &= \frac{1}{7} \left[\frac{u^2}{2} \right]_0^2 \left[\frac{v^2}{2} \right]_1^4 = \boxed{\frac{15}{7}}. \end{aligned}$$

Verification On the transformed rectangle, $u \geq 0$ and $v > 0$, so the positive answer has the required sign. The constant area scale $1/7$ also matches the inverse determinant.

Question 5

Let

$$R = \{(x, y) : 1 \leq x \leq 3, x \leq y \leq 2x\}.$$

Use $u = x$ and $v = y/x$ to evaluate

$$\iint_R \frac{1}{x} dA.$$

Tasks

1. Find the inverse map and transformed bounds.
2. Compute the Jacobian and evaluate.
3. Verify directly in Cartesian coordinates.

Question 5 – Solution

Strategy. The ratio $v = y/x$ turns the two sloping sides into constant-coordinate boundaries.

Step 1: Inverse and bounds Since $x > 0$ on R ,

$$x = u, \quad y = uv.$$

The region becomes the rectangle

$$1 \leq u \leq 3, \quad 1 \leq v \leq 2.$$

The inverse map is one-to-one because $u = x$ and $v = y/x$ recover each point uniquely.

Step 2: Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u.$$

Thus

$$I = \int_1^3 \int_1^2 \frac{1}{u}(u) \, dv \, du = \int_1^3 \int_1^2 1 \, dv \, du = \boxed{2}.$$

Verification Direct Cartesian integration gives

$$\int_1^3 \int_x^{2x} \frac{1}{x} \, dy \, dx = \int_1^3 \frac{2x - x}{x} \, dx = 2.$$

The two methods agree exactly, including the variable area scale u .

Question 6

Consider the map

$$T(u, v) = (u^2, v), \quad -1 \leq u \leq 1, \quad 0 \leq v \leq 1.$$

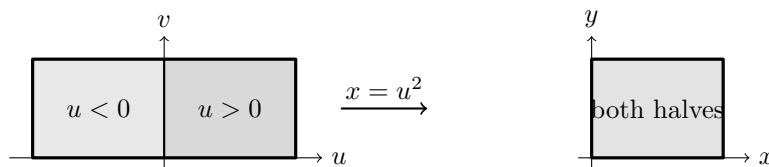
Tasks

1. Describe the image and determine whether T is one-to-one.
2. Explain why integrating $|J_T|$ over the full source gives the wrong image area.
3. Repair the change of variables calculation.

Question 6 – Solution

Strategy. Track multiplicity: the two halves of the source fold onto the same target rectangle.

Step 1: Image and folding



The image is $0 \leq x \leq 1$, $0 \leq y \leq 1$. Except along $u = 0$, the points (u, v) and $(-u, v)$ have the same image, so T is two-to-one.

Step 2: Diagnose the overcount Since

$$J_T = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u & 0 \\ 0 & 1 \end{vmatrix} = 2u,$$

integrating its absolute value over the full source gives

$$\int_{-1}^1 \int_0^1 |2u| \, dv \, du = 2,$$

twice the actual image area 1. The absolute value corrects orientation, not multiplicity.

Step 3: Repair Restrict to the one-to-one half $0 \leq u \leq 1$:

$$\text{area}(T(S)) = \int_0^1 \int_0^1 2u \, dv \, du = \boxed{1}.$$

Using $-1 \leq u \leq 0$ alone gives the same result after taking $|J_T|$.

Question 7

Let

$$R = \{(x, y) : 0 \leq x \leq 2, x^2 \leq y \leq x^2 + 3\}.$$

Use $u = x$ and $v = y - x^2$ to evaluate

$$\iint_R (y - x^2) dA.$$

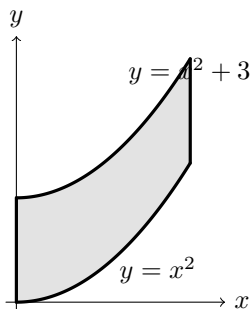
Tasks

1. Show how the change of variables straightens the parabolic strip.
2. Compute the Jacobian and integral.
3. Verify directly from the vertical thickness.

Question 7 – Solution

Strategy. Measure vertical displacement from the lower parabola; the curved strip then becomes a rectangle.

Step 1: Straighten the region



The inverse map is

$$x = u, \quad y = u^2 + v,$$

and the bounds become $0 \leq u \leq 2$, $0 \leq v \leq 3$.

Step 2: Jacobian and integral

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ 2u & 1 \end{vmatrix} = 1.$$

Because $y - x^2 = v$,

$$I = \int_0^2 \int_0^3 v \, dv \, du = (2) \left[\frac{v^2}{2} \right]_0^3 = \boxed{9}.$$

Verification For each fixed x , the displacement $t = y - x^2$ ranges from 0 to 3. Thus direct integration gives $\int_0^2 \int_0^3 t \, dt \, dx = 9$. The unit Jacobian reflects a vertical shear with no area scaling.

Question 8

The unit cube $0 \leq u, v, w \leq 1$ is mapped into xyz -space by

$$x = u + v, \quad y = v + w, \quad z = w + u.$$

Evaluate $\iiint_E (x + y + z) dV$, where E is the image parallelepiped.

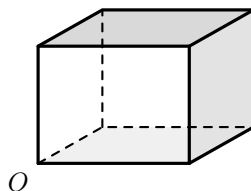
Tasks

1. Compute the three-dimensional Jacobian and establish invertibility.
2. Evaluate the transformed integral.
3. Verify using the image volume and centroid.

Question 8 – Solution

Strategy. A nonsingular linear map sends the unit cube to a parallelepiped with constant volume scale.

Step 1: Jacobian and geometry



$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2.$$

Because $J \neq 0$, the linear map is invertible and covers its image once.

Step 2: Transform the integrand We have $x + y + z = 2(u + v + w)$ and $dV = 2 du dv dw$. Therefore

$$\begin{aligned} I &= \int_0^1 \int_0^1 \int_0^1 4(u + v + w) dw dv du \\ &= 4 \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = \boxed{6}. \end{aligned}$$

Verification The image volume is $|J|$ times the cube volume, hence 2. The cube centroid $(1/2, 1/2, 1/2)$ maps to $(1, 1, 1)$, where $x + y + z = 3$. A linear function averages to its centroid value, so $I = (2)(3) = 6$.

Question 9

For $a, b > 0$, let

$$E_{a,b} = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

Suppose

$$\text{area}(E_{a,b}) = 6\pi, \quad \iint_{E_{a,b}} x^2 \, dA = \frac{27\pi}{2}.$$

Tasks

1. Use $x = au$, $y = bv$ to translate both measurements.
2. Recover a and b .
3. Verify both original conditions.

Question 9 – Solution

Strategy. Scale the ellipse to the unit disk; area determines ab , while the x^2 moment determines one additional power of a^2 .

Step 1: Area condition The map $x = au$, $y = bv$ has $dA = ab \, du \, dv$. Since the unit disk has area π ,

$$\pi ab = 6\pi, \quad \text{so} \quad ab = 6.$$

Step 2: Moment condition On the unit disk,

$$\iint_D u^2 \, du \, dv = \frac{\pi}{4}$$

by symmetry or a polar calculation. Therefore

$$\iint_{E_{a,b}} x^2 \, dA = a^2(ab) \frac{\pi}{4} = \frac{3\pi a^2}{2}.$$

Equating this with $27\pi/2$ gives $a^2 = 9$. Positivity then yields

$$\boxed{a = 3}, \quad \boxed{b = 2}.$$

Verification The recovered ellipse has area $\pi(3)(2) = 6\pi$. Its moment is $a^2(ab)\pi/4 = (9)(6)\pi/4 = 27\pi/2$, so both measurements are satisfied without sign ambiguity.

Question 10

Let

$$R = \{(x, y) : x \geq 0, y \geq 0, 1 \leq x + y \leq 3\}.$$

Use

$$u = x + y, \quad v = \frac{x}{x + y}$$

to evaluate $\iint_R (x + y)^{-1} dA$.

Tasks

1. Derive the inverse map and transformed rectangle.
2. Compute the Jacobian and integral.
3. Explain geometrically what the coordinate v measures.

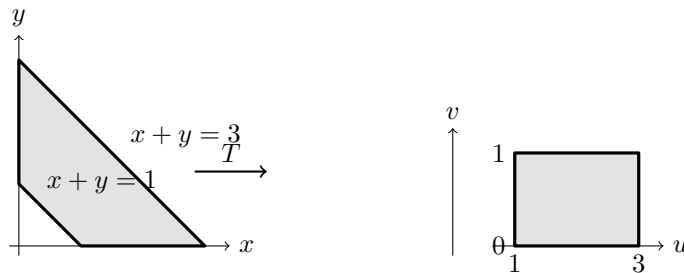
Question 10 – Solution

Strategy. Use total distance along the coordinate-sum direction and the fraction allocated to x .

Step 1: Inverse and geometry Since $u = x + y > 0$,

$$x = uv, \quad y = u(1 - v).$$

Nonnegativity gives $0 \leq v \leq 1$, while the two diagonal boundaries give $1 \leq u \leq 3$.



Step 2: Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} v & u \\ 1 - v & -u \end{vmatrix} = -u, \quad dA = u \, du \, dv.$$

Hence the integrand $1/(x + y) = 1/u$ cancels the area scale:

$$I = \int_1^3 \int_0^1 \frac{1}{u}(u) \, dv \, du = \boxed{2}.$$

Step 3: Interpretation and check On a segment $x + y = u$, the quantity $v = x/u$ is the fraction of the total coordinate sum assigned to x ; it runs once from the y -axis to the x -axis. Thus the inverse is one-to-one in the interior, and $u > 0$ confirms the absolute Jacobian used above.

Question 1

Find the area of the part of the plane

$$z = 2x - y$$

lying above the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 2$.

Tasks

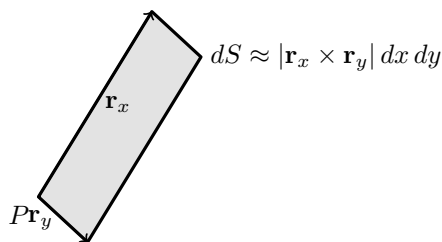
1. Derive the graph surface-area factor from tangent vectors.
2. Evaluate the surface area.
3. Verify by interpreting the constant area scale.

Question 1 – Solution

Strategy. Parametrize the graph by its projection and use the tangent-parallelogram area scale.

Step 1: Derive the element For a graph $z = f(x, y)$,

$$\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle, \quad \mathbf{r}_x = \langle 1, 0, f_x \rangle, \quad \mathbf{r}_y = \langle 0, 1, f_y \rangle.$$



Thus

$$|\mathbf{r}_x \times \mathbf{r}_y| = |\langle -f_x, -f_y, 1 \rangle| = \sqrt{1 + f_x^2 + f_y^2}.$$

Step 2: Apply to the plane Here $f_x = 2$ and $f_y = -1$, so

$$S = \int_0^3 \int_0^2 \sqrt{1 + 2^2 + (-1)^2} dy dx = \sqrt{6}(3)(2) = \boxed{6\sqrt{6}}.$$

Verification Every small projected area is enlarged by the same factor $\sqrt{6}$. Since the rectangle has area 6, the surface area must be $6\sqrt{6}$.

Question 2

Find the area of the paraboloid

$$z = x^2 + y^2$$

above the disk $x^2 + y^2 \leq 4$.

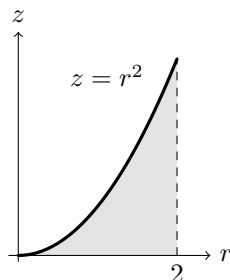
Tasks

1. Form the graph surface-area integrand.
2. Use polar coordinates to evaluate the integral.
3. Check the answer against the projected area.

Question 2 – Solution

Strategy. The squared gradient depends only on r , so polar coordinates reduce the area integral to one variable.

Step 1: Geometry and gradient



For $f = x^2 + y^2$,

$$f_x = 2x, \quad f_y = 2y, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 4r^2}.$$

Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta \\ &= 2\pi \left[\frac{(1 + 4r^2)^{3/2}}{12} \right]_0^2 = \boxed{\frac{\pi}{6} (17\sqrt{17} - 1)}. \end{aligned}$$

Verification The area factor is at least 1, so S must exceed the projected disk area 4π . The exact result does. Its integrand also increases with r , matching the increasing steepness of the paraboloid.

Question 3

Find the area of the conical surface

$$z = \sqrt{x^2 + y^2}$$

lying above the annulus $1 \leq x^2 + y^2 \leq 9$.

Tasks

1. Explain why the graph formula is valid on the annulus.
2. Evaluate the area.
3. Verify with the lateral-area formula for a conical frustum.

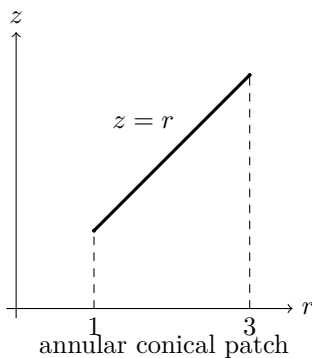
Question 3 – Solution

Strategy. The annulus avoids the cone's nondifferentiable vertex, and the cone has constant slope there.

Step 1: Gradient Write $r = \sqrt{x^2 + y^2}$. For $r > 0$,

$$f_x = \frac{x}{r}, \quad f_y = \frac{y}{r}, \quad f_x^2 + f_y^2 = 1.$$

The entire projection has $r \geq 1$, so no derivative singularity occurs.



Step 2: Area

$$S = \iint_D \sqrt{1 + 1} \, dA = \sqrt{2} \pi (3^2 - 1^2) = \boxed{8\sqrt{2} \pi}.$$

Verification The frustum has radii 1 and 3. Its slant height is $\sqrt{(3-1)^2 + (3-1)^2} = 2\sqrt{2}$, so the classical lateral area $\pi(R+r)\ell = \pi(3+1)(2\sqrt{2}) = 8\sqrt{2}\pi$ agrees.

Question 4

Use a surface-area integral to find the area of the upper hemisphere

$$z = \sqrt{9 - x^2 - y^2}.$$

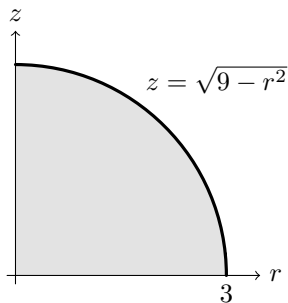
Tasks

1. Derive the surface-area factor over the open disk $x^2 + y^2 < 9$.
2. Evaluate the resulting improper integral.
3. Explain why the boundary singularity is integrable.

Question 4 – Solution

Strategy. Integrate over disks of radius $b < 3$, then take the limit as $b \rightarrow 3^-$.

Step 1: Area factor



With $r^2 = x^2 + y^2$,

$$f_x = -\frac{x}{\sqrt{9-r^2}}, \quad f_y = -\frac{y}{\sqrt{9-r^2}},$$

so

$$\sqrt{1 + f_x^2 + f_y^2} = \frac{3}{\sqrt{9-r^2}}.$$

Step 2: Improper integral

$$\begin{aligned} S &= \lim_{b \rightarrow 3} \int_0^{2\pi} \int_0^b \frac{3r}{\sqrt{9-r^2}} dr d\theta \\ &= \lim_{b \rightarrow 3} 6\pi \left(3 - \sqrt{9-b^2} \right) = \boxed{18\pi}. \end{aligned}$$

Verification The factor diverges as $r \rightarrow 3^-$ because the sphere becomes vertical, but its antiderivative has a finite limit. The result also equals half the full sphere area $4\pi(3^2) = 36\pi$.

Question 5

Find the area of the sideways paraboloid

$$x = y^2 + z^2$$

over the disk $y^2 + z^2 \leq 1$ in the yz -plane.

Tasks

1. State the surface-area formula for a graph $x = g(y, z)$.
2. Evaluate using polar coordinates in the yz -plane.
3. Check the result against the projected disk area.

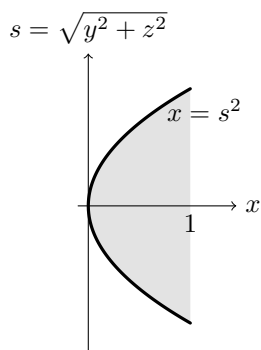
Question 5 – Solution

Strategy. Project onto the yz -plane, where both the region and squared gradient are radial.

Step 1: Projection formula For $x = g(y, z)$,

$$dS = \sqrt{1 + g_y^2 + g_z^2} dy dz.$$

Here $g = y^2 + z^2$, so with $s^2 = y^2 + z^2$ the factor is $\sqrt{1 + 4s^2}$.



Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4s^2} s ds d\theta \\ &= 2\pi \left[\frac{(1 + 4s^2)^{3/2}}{12} \right]_0^1 = \boxed{\frac{\pi}{6} (5\sqrt{5} - 1)}. \end{aligned}$$

Verification The graph projects one-to-one onto a disk of area π , and the area factor exceeds 1 except at its center. Accordingly, the computed surface area is strictly larger than π .

Question 6

Find the area of the portion of

$$z = 4 - x - y$$

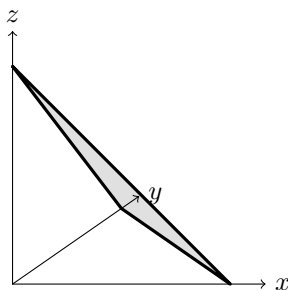
in the first octant.

Tasks

1. Determine the projection in the xy -plane.
2. Evaluate the graph surface-area integral.
3. Verify using the three intercept vertices.

Question 6 – Solution

Strategy. The coordinate-plane restrictions cut a triangular patch from a plane with constant slope.

Step 1: Region

In the first octant, $x \geq 0$, $y \geq 0$, and $z = 4 - x - y \geq 0$. Thus the projection is

$$D = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq 4 - x\},$$

a right triangle of area 8.

Step 2: Surface integral Since $f_x = f_y = -1$,

$$S = \iint_D \sqrt{1 + (-1)^2 + (-1)^2} dA = \sqrt{3}(8) = \boxed{8\sqrt{3}}.$$

Verification From $(4, 0, 0)$, use edge vectors $\mathbf{a} = \langle -4, 4, 0 \rangle$ and $\mathbf{b} = \langle -4, 0, 4 \rangle$. Then $|\mathbf{a} \times \mathbf{b}| = 16\sqrt{3}$, so the triangle area is $\frac{1}{2}(16\sqrt{3}) = 8\sqrt{3}$.

Question 7

Find the area of the surface

$$z = xy$$

above the quarter-disk $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.

Tasks

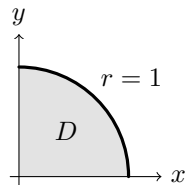
1. Show that the surface-area factor is radial even though $z = xy$ is not.
2. Evaluate the polar integral.
3. Verify the angular and radial bounds.

Question 7 – Solution

Strategy. Differentiate first: the two gradient components exchange x and y , so their squares combine to r^2 .

Step 1: Gradient and projection

$$f_x = y, \quad f_y = x, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + x^2 + y^2} = \sqrt{1 + r^2}.$$



The quarter-disk has $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq 1$.

Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{\pi/2} \int_0^1 \sqrt{1 + r^2} r \, dr \, d\theta \\ &= \frac{\pi}{2} \left[\frac{(1 + r^2)^{3/2}}{3} \right]_0^1 = \boxed{\frac{\pi}{6}(2\sqrt{2} - 1)}. \end{aligned}$$

Verification The angular width is one quarter of a full turn. The factor ranges from 1 to $\sqrt{2}$, so the result must lie between the projection area $\pi/4$ and $\sqrt{2}\pi/4$; it does.

Question 8

For constants a, b, c , the plane

$$z = ax + by + c$$

lies above the unit disk $x^2 + y^2 \leq 1$. Its surface area there is 3π .

Tasks

1. Determine all possible gradient pairs (a, b) .
2. Explain the role of c .
3. Give two distinct planes satisfying the condition and verify them.

Question 8 – Solution

Strategy. A plane has a constant area-enlargement factor determined only by the magnitude of its gradient.

Step 1: Area equation Since $f_x = a$ and $f_y = b$,

$$S = \iint_D \sqrt{1 + a^2 + b^2} \, dA = \pi \sqrt{1 + a^2 + b^2}.$$

The condition $S = 3\pi$ gives

$$1 + a^2 + b^2 = 9, \quad \boxed{a^2 + b^2 = 8}.$$

Thus all possible gradients form the circle of radius $2\sqrt{2}$ in the ab -plane.

Step 2: Vertical translation The constant c does not appear in either derivative, so

$$\boxed{c \text{ is arbitrary}}.$$

Changing c translates the patch vertically without stretching it.

Step 3: Examples and verification Two choices are

$$\boxed{z = 2\sqrt{2}x} \quad \text{and} \quad \boxed{z = 2x + 2y + 5}.$$

Their gradient-square sums are respectively $8 + 0$ and $4 + 4$. Each therefore has area factor $\sqrt{1 + 8} = 3$ and area 3π above the unit disk.

Question 9

Find the area of the saddle

$$z = \frac{x^2 - y^2}{2}$$

above the disk $x^2 + y^2 \leq 3$.

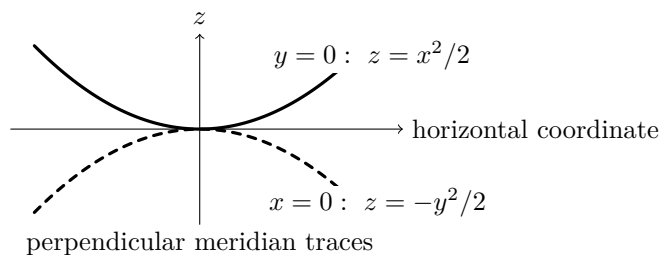
Tasks

1. Relate the two perpendicular meridian traces to the saddle shape.
2. Show that the surface-area factor is radial.
3. Evaluate the area exactly.

Question 9 – Solution

Strategy. Although the height changes sign by direction, the squared gradient depends only on distance from the origin.

Step 1: Geometry



Along $y = 0$ the trace opens upward, while along $x = 0$ it opens downward. Differentiation gives

$$f_x = x, \quad f_y = -y, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + r^2}.$$

Step 2: Polar integral

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + r^2} r \, dr \, d\theta \\ &= 2\pi \left[\frac{(1 + r^2)^{3/2}}{3} \right]_0^{\sqrt{3}} \\ &= \frac{2\pi}{3} (8 - 1) = \boxed{\frac{14\pi}{3}}. \end{aligned}$$

Verification The projected disk has area 3π , while the area factor ranges from 1 to 2. Hence $3\pi < S < 6\pi$, and $14\pi/3$ lies in that interval.

Question 10

Let f be continuously differentiable on a region D of area 12, and suppose

$$1 \leq |\nabla f(x, y)| \leq 2 \quad \text{throughout } D.$$

Let S be the area of the graph $z = f(x, y)$ above D .

Tasks

1. Derive the sharpest bounds guaranteed for S .
2. Prove the bounds from the surface-area integral.
3. Show that both endpoints can occur.

Question 10 – Solution

Strategy. Apply the monotonicity of $\sqrt{1+t^2}$ to the pointwise gradient bounds, then integrate.

Step 1: Pointwise comparison From $1 \leq |\nabla f| \leq 2$,

$$\sqrt{2} = \sqrt{1+1^2} \leq \sqrt{1+|\nabla f|^2} \leq \sqrt{1+2^2} = \sqrt{5}.$$

Step 2: Integrate The graph area is

$$S = \iint_D \sqrt{1+|\nabla f|^2} dA.$$

Integrating the pointwise inequalities and using $\text{area}(D) = 12$ yields

$$\boxed{12\sqrt{2} \leq S \leq 12\sqrt{5}}.$$

Step 3: Sharpness The lower endpoint occurs for any plane with constant gradient magnitude 1, for example $f(x, y) = x$. The upper endpoint occurs for a plane with constant gradient magnitude 2, for example $f(x, y) = 2x$. Indeed,

$$S_{f=x} = 12\sqrt{1+1} = 12\sqrt{2}, \quad S_{f=2x} = 12\sqrt{1+4} = 12\sqrt{5}.$$

Therefore no narrower interval can be guaranteed from the supplied data.

Question 1

Let $a > 0$. Find the area enclosed by the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

using the change of variables $x = u^3$, $y = v^3$.

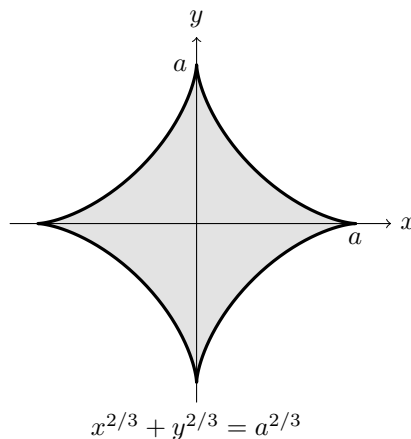
Tasks

1. Describe the transformed region and justify the substitution.
2. Compute the Jacobian and evaluate the area.
3. Check the scaling of the answer with respect to a .

Question 1 – Solution

Strategy. The cubic substitution converts the astroid into a disk, after which polar coordinates handle the transformed integral.

Step 1: Transform the region



Because the real cube function is one-to-one,

$$x^{2/3} = u^2, \quad y^{2/3} = v^2.$$

Thus the interior maps one-to-one to $u^2 + v^2 \leq a^{2/3}$, a disk of radius $a^{1/3}$.

Step 2: Jacobian and integral The absolute Jacobian is

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = |(3u^2)(3v^2)| = 9u^2v^2.$$

Writing $u = r \cos \theta$, $v = r \sin \theta$ gives

$$\begin{aligned} A &= 9 \int_0^{2\pi} \int_0^{a^{1/3}} r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta \\ &= 9 \left(\frac{a^2}{6} \right) \left(\frac{\pi}{4} \right) = \boxed{\frac{3\pi a^2}{8}}. \end{aligned}$$

Verification Replacing a by ka enlarges every length by k , so area must scale by k^2 . The result is proportional to a^2 , as required.

Question 2

Find the area common to the disks

$$x^2 + y^2 \leq 2x, \quad x^2 + y^2 \leq 2y.$$

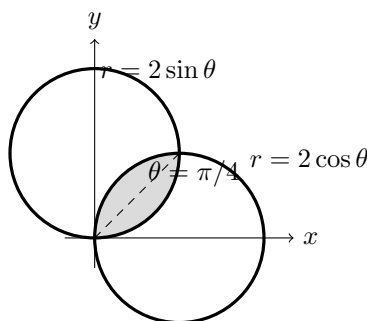
Tasks

1. Translate both inequalities into polar coordinates.
2. Identify where the radial boundary changes and evaluate the area.
3. Verify the symmetry used in the calculation.

Question 2 – Solution

Strategy. In the first quadrant the smaller of the two polar bounds switches at the line $\theta = \pi/4$.

Step 1: Polar description



For $r \geq 0$, the disks become

$$r \leq 2 \cos \theta, \quad r \leq 2 \sin \theta.$$

Their common region lies in $0 \leq \theta \leq \pi/2$. The bounds are equal at $\theta = \pi/4$.

Step 2: Evaluate Reflection across $y = x$ makes the two halves congruent, so

$$\begin{aligned} A &= 2 \int_0^{\pi/4} \int_0^{2 \sin \theta} r \, dr \, d\theta = 4 \int_0^{\pi/4} \sin^2 \theta \, d\theta \\ &= 4 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/4} = \boxed{\frac{\pi}{2} - 1}. \end{aligned}$$

Verification The circles have equal radius and centers $(1, 0)$ and $(0, 1)$, which are interchanged by reflecting across $y = x$. Their intersections, $(0, 0)$ and $(1, 1)$, also lie on that line.

Question 3

Find the volume enclosed between the paraboloids

$$z = x^2 + y^2 \quad \text{and} \quad z = 6 - x^2 - y^2.$$

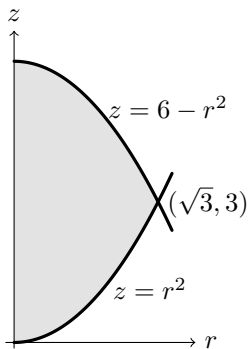
Tasks

1. Find the curve where the surfaces meet.
2. Set up and evaluate a cylindrical-coordinate integral.
3. Check that the height is nonnegative throughout the base.

Question 3 – Solution

Strategy. Subtract the lower paraboloid from the upper one over their circular projection.

Step 1: Intersection



Setting the surfaces equal gives

$$r^2 = 6 - r^2, \quad r = \sqrt{3}, \quad z = 3.$$

Hence the projection is the disk $0 \leq r \leq \sqrt{3}$.

Step 2: Volume The vertical height is $(6 - r^2) - r^2 = 6 - 2r^2$. Therefore

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\sqrt{3}} (6 - 2r^2)r \, dr \, d\theta \\ &= 2\pi \left[3r^2 - \frac{r^4}{2} \right]_0^{\sqrt{3}} = 2\pi \left(9 - \frac{9}{2} \right) = \boxed{9\pi}. \end{aligned}$$

Verification On $0 \leq r \leq \sqrt{3}$, the height is $2(3 - r^2) \geq 0$; it decreases from 6 at the axis to 0 at the intersection circle, matching the geometry.

Question 4

Let $a > 0$. Two perpendicular circular cylinders form the solid

$$x^2 + z^2 \leq a^2, \quad y^2 + z^2 \leq a^2.$$

Find its volume.

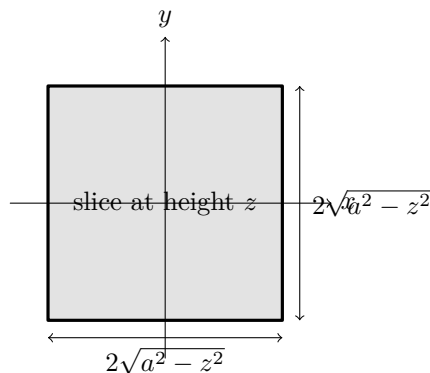
Tasks

1. Describe a horizontal cross-section at a fixed height z .
2. Integrate the cross-sectional area.
3. Check symmetry and dimensional scaling.

Question 4 – Solution

Strategy. At each fixed z , the two cylinder inequalities give independent bounds on x and y , producing a square.

Step 1: Cross-section



For $-a \leq z \leq a$,

$$|x| \leq \sqrt{a^2 - z^2}, \quad |y| \leq \sqrt{a^2 - z^2}.$$

Thus the slice is a square with area

$$A(z) = \left(2\sqrt{a^2 - z^2}\right)^2 = 4(a^2 - z^2).$$

Step 2: Integrate

$$\begin{aligned} V &= \int_{-a}^a 4(a^2 - z^2) dz = 8 \int_0^a (a^2 - z^2) dz \\ &= 8 \left[a^2 z - \frac{z^3}{3} \right]_0^a = \boxed{\frac{16a^3}{3}}. \end{aligned}$$

Verification The cross-sectional area is even in z , justifying the factor 2. The answer is proportional to a^3 , the correct scaling for a volume.

Question 5

A cylindrical hole of radius a is drilled through the center of a sphere of radius R , where $0 < a < R$. The remaining solid has height

$$h = 2\sqrt{R^2 - a^2}.$$

Find its volume and express the result using only h .

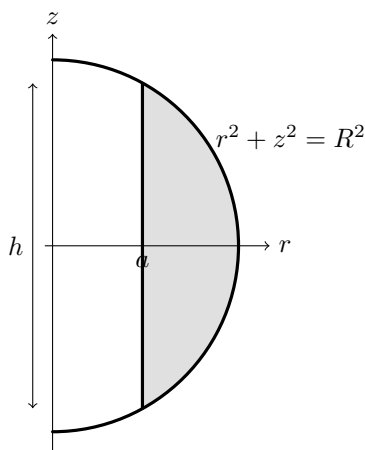
Tasks

1. Set up the volume in cylindrical coordinates.
2. Evaluate it and eliminate R and a in favor of h .
3. Explain the resulting “napkin-ring” invariance.

Question 5 – Solution

Strategy. Use cylindrical shells with $a \leq r \leq R$; the sphere supplies the upper and lower z -bounds.

Step 1: Bounds



The sphere is $r^2 + z^2 \leq R^2$. Hence

$$0 \leq \theta \leq 2\pi, \quad a \leq r \leq R, \quad -\sqrt{R^2 - r^2} \leq z \leq \sqrt{R^2 - r^2}.$$

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_a^R 2\sqrt{R^2 - r^2} r \, dr \, d\theta \\ &= 4\pi \left[-\frac{(R^2 - r^2)^{3/2}}{3} \right]_a^R = \frac{4\pi}{3} (R^2 - a^2)^{3/2}. \end{aligned}$$

Since $R^2 - a^2 = h^2/4$,

$$\boxed{V = \frac{\pi h^3}{6}}.$$

Verification The final formula contains only the remaining height. Thus every drilled sphere with the same height h leaves the same volume, regardless of its original R and hole radius a .

Question 6

Find the volume of the solid torus

$$\left(\sqrt{x^2 + y^2} - 3\right)^2 + z^2 \leq 1$$

using cylindrical coordinates.

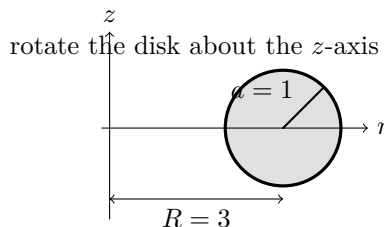
Tasks

1. Describe the bounds in the meridian (r, z) -plane.
2. Evaluate the triple integral, using symmetry where helpful.
3. Give a geometric check of the answer.

Question 6 – Solution

Strategy. In cylindrical coordinates the torus is obtained by rotating a radius-1 disk centered at $(r, z) = (3, 0)$.

Step 1: Bounds



The radial variable ranges from 2 to 4, and

$$-\sqrt{1 - (r - 3)^2} \leq z \leq \sqrt{1 - (r - 3)^2}, \quad 0 \leq \theta \leq 2\pi.$$

Step 2: Evaluate With $u = r - 3$,

$$\begin{aligned} V &= 2\pi \int_2^4 2r \sqrt{1 - (r - 3)^2} dr \\ &= 4\pi \int_{-1}^1 (u + 3) \sqrt{1 - u^2} du. \end{aligned}$$

The u -term is odd, while $\int_{-1}^1 \sqrt{1 - u^2} du = \pi/2$. Therefore

$$V = 4\pi(3) \frac{\pi}{2} = \boxed{6\pi^2}.$$

Verification The generating disk has area π , and its center travels a circle of circumference $2\pi(3) = 6\pi$. Their product is $6\pi^2$, agreeing with the integral.

Question 7

Consider the solids

$$S_1 = \{(x, y, z) : 0 \leq z \leq 1, x^2 + y^2 \leq z^2\}$$

and

$$S_2 = \{(x, y, z) : 0 \leq z \leq 1, 0 \leq x \leq z^2, 0 \leq y \leq \pi\}.$$

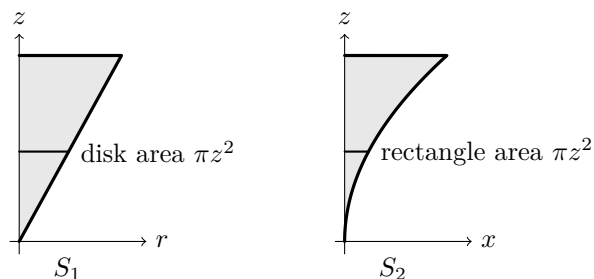
Tasks

1. Find the area of each horizontal cross-section.
2. Use Cavalieri's principle to compare the volumes.
3. Compute their common volume.

Question 7 – Solution

Strategy. Although the slices have different shapes, compare their areas at the same height.

Step 1: Slice areas



At height z , the slice of S_1 is a disk of radius z , so

$$A_1(z) = \pi z^2.$$

The slice of S_2 is a rectangle of side lengths z^2 and π , so

$$A_2(z) = z^2\pi = \pi z^2.$$

Step 2: Compare and integrate Both solids occupy $0 \leq z \leq 1$ and have equal cross-sectional area at every height. Cavalieri's principle therefore gives equal volumes, and

$$V_1 = V_2 = \int_0^1 \pi z^2 dz = \pi \left[\frac{z^3}{3} \right]_0^1 = \boxed{\frac{\pi}{3}}.$$

Verification The conclusion depends on slice areas, not slice shapes: circular slices in S_1 and rectangular slices in S_2 accumulate the same volume height by height.

Question 8

For $A > 0$, let E_A be the solid bounded below by

$$z = x^2 + y^2$$

and above by the plane $z = A$. If E_A has volume 18π , determine A , the base radius, and the height.

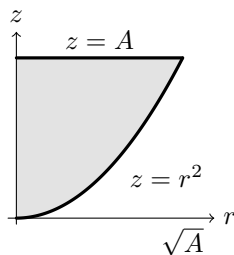
Tasks

1. Derive a formula for $V(A)$.
2. Solve the inverse volume condition.
3. Verify the dimensions and positivity of the result.

Question 8 – Solution

Strategy. First express the volume as a function of the unknown cutting height A , then solve the resulting equation.

Step 1: Geometry and volume



The plane and paraboloid meet when $r^2 = A$, so $0 \leq r \leq \sqrt{A}$. Thus

$$\begin{aligned} V(A) &= \int_0^{2\pi} \int_0^{\sqrt{A}} (A - r^2)r \, dr \, d\theta \\ &= 2\pi \left[\frac{Ar^2}{2} - \frac{r^4}{4} \right]_0^{\sqrt{A}} = \frac{\pi A^2}{2}. \end{aligned}$$

Step 2: Recover the dimensions The condition $V(A) = 18\pi$ yields

$$\frac{\pi A^2}{2} = 18\pi, \quad A^2 = 36.$$

Since $A > 0$,

$$\boxed{A = 6}, \quad \boxed{\text{base radius} = \sqrt{6}}, \quad \boxed{\text{height} = 6}.$$

Verification Substitution gives $V(6) = \pi(36)/2 = 18\pi$. The positive root is required because A is a height, and $r = \sqrt{A}$ follows from the intersection equation.

Question 9

Find the volume of the ellipsoidal shell

$$1 \leq \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \leq 4.$$

Use the change of variables

$$x = 2u, \quad y = 3v, \quad z = 4w.$$

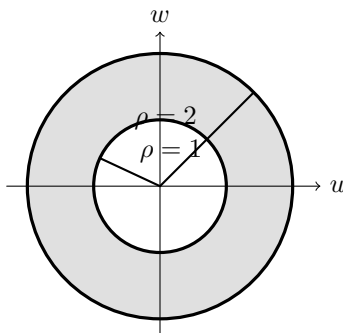
Tasks

1. Describe the transformed shell.
2. Compute the three-dimensional Jacobian and the volume.
3. Check the scaling factor geometrically.

Question 9 – Solution

Strategy. The coordinate scaling turns the ellipsoids into concentric spheres and multiplies every volume by a constant Jacobian.

Step 1: Transform the shell



scaled spherical shell

The inequality becomes

$$1 \leq u^2 + v^2 + w^2 \leq 4,$$

so the transformed region is the spherical shell $1 \leq \rho \leq 2$.

Step 2: Jacobian and volume The derivative matrix is diagonal, so

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = |(2)(3)(4)| = 24.$$

Consequently,

$$V = 24 \left[\frac{4\pi}{3} (2^3 - 1^3) \right] = 24 \left(\frac{28\pi}{3} \right) = \boxed{224\pi}.$$

Verification The map stretches lengths by factors 2, 3, and 4 along the coordinate axes, so it must stretch volume by their product 24, exactly the Jacobian used above.

Question 10

A sphere of radius R has surface-area-to-volume ratio

$$\frac{S}{V} = \frac{3}{2}.$$

Determine its radius, surface area, and volume.

Tasks

1. Derive V with a spherical-coordinate triple integral.
2. Derive S by viewing the two hemispheres as graphs.
3. Use the given ratio and verify its units.

Question 10 – Solution

Strategy. Rebuild both standard sphere formulas from integration, then use their ratio to recover the radius.

Step 1: Volume In spherical coordinates,

$$V = \int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi(2) \frac{R^3}{3} = \frac{4\pi R^3}{3}.$$

Step 2: Surface area For the upper graph $z = \sqrt{R^2 - r^2}$, the area factor is $R/\sqrt{R^2 - r^2}$. Its area is

$$S_{\text{upper}} = \int_0^{2\pi} \int_0^R \frac{Rr}{\sqrt{R^2 - r^2}} \, dr \, d\theta = 2\pi R^2.$$

Doubling for both hemispheres gives $S = 4\pi R^2$.

Step 3: Use the ratio

$$\frac{S}{V} = \frac{4\pi R^2}{4\pi R^3/3} = \frac{3}{R} = \frac{3}{2}, \quad \boxed{R = 2}.$$

Therefore

$$\boxed{S = 16\pi}, \quad \boxed{V = \frac{32\pi}{3}}.$$

Verification Surface area divided by volume has units of inverse length, and $3/R$ has those units. Substituting $R = 2$ reproduces the stated ratio.