

Question 1

For the graph

$$z = f(x, y) = x^2 e^y,$$

consider the point above $(1, 0)$.

Tasks

1. Find the tangent plane and the linearization $L(x, y)$ at $(1, 0)$.
2. Use L to estimate $f(1.02, -0.03)$.
3. Compare the estimate with the calculator value $1.0404e^{-0.03}$.

Question 1 – Solution

Strategy. Use the value and both first partial derivatives at the base point to build the unique affine approximation.

Step 1: Local data

$$f(1, 0) = 1, \quad f_x = 2xe^y, \quad f_y = x^2e^y.$$

Thus $f_x(1, 0) = 2$ and $f_y(1, 0) = 1$.

Step 2: Plane and linearization The tangent-plane formula gives

$$z - 1 = 2(x - 1) + (y - 0).$$

Consequently,

$$\boxed{z = 1 + 2(x - 1) + y}, \quad \boxed{L(x, y) = 1 + 2(x - 1) + y}.$$

Step 3: Estimate Here $\Delta x = 0.02$ and $\Delta y = -0.03$, so

$$f(1.02, -0.03) \approx L(1.02, -0.03) = 1 + 2(0.02) - 0.03 = \boxed{1.0100}.$$

Verification Direct evaluation gives

$$1.0404e^{-0.03} \approx 1.0096515.$$

The linearization is high by about 0.0003485, a small error consistent with the small displacement from $(1, 0)$.

Question 2

The upper hemisphere

$$z = \sqrt{25 - x^2 - y^2}$$

is sampled near the point $(3, 0, 4)$.

Tasks

1. Derive the linearization at $(x, y) = (3, 0)$.
2. Estimate the height above $(3.04, 0.06)$.
3. Explain geometrically why the first-order estimate has no y -correction.

Question 2 – Solution

Strategy. Differentiate the height function, then evaluate its first-order change from the base point.

Step 1: Partial derivatives For $f(x, y) = (25 - x^2 - y^2)^{1/2}$,

$$f_x = -\frac{x}{\sqrt{25 - x^2 - y^2}}, \quad f_y = -\frac{y}{\sqrt{25 - x^2 - y^2}}.$$

At $(3, 0)$, these become $f_x = -3/4$ and $f_y = 0$.

Step 2: Linearization

$$L(x, y) = 4 - \frac{3}{4}(x - 3).$$

At the requested point,

$$f(3.04, 0.06) \approx 4 - \frac{3}{4}(0.04) = \boxed{3.9700}.$$

Step 3: Geometry The sphere is symmetric under $y \mapsto -y$. At $y = 0$, moving a small distance in either y -direction changes the height only to second order, so the tangent plane has no y -tilt.

Verification The exact height is

$$\sqrt{25 - 3.04^2 - 0.06^2} = \sqrt{15.7548} \approx 3.9692317.$$

The estimate is high by about 0.0007683.

Question 3

A differentiable function f has tangent plane

$$z = 7 + 3(x - 2) - 4(y + 1)$$

at the point whose input coordinates are $(2, -1)$.

Tasks

1. Recover $f(2, -1)$, $f_x(2, -1)$, and $f_y(2, -1)$.
2. Predict the first-order change along $x = 2 + t$, $y = -1 + 2t$.
3. Construct one nonlinear polynomial f having exactly this tangent plane.

Question 3 – Solution

Strategy. Read the constant and slope coefficients from the standard linearization form, then add terms that vanish to first order.

Step 1: Recover the data Comparing with

$$L = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

at $(a, b) = (2, -1)$ gives

$$\boxed{f(2, -1) = 7}, \quad \boxed{f_x(2, -1) = 3}, \quad \boxed{f_y(2, -1) = -4}.$$

Step 2: Change along the path The input changes are $\Delta x = t$ and $\Delta y = 2t$. Hence

$$\Delta f \approx 3t - 4(2t) = \boxed{-5t}.$$

Step 3: Construct an example One choice is

$$\boxed{f(x, y) = 7 + 3(x - 2) - 4(y + 1) + (x - 2)^2 + (y + 1)^2}.$$

The two quadratic terms and their first partial derivatives vanish at $(2, -1)$, so they do not alter the tangent plane.

Verification Substitution gives $f(2, -1) = 7$. Differentiation gives $f_x = 3 + 2(x - 2)$ and $f_y = -4 + 2(y + 1)$, which reproduce the recovered slopes at the base point.

Question 4

For the graph

$$z = f(x, y) = x^3 - 3x + y^2 - 4y,$$

locate every point at which the tangent plane is horizontal.

Tasks

1. Translate “horizontal tangent plane” into conditions on the partial derivatives.
2. Find all corresponding points on the graph.
3. Write each horizontal tangent plane and show that no others exist.

Question 4 – Solution

Strategy. A tangent plane to a graph is horizontal precisely when both of its slope coefficients vanish.

Step 1: Slope conditions

$$f_x = 3x^2 - 3, \quad f_y = 2y - 4.$$

Thus a horizontal plane requires

$$3x^2 - 3 = 0, \quad 2y - 4 = 0.$$

Therefore $x = \pm 1$ and $y = 2$.

Step 2: Heights

$$f(1, 2) = 1 - 3 + 4 - 8 = -6,$$

$$f(-1, 2) = -1 + 3 + 4 - 8 = -2.$$

The two points are

$$\boxed{(1, 2, -6) \quad \text{and} \quad (-1, 2, -2)}.$$

Step 3: Planes and completeness Because both slope coefficients are zero, the planes are

$$\boxed{z = -6} \quad \text{and} \quad \boxed{z = -2}.$$

The equations $x^2 = 1$ and $y = 2$ exhaust every simultaneous zero of f_x and f_y , so there are no other horizontal tangent planes.

Question 5

Define

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute both partial derivatives at the origin from their definitions.
2. Identify the candidate tangent plane suggested by those partial derivatives.
3. Test the linear-approximation remainder along $y = x^2$ and decide whether a tangent plane exists there in the differentiability sense.

Question 5 – Solution

Strategy. Existing partial derivatives only suggest a plane; differentiability requires the remainder to be negligible compared with the distance to the base point.

Step 1: Axis partial derivatives Along either coordinate axis, f is zero. Hence

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - 0}{h} = 0, \quad f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - 0}{k} = 0.$$

Step 2: Candidate plane The value and both proposed slopes are zero, so the formal candidate is

$$\boxed{z = 0}.$$

Step 3: Remainder test Along $(x, y) = (t, t^2)$ with $t \neq 0$,

$$f(t, t^2) = \frac{t^4}{t^4 + t^4} = \frac{1}{2}.$$

For the candidate linearization $L = 0$, the required quotient is

$$\frac{|f(t, t^2) - L(t, t^2)|}{\sqrt{t^2 + t^4}} = \frac{1/2}{|t|\sqrt{1 + t^2}} \rightarrow \infty.$$

Therefore the error is not $o(\sqrt{x^2 + y^2})$.

Conclusion The function is not even continuous at the origin along this curve, so it is not differentiable there. Thus the graph has no tangent plane at the origin in the linear-approximation sense.

Question 6

Let

$$f(x, y) = e^{x+2y}.$$

Use the linearization at $(0, 0)$ to estimate $f(0.02, 0.01)$ and rigorously bound the absolute error using the one-variable Taylor theorem for e^t .

Tasks

1. Find the linearization and the estimate.
2. Obtain a uniform error bound for $|x| \leq 0.02$, $|y| \leq 0.02$.
3. Compare the actual error at $(0.02, 0.01)$ with that bound.

Question 6 – Solution

Strategy. Write $t = x + 2y$ so the two-variable approximation reduces to the first-order Taylor approximation of e^t at zero.

Step 1: Linearization At the origin, $f = 1$, $f_x = 1$, and $f_y = 2$. Therefore

$$\boxed{L(x, y) = 1 + x + 2y}.$$

At $(0.02, 0.01)$, $t = 0.04$, so

$$f(0.02, 0.01) \approx 1.04.$$

Step 2: Uniform bound Taylor's theorem gives

$$e^t = 1 + t + \frac{e^\xi}{2}t^2$$

for some ξ between 0 and t . In the given rectangle,

$$|t| = |x + 2y| \leq 0.06,$$

so $e^\xi \leq e^{0.06}$. Consequently,

$$\boxed{|f - L| \leq \frac{e^{0.06}}{2}(0.06)^2 < 0.001912}.$$

Step 3: Actual error

$$|e^{0.04} - 1.04| \approx \boxed{0.0008108},$$

which lies below the uniform bound. The bound is larger because it must cover the entire rectangle, not only the requested point.

Question 7

For the paraboloid

$$z = x^2 + 2y^2,$$

find the tangent plane parallel to the plane

$$2x - 4y + z = 7.$$

Tasks

1. Determine the point of tangency.
2. Write the tangent plane in standard form.
3. Find the perpendicular distance between the two parallel planes.

Question 7 – Solution

Strategy. Match the x - and y -slope coefficients of the tangent plane to those of the given plane.

Step 1: Match slopes The given plane is $z = 7 - 2x + 4y$, so its slopes are -2 and 4 . For $f = x^2 + 2y^2$,

$$f_x = 2x, \quad f_y = 4y.$$

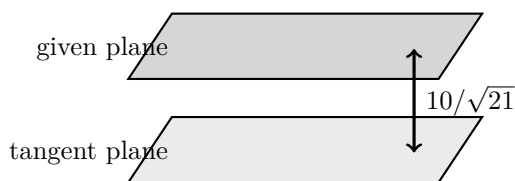
Thus $2x = -2$ and $4y = 4$, giving $(x, y) = (-1, 1)$. The height is $z = 1 + 2 = 3$.

Step 2: Tangent plane At $(-1, 1, 3)$,

$$z - 3 = -2(x + 1) + 4(y - 1),$$

or

$$\boxed{2x - 4y + z = -3}.$$



Step 3: Separation For parallel planes $2x - 4y + z = 7$ and $2x - 4y + z = -3$, the distance is

$$\boxed{\frac{|7 - (-3)|}{\sqrt{2^2 + (-4)^2 + 1^2}} = \frac{10}{\sqrt{21}}}.$$

The identical left sides verify parallelism, and substituting $(-1, 1, 3)$ into the tangent plane gives -3 .

Question 8

Suppose a function f is differentiable at (a, b) . An affine function has the form

$$A(x, y) = c + p(x - a) + q(y - b).$$

Tasks

1. Prove that if A is a linear approximation to f at (a, b) , then $c = f(a, b)$.
2. Use displacements along the coordinate axes to prove $p = f_x(a, b)$ and $q = f_y(a, b)$.
3. Conclude that the tangent-plane linearization is unique.

Question 8 – Solution

Strategy. Apply the defining small-error condition first at zero displacement and then on each coordinate axis.

Step 1: Constant term For A to approximate f at the base point, it must agree there. Since $A(a, b) = c$,

$$\boxed{c = f(a, b)}.$$

Equivalently, the remainder must vanish when the displacement is zero.

Step 2: The x coefficient The approximation condition along $(h, 0)$ is

$$f(a + h, b) = f(a, b) + ph + o(|h|).$$

Subtract $f(a, b)$, divide by $h \neq 0$, and let $h \rightarrow 0$. Since $o(|h|)/h \rightarrow 0$ from both sides,

$$p = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h} = \boxed{f_x(a, b)}.$$

Step 3: The y coefficient The same argument along $(0, k)$ yields

$$q = \lim_{k \rightarrow 0} \frac{f(a, b + k) - f(a, b)}{k} = \boxed{f_y(a, b)}.$$

Conclusion Every affine first-order approximation must therefore be

$$\boxed{L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}.$$

All three coefficients are forced, proving uniqueness.

Question 9

A cylinder has volume

$$V(r, h) = \pi r^2 h.$$

Its nominal dimensions are $r = 10$ cm and $h = 20$ cm. A measured cylinder instead has $\Delta r = 0.03$ cm and $\Delta h = -0.08$ cm.

Tasks

1. Use linearization to estimate the change in volume.
2. Estimate the relative and percentage changes.
3. Compute the exact change and quantify the linearization error.

Question 9 – Solution

Strategy. Evaluate the tangent-plane increment $\Delta V \approx V_r \Delta r + V_h \Delta h$ at the nominal dimensions.

Step 1: Volume change

$$V_r = 2\pi r h, \quad V_h = \pi r^2.$$

At (10, 20),

$$dV = (400\pi)(0.03) + (100\pi)(-0.08) = 12\pi - 8\pi.$$

Thus

$$\boxed{\Delta V \approx 4\pi \text{ cm}^3 \approx 12.566 \text{ cm}^3}.$$

Step 2: Relative change Dividing the differential by $V = \pi r^2 h$ gives

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h} = 2\frac{0.03}{10} - \frac{0.08}{20} = 0.002.$$

The estimated percentage increase is $\boxed{0.2\%}$.

Step 3: Exact comparison

$$\Delta V_{\text{exact}} = \pi(10.03)^2(19.92) - 2000\pi = \boxed{3.969928\pi \text{ cm}^3}.$$

The linear estimate exceeds the exact change by

$$(4 - 3.969928)\pi = \boxed{0.030072\pi \text{ cm}^3 \approx 0.09447 \text{ cm}^3}.$$

Question 10

Consider the family

$$f_{a,b}(x, y) = ax^2 + bxy + y^2.$$

At $(1, 1)$, its tangent plane passes through $(2, 0, 5)$, and its linearization predicts the value 3.06 at $(1.02, 0.99)$.

Tasks

1. Determine the parameters a and b .
2. Write the resulting tangent plane at $(1, 1)$.
3. Use it to estimate $f(0.98, 1.03)$ and verify both original constraints.

Question 10 – Solution

Strategy. Express the value and slopes at $(1, 1)$ in terms of a, b , then turn the two tangent-plane conditions into a linear system.

Step 1: General linearization At $(1, 1)$,

$$f = a + b + 1, \quad f_x = 2a + b, \quad f_y = b + 2.$$

Therefore

$$L(x, y) = a + b + 1 + (2a + b)(x - 1) + (b + 2)(y - 1).$$

Passing through $(2, 0, 5)$ requires

$$3a + b - 1 = 5, \quad \text{so} \quad 3a + b = 6.$$

The prediction at $(1.02, 0.99)$ requires

$$1.04a + 1.01b + 0.98 = 3.06.$$

Solving the two equations gives

$$\boxed{a = 2, \quad b = 0}.$$

Step 2: Tangent plane Now $f(1, 1) = 3$, $f_x(1, 1) = 4$, and $f_y(1, 1) = 2$. Hence

$$\boxed{z = 3 + 4(x - 1) + 2(y - 1)}.$$

Step 3: New estimate and verification At $(0.98, 1.03)$,

$$f(0.98, 1.03) \approx 3 + 4(-0.02) + 2(0.03) = \boxed{2.98}.$$

Also $L(2, 0) = 3 + 4 - 2 = 5$, and $L(1.02, 0.99) = 3 + 0.08 - 0.02 = 3.06$, verifying both supplied constraints.

Question 1

Consider the ellipsoid

$$F(x, y, z) = x^2 + 2y^2 + 3z^2 = 6$$

at $P = (1, 1, 1)$.

Tasks

1. Compute the gradient and verify that P is a regular point.
2. Find the tangent plane at P .
3. Find a parametric equation of the normal line and verify its direction.

Question 1 – Solution

Strategy. For a regular level surface, the gradient at the point is normal to the tangent plane.

Step 1: Gradient and regularity

$$\nabla F = \langle 2x, 4y, 6z \rangle, \quad \nabla F(1, 1, 1) = \langle 2, 4, 6 \rangle.$$

This vector is nonzero, so P is regular. Also $F(1, 1, 1) = 1 + 2 + 3 = 6$, confirming that P lies on the surface.

Step 2: Tangent plane Using $\nabla F(P)$ as a normal,

$$2(x - 1) + 4(y - 1) + 6(z - 1) = 0.$$

After division by 2,

$$\boxed{x + 2y + 3z = 6}.$$

Step 3: Normal line A simplified direction parallel to the gradient is $\langle 1, 2, 3 \rangle$. Hence

$$\boxed{(x, y, z) = (1, 1, 1) + t(1, 2, 3)},$$

or $x = 1 + t$, $y = 1 + 2t$, $z = 1 + 3t$.

Verification The line passes through P at $t = 0$, and $\langle 1, 2, 3 \rangle$ is perpendicular to every displacement $\langle u, v, w \rangle$ in the plane because the plane equation requires $u + 2v + 3w = 0$.

Question 2

Let the level surface

$$F(x, y, z) = xye^z = 2$$

contain $P = (1, 2, 0)$.

Tasks

1. Find the two unit normal vectors at P .
2. Write the tangent plane in standard form.
3. Give symmetric or parametric equations for the normal line.

Question 2 – Solution

Strategy. Differentiate the defining level function; its gradient supplies both the plane normal and the normal-line direction.

Step 1: Gradient

$$\nabla F = \langle ye^z, xe^z, xye^z \rangle.$$

At $P = (1, 2, 0)$,

$$\nabla F(P) = \langle 2, 1, 2 \rangle, \quad \|\nabla F(P)\| = 3.$$

Therefore the two unit normals are

$$\pm \frac{1}{3} \langle 2, 1, 2 \rangle.$$

Step 2: Tangent plane

$$2(x - 1) + (y - 2) + 2(z - 0) = 0,$$

so

$$2x + y + 2z = 4.$$

Step 3: Normal line Using the unsimplified integer direction,

$$(x, y, z) = (1, 2, 0) + t(2, 1, 2).$$

Equivalently,

$$\frac{x - 1}{2} = y - 2 = \frac{z}{2}.$$

Verification The point satisfies $1 \cdot 2 \cdot e^0 = 2$. The line direction equals the plane's coefficient vector, so it is perpendicular to the plane.

Question 3

The sphere

$$x^2 + y^2 + z^2 = 9$$

and the plane

$$x + y + z = 5$$

intersect in a curve through $P = (1, 2, 2)$.

Tasks

1. Find a direction tangent to the intersection curve at P .
2. Write the tangent line to the curve.
3. Verify that the direction is tangent to both surfaces.

Question 3 – Solution

Strategy. A curve lying on both surfaces has a tangent perpendicular to both surface normals, so use their cross product.

Step 1: Surface normals Let

$$F = x^2 + y^2 + z^2, \quad G = x + y + z.$$

At P ,

$$\nabla F(P) = \langle 2, 4, 4 \rangle, \quad \nabla G(P) = \langle 1, 1, 1 \rangle.$$

Their cross product is

$$\nabla F(P) \times \nabla G(P) = \langle 0, 2, -2 \rangle,$$

so a simplified tangent direction is $\langle 0, 1, -1 \rangle$.

Step 2: Tangent line

$$\boxed{(x, y, z) = (1, 2, 2) + t(0, 1, -1)}.$$

Step 3: Verification First, $1^2 + 2^2 + 2^2 = 9$ and $1 + 2 + 2 = 5$, so P lies on both surfaces. Next,

$$\langle 2, 4, 4 \rangle \cdot \langle 0, 1, -1 \rangle = 0, \quad \langle 1, 1, 1 \rangle \cdot \langle 0, 1, -1 \rangle = 0.$$

Thus the direction lies in both tangent planes, as required.

Question 4

On the ellipsoid

$$x^2 + 2y^2 + z^2 = 9,$$

find every point whose tangent plane is parallel to

$$2x - 4y + 2z = 7.$$

Tasks

1. Determine all points of tangency.
2. Write the tangent plane at each point.
3. Give a normal line at each point and verify completeness.

Question 4 – Solution

Strategy. Parallel tangent planes have parallel normal vectors; impose that condition together with the surface equation.

Step 1: Parallel-gradient condition For $F = x^2 + 2y^2 + z^2$,

$$\nabla F = \langle 2x, 4y, 2z \rangle.$$

The given plane has normal parallel to $\langle 1, -2, 1 \rangle$. Set

$$\langle 2x, 4y, 2z \rangle = \lambda \langle 1, -2, 1 \rangle.$$

Then $x = \lambda/2$, $y = -\lambda/2$, and $z = \lambda/2$. Substitution into the ellipsoid gives $\lambda^2 = 9$.

Thus the two points are

$$P_+ = \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2} \right), \quad P_- = -P_+.$$

Step 2: Tangent planes Using the simplified common normal,

$$x - 2y + z = 6 \quad \text{at } P_+,$$

$$x - 2y + z = -6 \quad \text{at } P_-.$$

Step 3: Normal lines

$$\mathbf{r} = P_+ + t \langle 1, -2, 1 \rangle, \quad \mathbf{r} = P_- + t \langle 1, -2, 1 \rangle.$$

The equations $\lambda = \pm 3$ exhaust the surface constraint, proving there are no additional points.

Question 5

For the surface

$$xy + yz + zx = 3,$$

consider $P = (1, 1, 1)$.

Tasks

1. Find the normal line at P .
2. Find every point where that normal line intersects the sphere $x^2 + y^2 + z^2 = 12$.
3. Verify both the surface-normal direction and the sphere intersections.

Question 5 – Solution

Strategy. Parameterize the normal line using the gradient, then substitute that line into the sphere equation.

Step 1: Normal line Let $F = xy + yz + zx$. Then

$$\nabla F = \langle y + z, x + z, x + y \rangle, \quad \nabla F(P) = \langle 2, 2, 2 \rangle.$$

Using the simplified direction $\langle 1, 1, 1 \rangle$,

$$\boxed{(x, y, z) = (1 + t, 1 + t, 1 + t)}.$$

Step 2: Intersections with the sphere Substitution gives

$$3(1 + t)^2 = 12,$$

so $(1 + t)^2 = 4$ and

$$t = 1 \quad \text{or} \quad t = -3.$$

The intersection points are therefore

$$\boxed{(2, 2, 2) \quad \text{and} \quad (-2, -2, -2)}.$$

Step 3: Verification The base point satisfies $1 + 1 + 1 = 3$, and the line direction is parallel to $\nabla F(P)$, so the line is normal to the original surface. Finally,

$$2^2 + 2^2 + 2^2 = 12, \quad (-2)^2 + (-2)^2 + (-2)^2 = 12,$$

confirming both sphere intersections.

Question 6

Consider the double cone

$$F(x, y, z) = x^2 + y^2 - z^2 = 0.$$

Tasks

1. Find the tangent plane and normal line at the regular point $P = (1, 0, 1)$.
2. Explain why the gradient formula fails to select a plane at the origin.
3. Use generator curves through the origin to prove that no single tangent plane contains all tangent directions there.

Question 6 – Solution

Strategy. Contrast a regular point, where the gradient is nonzero, with the cone vertex, where many generator directions meet and the gradient vanishes.

Step 1: The regular point

$$\nabla F = \langle 2x, 2y, -2z \rangle, \quad \nabla F(1, 0, 1) = \langle 2, 0, -2 \rangle.$$

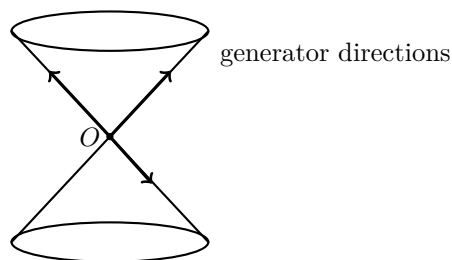
Thus

$$2(x - 1) - 2(z - 1) = 0, \quad \boxed{x - z = 0},$$

and a normal line is

$$\boxed{(x, y, z) = (1, 0, 1) + t(1, 0, -1)}.$$

Step 2: The vertex At $O = (0, 0, 0)$, $\nabla F(O) = \langle 0, 0, 0 \rangle$, so the usual plane equation reduces to $0 = 0$ and supplies no normal.



Step 3: Generator argument For every θ , the curve

$$\mathbf{r}_\theta(s) = \langle s \cos \theta, s \sin \theta, s \rangle$$

lies on the cone and has tangent $\langle \cos \theta, \sin \theta, 1 \rangle$ at O . The directions for $\theta = 0, \pi$, and $\pi/2$ span all of $\mathbb{R}^3$: the first two yield the x - and z -directions by differences and sums, and the third then yields the y -direction. No two-dimensional plane can contain them all. Hence $\boxed{\text{there is no unique tangent plane at the vertex}}.$

Question 7

An unknown member of the family

$$F(x, y, z) = x^2 + ay^2 + bz^2 = c$$

passes through $P = (1, 1, 1)$ and has tangent plane

$$x + 2y + 3z = 6$$

there.

Tasks

1. Determine a , b , and c .
2. Confirm that P is regular for the resulting surface.
3. Write its normal line at P and verify the prescribed tangent plane.

Question 7 – Solution

Strategy. Match the unknown gradient to the plane's normal; the first gradient component fixes the proportionality factor.

Step 1: Match normals

$$\nabla F(P) = \langle 2, 2a, 2b \rangle.$$

This must equal $\lambda \langle 1, 2, 3 \rangle$. The first component gives $\lambda = 2$. Therefore

$$2a = 4, \quad 2b = 6,$$

so

$$\boxed{a = 2, \quad b = 3}.$$

Since P lies on the surface,

$$c = 1^2 + 2(1)^2 + 3(1)^2 = \boxed{6}.$$

Step 2: Regularity For the resulting surface,

$$\nabla F(P) = \langle 2, 4, 6 \rangle \neq \langle 0, 0, 0 \rangle,$$

so P is regular.

Step 3: Line and plane verification The normal line is

$$\boxed{(x, y, z) = (1, 1, 1) + t(1, 2, 3)}.$$

The plane obtained from the gradient is

$$2(x - 1) + 4(y - 1) + 6(z - 1) = 0.$$

Dividing by 2 produces $x + 2y + 3z = 6$, exactly the prescribed plane.

Question 8

For the level surface

$$xyz = 8,$$

let $P = (2, 2, 2)$.

Tasks

1. Find the tangent plane and normal line at P .
2. Show that the normal line passes through the origin.
3. Prove that P is the perpendicular foot from the origin to the tangent plane and find that distance in two ways.

Question 8 – Solution

Strategy. Use the gradient for the plane and line, then compare the point-to-plane formula with the length of the normal segment.

Step 1: Plane and line For $F = xyz$,

$$\nabla F = \langle yz, xz, xy \rangle, \quad \nabla F(P) = \langle 4, 4, 4 \rangle.$$

Thus the tangent plane is

$$4(x - 2) + 4(y - 2) + 4(z - 2) = 0,$$

or

$$\boxed{x + y + z = 6}.$$

A normal line is

$$\boxed{(x, y, z) = (2, 2, 2) + t(1, 1, 1)}.$$

Step 2: Origin on the line At $t = -2$, all three coordinates are zero. Thus the segment from the origin to P follows the plane's normal direction, making P the perpendicular foot.

Step 3: Distance checks The normal-segment length is

$$\| \langle 2, 2, 2 \rangle \| = \boxed{2\sqrt{3}}.$$

The point-to-plane formula independently gives

$$\frac{|0 + 0 + 0 - 6|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{6}{\sqrt{3}} = \boxed{2\sqrt{3}}.$$

The agreement verifies both the geometry and the scale of the normal vector.

Question 9

Let S be a smooth level surface $F(x, y, z) = c$, and let P be a point of S with $\nabla F(P) \neq 0$.

Tasks

1. For any differentiable curve $\mathbf{r}(t)$ on S through P , prove that its tangent at P is perpendicular to $\nabla F(P)$.
2. Deduce the tangent-plane equation at $P = (x_0, y_0, z_0)$.
3. Deduce the normal-line equation and explain the role of the regularity condition.

Question 9 – Solution

Strategy. Differentiate the constant identity $F(\mathbf{r}(t)) = c$ and interpret the chain rule geometrically.

Step 1: Tangent vectors Suppose $\mathbf{r}(t_0) = P$ and the curve lies on S . Then

$$F(\mathbf{r}(t)) = c.$$

Differentiating at t_0 by the multivariable chain rule gives

$$0 = \left. \frac{d}{dt} F(\mathbf{r}(t)) \right|_{t=t_0} = \nabla F(P) \cdot \mathbf{r}'(t_0).$$

Thus every surface-tangent velocity is perpendicular to $\nabla F(P)$.

Step 2: Tangent plane Because $\nabla F(P)$ is nonzero, it supplies a valid normal. Therefore

$$\boxed{F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0}.$$

Step 3: Normal line The line through P in the gradient direction is

$$\boxed{\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \nabla F(P)}.$$

The condition $\nabla F(P) \neq 0$ ensures a genuine normal direction and a two-dimensional tangent plane. At a singular point, the zero vector cannot determine either object; additional geometric analysis is required.

Question 10

On the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6,$$

find every point at which the normal line passes through the origin.

Tasks

1. Derive the condition relating a point to its gradient.
2. Prove that at most one coordinate of such a point can be nonzero.
3. List all points, tangent planes, and normal lines, and verify completeness.

Question 10 – Solution

Strategy. If the origin lies on the normal line at P , then the vector from P to the origin must be parallel to $\nabla F(P)$.

Step 1: Parallelism condition Write $P = (p, q, r)$. Since

$$\nabla F(P) = \langle 2p, 4q, 6r \rangle,$$

the origin lies on the normal line exactly when some t satisfies

$$\langle -p, -q, -r \rangle = t \langle 2p, 4q, 6r \rangle.$$

If $p \neq 0$, then $t = -1/2$; if $q \neq 0$, then $t = -1/4$; if $r \neq 0$, then $t = -1/6$. One value of t cannot meet two of these conditions, so at most one coordinate is nonzero.

Step 2: Points and planes Using the ellipsoid equation on each coordinate axis gives exactly six points:

$$(\pm\sqrt{6}, 0, 0), \quad (0, \pm\sqrt{3}, 0), \quad (0, 0, \pm\sqrt{2}).$$

Their tangent planes are, respectively,

$$\boxed{x = \pm\sqrt{6}}, \quad \boxed{y = \pm\sqrt{3}}, \quad \boxed{z = \pm\sqrt{2}}.$$

Step 3: Normal lines and verification The corresponding normal lines are

$$\boxed{\mathbf{r} = (\pm\sqrt{6}, 0, 0) + t \langle 1, 0, 0 \rangle},$$
$$\boxed{\mathbf{r} = (0, \pm\sqrt{3}, 0) + t \langle 0, 1, 0 \rangle}, \quad \boxed{\mathbf{r} = (0, 0, \pm\sqrt{2}) + t \langle 0, 0, 1 \rangle}.$$

These are the coordinate axes written through their stated points. Every one passes through the origin and is perpendicular to its listed coordinate plane. The parallelism argument rules out mixed-coordinate points, while the surface equation rules out the origin itself, so the list is complete.

Question 1

For

$$f(x, y) = x^2 + 2y^2 - 4x + 8y + 1,$$

analyze all relative extrema.

Tasks

1. Find every critical point.
2. Classify each point with the second derivative test.
3. Complete the square to verify the classification and find the extremal value.

Question 1 – Solution

Strategy. Solve the first-derivative equations, classify with the Hessian determinant, and then use an exact algebraic form as an independent check.

Step 1: Critical point

$$f_x = 2x - 4, \quad f_y = 4y + 8.$$

Both vanish only when

$$\boxed{(x, y) = (2, -2)}.$$

Step 2: Second derivative test

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = 0.$$

The Hessian determinant is

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 8 > 0.$$

Since $f_{xx} > 0$, the point is a strict relative minimum. Its value is

$$f(2, -2) = \boxed{-11}.$$

Step 3: Algebraic verification Completing squares gives

$$f(x, y) = (x - 2)^2 + 2(y + 2)^2 - 11.$$

Both squared terms are nonnegative and vanish simultaneously only at $(2, -2)$. This directly confirms the strict local minimum and rules out any other critical point or relative extremum.

Question 2

For

$$f(x, y) = x^3 + y^3 - 3xy,$$

find and classify every critical point.

Tasks

1. Solve the nonlinear critical-point equations without losing solutions.
2. Apply the second derivative test at each point.
3. Verify the saddle classification by displaying nearby values of opposite signs relative to the critical value.

Question 2 – Solution

Strategy. Use the coupled equations to reduce the possible coordinates, then apply the Hessian determinant separately at each solution.

Step 1: Critical points

$$f_x = 3x^2 - 3y, \quad f_y = 3y^2 - 3x.$$

Thus $y = x^2$ and $x = y^2$. Substitution gives

$$x = x^4, \quad x(x^3 - 1) = 0.$$

The real possibilities are $x = 0$ and $x = 1$, giving

$$\boxed{(0, 0) \quad \text{and} \quad (1, 1)}.$$

Step 2: Hessian test

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3,$$

so $D = 36xy - 9$. At $(0, 0)$, $D = -9 < 0$, hence it is a saddle point. At $(1, 1)$, $D = 27 > 0$ and $f_{xx} = 6 > 0$, so it is a strict relative minimum with

$$f(1, 1) = \boxed{-1}.$$

Step 3: Saddle verification At the origin the critical value is 0. Along $y = 0$,

$$f(x, 0) = x^3,$$

which is positive for small $x > 0$ and negative for small $x < 0$. Thus every neighborhood contains values above and below 0, independently verifying the saddle classification.

Question 3

Analyze the origin for

$$f(x, y) = x^2 + 4xy + y^2.$$

Tasks

1. Verify that the origin is the only critical point.
2. Classify it using the second derivative test.
3. Give two explicit paths that make the saddle geometry visible.

Question 3 – Solution

Strategy. The quadratic form itself records the local geometry; the Hessian test and carefully chosen diagonal paths should agree.

Step 1: Critical point

$$f_x = 2x + 4y, \quad f_y = 4x + 2y.$$

Solving $x + 2y = 0$ and $2x + y = 0$ gives $x = y = 0$. Therefore the origin is the unique critical point.

Step 2: Hessian test

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 4.$$

Hence

$$D = (2)(2) - 4^2 = -12 < 0.$$

Therefore

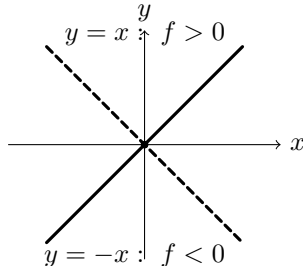
$(0, 0)$ is a saddle point.

Step 3: Path verification Along $y = x$,

$$f(x, x) = x^2 + 4x^2 + x^2 = 6x^2 > 0$$

for $x \neq 0$. Along $y = -x$,

$$f(x, -x) = x^2 - 4x^2 + x^2 = -2x^2 < 0.$$



Both paths approach the origin, whose value is zero, so arbitrarily nearby values occur on both sides of the critical value.

Question 4

Consider the “monkey saddle”

$$f(x, y) = x^3 - 3xy^2.$$

Tasks

1. Find all critical points.
2. Show precisely why the second derivative test is inconclusive there.
3. Classify the point using both simple paths and the polar form of the function.

Question 4 – Solution

Strategy. When the Hessian degenerates, return to the definition and compare the function along paths through the critical point.

Step 1: Critical point

$$f_x = 3x^2 - 3y^2, \quad f_y = -6xy.$$

The second equation requires $x = 0$ or $y = 0$. In either case the first equation then forces the other coordinate to be zero. Thus the only critical point is $(0, 0)$.

Step 2: Failed Hessian test

$$f_{xx} = 6x, \quad f_{yy} = -6x, \quad f_{xy} = -6y.$$

All three second derivatives vanish at the origin, so

$$D(0, 0) = 0.$$

The test gives no classification.

Step 3: Direct classification

Along $y = 0$,

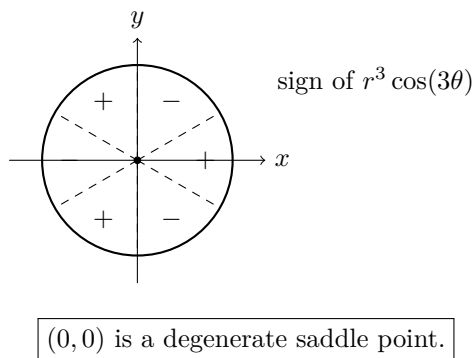
$$f(x, 0) = x^3,$$

which takes both signs arbitrarily close to zero. Hence the origin is a saddle.

For a structural check, let $x = r \cos \theta$ and $y = r \sin \theta$. Then

$$f = r^3 (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) = r^3 \cos(3\theta).$$

The factor $\cos(3\theta)$ alternates sign in six angular sectors. Therefore



Question 5

Let

$$f(x, y) = (x^2 + y^2)e^{-(x^2 + y^2)}.$$

Tasks

1. Find the complete critical set.
2. Classify the origin and every point on the critical circle as relative extrema.
3. Explain why the circular extrema are not strict and why the ordinary Hessian test degenerates there.

Question 5 – Solution

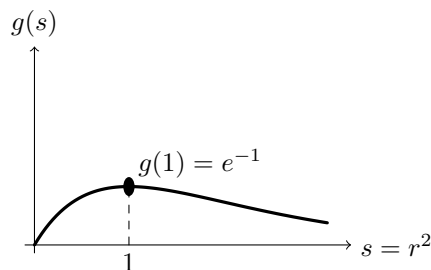
Strategy. Set $s = x^2 + y^2$ and analyze the one-variable radial profile $g(s) = se^{-s}$.

Step 1: Critical set Since

$$f_x = 2xe^{-s}(1-s), \quad f_y = 2ye^{-s}(1-s),$$

both derivatives vanish when $(x, y) = (0, 0)$ or when $s = 1$. Thus

$$\boxed{\text{critical set} = \{(0, 0)\} \cup \{x^2 + y^2 = 1\}}.$$



Step 2: Radial classification For $s \geq 0$,

$$g'(s) = e^{-s}(1-s).$$

It is positive for $0 \leq s < 1$ and negative for $s > 1$. Thus $g(0) = 0$ increases immediately away from the origin, so

$$\boxed{(0, 0) \text{ is a strict relative minimum.}}$$

The radial profile peaks at $s = 1$, so every point of the unit circle is a relative maximum with value

$$\boxed{e^{-1}}.$$

Step 3: Non-strictness and degeneracy Moving tangentially along the circle leaves s and f unchanged, so none of these maxima is strict. That zero-curvature tangential direction makes the Hessian determinant zero on the critical circle, which is why the standard second derivative test is inconclusive there.

Question 6

Consider the parameterized quadratic

$$f_{a,b}(x, y) = x^2 + axy + by^2 - 4x + 2y.$$

The point $P = (1, -1)$ is known to be critical.

Tasks

1. Determine a and b from the critical-point condition.
2. Classify P for the resulting function.
3. Verify the classification with two lines through P .

Question 6 – Solution

Strategy. Use the two first-derivative equations to reconstruct the parameters, then test the resulting quadratic form around P .

Step 1: Determine the parameters

$$f_x = 2x + ay - 4, \quad f_y = ax + 2by + 2.$$

At $(1, -1)$, the equations $f_x = f_y = 0$ become

$$-2 - a = 0, \quad a - 2b + 2 = 0.$$

Therefore

$$\boxed{a = -2, \quad b = 0}.$$

Step 2: Hessian classification For the resulting function,

$$f_{xx} = 2, \quad f_{yy} = 0, \quad f_{xy} = -2.$$

Thus

$$D = (2)(0) - (-2)^2 = -4 < 0,$$

so P is a saddle point.

Step 3: Direct verification Write $x = 1 + h$, $y = -1 + k$. Since the function is quadratic and P is critical,

$$f(1 + h, -1 + k) - f(1, -1) = h^2 - 2hk.$$

Along $k = 0$, the change is $h^2 > 0$. Along $k = h$, it is $-h^2 < 0$. Hence every neighborhood of P contains values above and below $f(P)$, confirming

$$\boxed{P = (1, -1) \text{ is a saddle point.}}$$

Question 7

A twice differentiable function has the local expansion near a critical point P :

$$f(P + (h, k)) = 5 + 3h^2 + 4hk - k^2 + o(h^2 + k^2).$$

Tasks

1. Reconstruct the Hessian matrix at P .
2. Apply the second derivative test.
3. Use the remainder term carefully to prove the classification along two coordinate directions.

Question 7 – Solution

Strategy. Match the quadratic terms with $\frac{1}{2}(h, k)H(h, k)^T$, then show that the leading signs dominate the smaller remainder.

Step 1: Hessian The second-order Taylor term is

$$\frac{1}{2} (f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2).$$

Matching coefficients gives

$$f_{xx} = 6, \quad f_{xy} = 4, \quad f_{yy} = -2,$$

so

$$H(P) = \begin{pmatrix} 6 & 4 \\ 4 & -2 \end{pmatrix}.$$

Step 2: Second derivative test

$$D = (6)(-2) - 4^2 = -28 < 0.$$

Therefore P is a saddle point.

Step 3: Remainder verification Along $k = 0$,

$$f(P + (h, 0)) - 5 = 3h^2 + o(h^2),$$

which is positive for all sufficiently small nonzero h . Along $h = 0$,

$$f(P + (0, k)) - 5 = -k^2 + o(k^2),$$

which is negative for all sufficiently small nonzero k . The little- o terms are eventually smaller in magnitude than, for example, half of the displayed quadratic terms. Thus nearby values occur on both sides of 5, rigorously confirming the saddle.

Question 8

Let f be differentiable at an interior point $P = (a, b)$ of its domain.

Tasks

1. Prove that if f has a relative maximum or minimum at P , then $\nabla f(P) = \mathbf{0}$.
2. Explain why the converse is false.
3. Apply the result to show that $g(x, y) = x + e^y$ has no relative extrema on $\mathbb{R}^2$.

Question 8 – Solution

Strategy. Restrict the function to coordinate lines through the interior point and apply the one-variable necessary condition.

Step 1: Necessary condition Suppose f has a relative extremum at (a, b) . The one-variable restriction

$$\phi(x) = f(x, b)$$

has a relative extremum at $x = a$. Differentiability gives

$$f_x(a, b) = \phi'(a) = 0.$$

Likewise, $\psi(y) = f(a, y)$ has a relative extremum at b , so

$$f_y(a, b) = \psi'(b) = 0.$$

Therefore

$$\boxed{\nabla f(P) = \mathbf{0}}.$$

Step 2: The converse A zero gradient makes P a critical point, not necessarily an extremum. For instance, $f(x, y) = x^2 - y^2$ has gradient zero at the origin but takes both positive and negative values nearby, so the origin is a saddle.

Step 3: Application For $g(x, y) = x + e^y$,

$$g_x = 1, \quad g_y = e^y.$$

In particular, the gradient is never zero. Since every point of $\mathbb{R}^2$ is interior and g is differentiable everywhere, the necessary condition rules out a relative maximum or minimum at every point. Hence

$$\boxed{g \text{ has no relative extrema on } \mathbb{R}^2}.$$

Question 9

Consider

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Tasks

1. Prove directly that the origin is a strict relative minimum.
2. Determine whether $f_x(0, 0)$ and $f_y(0, 0)$ exist.
3. Explain how this example refines the usual critical-point search procedure.

Question 9 – Solution

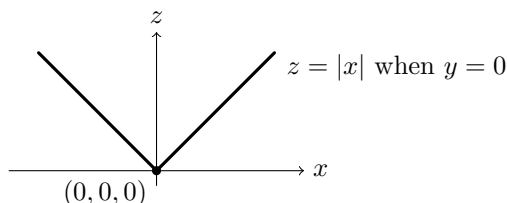
Strategy. Use the definition of a relative minimum first, then test each partial derivative with a signed two-sided limit.

Step 1: Direct extremum test At the origin, $f(0, 0) = 0$. For every $(x, y) \neq (0, 0)$,

$$f(x, y) = \sqrt{x^2 + y^2} > 0 = f(0, 0).$$

Therefore

$(0, 0)$ is a strict relative minimum.



Step 2: Partial derivatives Along the x -axis,

$$\frac{f(h, 0) - f(0, 0)}{h} = \frac{|h|}{h},$$

whose right-hand limit is 1 and left-hand limit is -1 . Thus $f_x(0, 0)$ does not exist. The identical calculation on the y -axis shows that $f_y(0, 0)$ does not exist.

Step 3: Methodological conclusion Fermat's theorem says that a *differentiable* interior extremum has zero gradient. It does not say every extremum is differentiable. Consequently, a complete candidate search must include interior points where first partial derivatives vanish *and* points where they fail to exist. This example supplies the latter case.

Question 10

A local profit model, in suitable units, is

$$P(x, y) = 80x + 60y - 2x^2 - xy - y^2.$$

Assume the critical point found below lies in the interior of the operational region.

Tasks

1. Find the critical production levels.
2. Classify the critical point using the Hessian.
3. Complete the square in translated coordinates to verify the local classification and compute the modeled profit there.

Question 10 – Solution

Strategy. Solve the marginal-profit equations, then verify the Hessian conclusion by expressing every nearby change as a negative quadratic form.

Step 1: Critical production levels

$$P_x = 80 - 4x - y, \quad P_y = 60 - x - 2y.$$

The equations $4x + y = 80$ and $x + 2y = 60$ yield

$$\boxed{x = \frac{100}{7}, \quad y = \frac{160}{7}}.$$

Step 2: Hessian test

$$P_{xx} = -4, \quad P_{yy} = -2, \quad P_{xy} = -1.$$

Thus

$$D = (-4)(-2) - (-1)^2 = 7 > 0.$$

Because $P_{xx} < 0$, the critical point is a strict relative maximum. Direct substitution gives

$$\boxed{P\left(\frac{100}{7}, \frac{160}{7}\right) = \frac{8800}{7}}.$$

Step 3: Translated verification Let $h = x - 100/7$ and $k = y - 160/7$. Since P is quadratic and its linear terms vanish at the critical point,

$$P(x, y) - \frac{8800}{7} = -2h^2 - hk - k^2.$$

Complete the square:

$$-2h^2 - hk - k^2 = -2\left(h + \frac{k}{4}\right)^2 - \frac{7}{8}k^2 < 0$$

for every $(h, k) \neq (0, 0)$. This independently verifies the strict relative maximum.

Question 1

Find the absolute extrema of

$$f(x, y) = x^2 + y^2 - 2x - 4y$$

on the rectangle $R = [0, 3] \times [0, 4]$.

Tasks

1. Find all interior critical points.
2. Analyze all four boundary segments, including their endpoints.
3. Compare the complete candidate list and report every absolute extremum.

Question 1 – Solution

Strategy. On a closed, bounded rectangle, compare values at interior critical points, one-variable boundary critical points, and corners.

Step 1: Interior

$$f_x = 2x - 2, \quad f_y = 2y - 4.$$

The sole interior critical point is $(1, 2)$, where

$$f(1, 2) = 1 + 4 - 2 - 8 = \boxed{-5}.$$

Step 2: Boundaries On $x = 0$ and $x = 3$,

$$f(0, y) = y^2 - 4y, \quad f(3, y) = y^2 - 4y + 3.$$

Both have a boundary critical value at $y = 2$, giving -4 and -1 . On $y = 0$ and $y = 4$,

$$f(x, 0) = f(x, 4) = x^2 - 2x,$$

whose critical point $x = 1$ gives -1 . The four corner values are

$$f(0, 0) = 0, \quad f(3, 0) = 3, \quad f(0, 4) = 0, \quad f(3, 4) = 3.$$

Step 3: Comparison The smallest candidate value is -5 and the largest is 3 . Hence

$$\boxed{f_{\min} = -5 \text{ at } (1, 2)},$$

$$\boxed{f_{\max} = 3 \text{ at } (3, 0) \text{ and } (3, 4)}.$$

Completing the square, $f = (x - 1)^2 + (y - 2)^2 - 5$, independently confirms the comparison.

Question 2

Find the absolute extrema of

$$f(x, y) = x^2 + y^2 - 2x$$

on the closed unit disk $D : x^2 + y^2 \leq 1$.

Tasks

1. Check for interior critical points.
2. Parameterize the circular boundary and find all boundary candidates.
3. Compare the candidates and verify the result geometrically by completing the square.

Question 2 – Solution

Strategy. Treat the interior and the circular boundary separately; on the boundary the radial term is constant.

Step 1: Interior

$$f_x = 2x - 2, \quad f_y = 2y.$$

The only solution is $(1, 0)$, which lies on the boundary rather than in the interior. Thus there are no interior critical points.

Step 2: Boundary Set $x = \cos \theta$, $y = \sin \theta$. Then

$$f(\cos \theta, \sin \theta) = 1 - 2 \cos \theta.$$

The smallest value occurs when $\cos \theta = 1$ and the largest when $\cos \theta = -1$. Therefore

$$\boxed{f_{\min} = -1 \text{ at } (1, 0)}, \quad \boxed{f_{\max} = 3 \text{ at } (-1, 0)}.$$

Step 3: Geometric verification

$$f(x, y) = (x - 1)^2 + y^2 - 1.$$

Thus $f + 1$ is squared distance from (x, y) to $(1, 0)$. Within the disk, that distance is smallest at $(1, 0)$ and largest at the diametrically opposite point $(-1, 0)$. This confirms both locations and values.

Question 3

On the closed triangle

$$T = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 4\},$$

find the absolute extrema of

$$f(x, y) = xy(4 - x - y).$$

Tasks

1. Find every interior critical point.
2. Analyze all three edges and vertices.
3. Compare the candidates and interpret why the minimum occurs along an entire boundary.

Question 3 – Solution

Strategy. The compact triangle guarantees extrema; compare the interior stationary value with the three edge restrictions.

Step 1: Interior

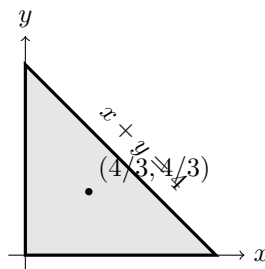
$$f_x = y(4 - 2x - y), \quad f_y = x(4 - x - 2y).$$

In the interior $x, y > 0$, so

$$2x + y = 4, \quad x + 2y = 4.$$

Thus $x = y = 4/3$, and

$$f\left(\frac{4}{3}, \frac{4}{3}\right) = \frac{4}{3} \frac{4}{3} \frac{4}{3} = \boxed{\frac{64}{27}}.$$



Step 2: Boundary On $x = 0$ or $y = 0$, one factor vanishes. On $x + y = 4$, the third factor vanishes. Hence

$$f = 0 \quad \text{on all three edges, including the vertices.}$$

Step 3: Comparison Inside T , all three factors x , y , and $4 - x - y$ are positive, so $f > 0$. Therefore

$$\boxed{f_{\min} = 0 \text{ at every boundary point of } T},$$

$$\boxed{f_{\max} = \frac{64}{27} \text{ at } (4/3, 4/3)}.$$

The candidate list is complete because the polynomial is continuous and T is closed and bounded.

Question 4

Find the absolute extrema of

$$f(x, y) = x + 2y$$

on and inside the ellipse

$$x^2 + 4y^2 \leq 4.$$

Do not use Lagrange multipliers.

Tasks

1. Transform the ellipse to a disk with a change of variables.
2. Use an inequality to derive sharp upper and lower bounds.
3. Find all equality points and verify they lie on the ellipse.

Question 4 – Solution

Strategy. Let $u = x$ and $v = 2y$; then the region is a radius-2 disk and the objective is $u + v$.

Step 1: Transform Set

$$u = x, \quad v = 2y.$$

The constraint becomes $u^2 + v^2 \leq 4$, while

$$f = u + v = \langle u, v \rangle \cdot \langle 1, 1 \rangle.$$

Step 2: Sharp bounds By Cauchy–Schwarz,

$$|u + v| \leq \sqrt{u^2 + v^2} \sqrt{2} \leq 2\sqrt{2}.$$

Thus $-2\sqrt{2} \leq f \leq 2\sqrt{2}$.

Equality on the upper side requires $\langle u, v \rangle$ to point with $\langle 1, 1 \rangle$ and have length 2, giving

$$(u, v) = (\sqrt{2}, \sqrt{2}).$$

Lower equality gives its negative.

Step 3: Return to (x, y) Because $x = u$ and $y = v/2$,

$$f_{\max} = 2\sqrt{2} \text{ at } \left(\sqrt{2}, \frac{\sqrt{2}}{2} \right),$$

$$f_{\min} = -2\sqrt{2} \text{ at } \left(-\sqrt{2}, -\frac{\sqrt{2}}{2} \right).$$

For either point, $x^2 + 4y^2 = 2 + 2 = 4$, verifying boundary membership and sharpness.

Question 5

Find the absolute extrema of

$$f(x, y) = x^2 + y^2 - 3x$$

on the closed annulus

$$1 \leq x^2 + y^2 \leq 4.$$

Tasks

1. Find all interior critical points that belong to the annulus.
2. Analyze both circular boundary components.
3. Compare every candidate and identify the absolute extrema.

Question 5 – Solution

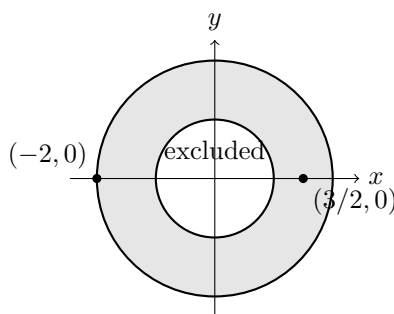
Strategy. Check the annulus interior, then write each boundary circle as $x = r \cos \theta$, $y = r \sin \theta$ with fixed r .

Step 1: Interior

$$f_x = 2x - 3, \quad f_y = 2y.$$

The critical point is $(3/2, 0)$, whose radius $3/2$ lies strictly between 1 and 2. Its value is

$$f(3/2, 0) = \frac{9}{4} - \frac{9}{2} = \boxed{-\frac{9}{4}}.$$



Step 2: Circular boundaries On a circle of radius r ,

$$f = r^2 - 3r \cos \theta.$$

For $r = 1$, its range is $[-2, 4]$. For $r = 2$, its range is $[-2, 10]$. The value 10 occurs on the outer circle when $\cos \theta = -1$, at $(-2, 0)$.

Step 3: Comparison Comparing $-9/4$, the boundary minimum -2 , and the boundary maximum 10 gives

$$\boxed{f_{\min} = -\frac{9}{4} \text{ at } (3/2, 0)}, \quad \boxed{f_{\max} = 10 \text{ at } (-2, 0)}.$$

Both boundary components were checked, so the hole in the region causes no missing candidates.

Question 6

Let

$$D = \{(x, y) : x^2 + y^2 < 1\}$$

be the open unit disk, and let $f(x, y) = x + y$.

Tasks

1. Determine whether f has an absolute maximum or minimum on D .
2. Find its supremum and infimum and give sequences approaching them.
3. Explain exactly which hypothesis of the Extreme Value Theorem is absent.

Question 6 – Solution

Strategy. Bound the linear function with Cauchy–Schwarz, then determine whether equality points belong to the open disk.

Step 1: Bounds For $(x, y) \in D$,

$$|x + y| \leq \sqrt{x^2 + y^2} \sqrt{2} < \sqrt{2}.$$

Therefore every function value lies strictly between $-\sqrt{2}$ and $\sqrt{2}$.

Step 2: Approaching the bounds For $0 < r < 1$, the point

$$\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}} \right)$$

lies in D and gives $f = r\sqrt{2} \rightarrow \sqrt{2}$ as $r \rightarrow 1^-$. Its negative gives values tending to $-\sqrt{2}$. Thus

$$\boxed{\sup_D f = \sqrt{2}}, \quad \boxed{\inf_D f = -\sqrt{2}}.$$

Neither value is attained, so f has no absolute maximum or minimum on D .

Step 3: The theorem The function is continuous and the disk is bounded, but D is not closed: its boundary circle is omitted. Hence the domain is not compact, and the Extreme Value Theorem does not guarantee attained extrema. The missing equality points are precisely

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$

Question 7

On the unbounded domain $\mathbb{R}^2$, consider

$$f(x, y) = x^2 + y^2 + \frac{1}{1 + x^2 + y^2}.$$

Tasks

1. Determine all absolute minima and their value.
2. Decide whether an absolute maximum exists.
3. Explain why this example shows that compactness is sufficient, but not necessary, for an absolute minimum to exist.

Question 7 – Solution

Strategy. Reduce the function to the radial variable $s = x^2 + y^2 \geq 0$ and study a one-variable function on an unbounded interval.

Step 1: Radial reduction Let

$$g(s) = s + \frac{1}{1+s}, \quad s \geq 0.$$

Then

$$g'(s) = 1 - \frac{1}{(1+s)^2}.$$

At $s = 0$, $g'(0) = 0$, and for every $s > 0$, $(1+s)^2 > 1$, so $g'(s) > 0$. Therefore g increases strictly away from $s = 0$.

It follows that

$$\boxed{f_{\min} = 1 \text{ at } (0,0)}.$$

Step 2: No maximum As $x^2 + y^2 = s \rightarrow \infty$,

$$g(s) = s + \frac{1}{1+s} \rightarrow \infty.$$

Thus f is unbounded above and has $\boxed{\text{no absolute maximum}}$.

Step 3: Interpretation The domain $\mathbb{R}^2$ is closed but not bounded, so the Extreme Value Theorem does not apply. Nevertheless, direct analysis proves that an absolute minimum is attained. Compactness guarantees both extrema for every continuous function, but a particular continuous function may still attain one or both extrema on a noncompact domain.

Question 8

For a real parameter a , define

$$f_a(x, y) = (x - a)^2 + y^2$$

on the square $S = [-1, 1] \times [-1, 1]$.

Tasks

1. Determine the absolute minimum value and all minimizing points for every a .
2. Determine the absolute maximum value and all maximizing points for every a .
3. Treat the tie case $a = 0$ explicitly and interpret the answer as squared distance.

Question 8 – Solution

Strategy. The function is squared distance from (x, y) to $(a, 0)$; minimize and maximize the two nonnegative coordinate contributions independently.

Step 1: Minimum The y^2 term is minimized uniquely at $y = 0$. The closest $x \in [-1, 1]$ to a is the clipped value

$$x_* = \begin{cases} -1, & a < -1, \\ a, & -1 \leq a \leq 1, \\ 1, & a > 1. \end{cases}$$

Hence the unique minimizing point is $(x_*, 0)$, and

$$f_{\min} = (\max\{|a| - 1, 0\})^2.$$

Step 2: Maximum The largest possible y^2 is 1, attained at $y = \pm 1$. The endpoint of $[-1, 1]$ farthest from a is at distance $|a| + 1$. Therefore

$$f_{\max} = (|a| + 1)^2 + 1.$$

If $a > 0$, the maximizers are $(-1, \pm 1)$; if $a < 0$, they are $(1, \pm 1)$.

Step 3: Tie case When $a = 0$, both x -endpoints are equally far away, so all four corners $(\pm 1, \pm 1)$ maximize f_0 with value 2. The point $(0, 0)$ uniquely minimizes it with value 0. This agrees exactly with the squared-distance interpretation.

Question 9

Let R be the closed region between

$$y = x^2 \quad \text{and} \quad y = 2 - x.$$

Find the absolute extrema of

$$f(x, y) = x^2 + y$$

on R .

Tasks

1. Find the boundary intersection points and describe the compact region.
2. Check the interior and each curved boundary piece.
3. Compare all candidates and verify the result from the geometry of the region.

Question 9 – Solution

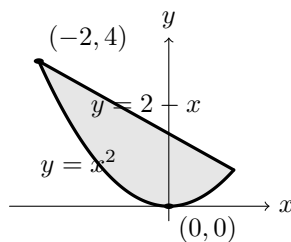
Strategy. Determine the finite interval where the line lies above the parabola, then reduce each boundary to one variable.

Step 1: Region The boundaries meet when

$$x^2 = 2 - x,$$

so $(x + 2)(x - 1) = 0$. The intersection points are $(-2, 4)$ and $(1, 1)$, and

$$-2 \leq x \leq 1, \quad x^2 \leq y \leq 2 - x.$$



Step 2: Candidate search Since $f_y = 1$, there are no interior critical points. On the lower boundary $y = x^2$,

$$f = 2x^2,$$

whose minimum is 0 at $x = 0$ and whose largest endpoint value is 8 at $x = -2$. On the upper boundary $y = 2 - x$,

$$f = x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + \frac{7}{4}.$$

Its boundary minimum is $7/4$ at $x = 1/2$; its endpoint values are 8 and 2.

Step 3: Comparison Therefore

$$\boxed{f_{\min} = 0 \text{ at } (0, 0)}, \quad \boxed{f_{\max} = 8 \text{ at } (-2, 4)}.$$

Because $f = x^2 + y$ increases with y , it is also geometrically consistent that vertical comparisons reduce to the lower curve for the minimum and upper curve for the maximum.

Question 10

Find the absolute extrema of the nondifferentiable function

$$f(x, y) = |x| + y^2$$

on the closed unit disk $x^2 + y^2 \leq 1$.

Tasks

1. Analyze the interior, including the crease $x = 0$ where f_x does not exist.
2. Reduce the circular boundary problem to a single variable $u = |x|$.
3. Find and verify all absolute minimizing and maximizing points.

Question 10 – Solution

Strategy. A complete candidate search must include nondifferentiable points. On the circle, replace y^2 by $1 - x^2$.

Step 1: Interior and crease Where $x \neq 0$,

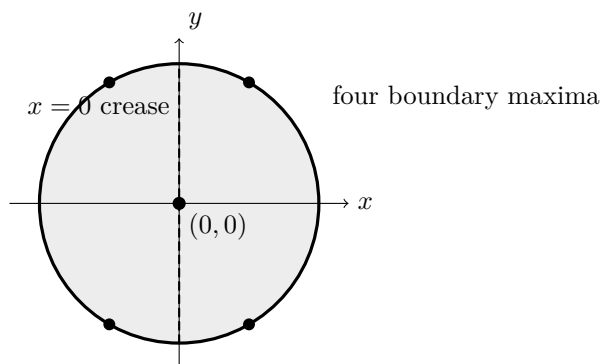
$$f_x = \operatorname{sgn}(x), \quad f_y = 2y,$$

so there are no differentiable interior critical points. On the crease $x = 0$,

$$f(0, y) = y^2.$$

Its smallest value is 0 at the origin. Since $|x| \geq 0$ and $y^2 \geq 0$ everywhere,

$$\boxed{f_{\min} = 0 \text{ at } (0, 0)}.$$



Step 2: Circular boundary Let $u = |x| \in [0, 1]$. On $x^2 + y^2 = 1$,

$$f = u + 1 - u^2 = \frac{5}{4} - \left(u - \frac{1}{2}\right)^2.$$

Thus the largest boundary value is $5/4$, attained when $|x| = 1/2$. Then

$$y^2 = 1 - \frac{1}{4} = \frac{3}{4},$$

so $y = \pm\sqrt{3}/2$.

Step 3: Final comparison For a fixed x , increasing y^2 increases f , so an absolute maximum must lie on the circle; the boundary calculation is therefore decisive. Hence

$$\boxed{f_{\max} = \frac{5}{4} \text{ at } \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)},$$

where the two signs are independent, giving four points. Direct substitution returns $1/2 + 3/4 = 5/4$ at each.

Question 1

Find the constrained extrema of

$$f(x, y) = x + y$$

subject to

$$x^2 + y^2 = 1.$$

Tasks

1. Write and solve the Lagrange multiplier equations.
2. Determine the maximum and minimum values and their locations.
3. Interpret the multiplier condition as tangency of level curves.

Question 1 – Solution

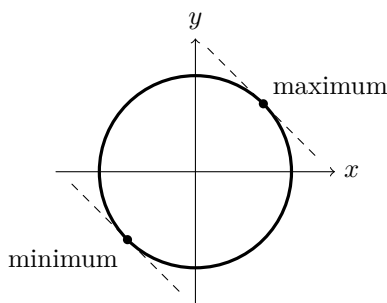
Strategy. Solve $\nabla f = \lambda \nabla g$ together with the circular constraint, then compare the finite candidate set.

Step 1: Lagrange system Let $g = x^2 + y^2$. Then

$$\langle 1, 1 \rangle = \lambda \langle 2x, 2y \rangle, \quad x^2 + y^2 = 1.$$

The first two equations imply $x = y$. Hence $2x^2 = 1$, so

$$(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad \text{or} \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$



Step 2: Values Direct evaluation gives

$$f_{\max} = \sqrt{2} \text{ at } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),$$

$$f_{\min} = -\sqrt{2} \text{ at } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$

Step 3: Geometry and verification The level curves $x + y = c$ are parallel lines with normal ∇f . At an extreme one such line is tangent to the circle, so its normal is parallel to the radial normal ∇g . The circle is compact and the two candidates exhaust the multiplier system, completing the verification.

Question 2

Find the maximum and minimum of

$$f(x, y) = xy$$

subject to the ellipse

$$x^2 + 4y^2 = 4.$$

Tasks

1. Solve the multiplier equations, treating possible zero coordinates carefully.
2. List all constrained extrema and values.
3. Verify the result with a substitution that converts the constraint to a circle.

Question 2 – Solution

Strategy. Solve the component equations without dividing until zero-coordinate cases have been checked.

Step 1: Multiplier equations With $g = x^2 + 4y^2$,

$$\langle y, x \rangle = \lambda \langle 2x, 8y \rangle, \quad x^2 + 4y^2 = 4.$$

If $x = 0$, the first multiplier equation forces $y = 0$, violating the constraint; similarly $y = 0$ is impossible. Thus $xy \neq 0$, and

$$y = 2\lambda x, \quad x = 8\lambda y.$$

Consequently $16\lambda^2 = 1$ and $y = \pm x/2$. The constraint then gives $2x^2 = 4$, so $x = \pm\sqrt{2}$.

Step 2: Candidates and values Equal signs for x and y give $xy = 1$; opposite signs give $xy = -1$. Therefore

$$f_{\max} = 1 \text{ at } (\sqrt{2}, \sqrt{2}/2), (-\sqrt{2}, -\sqrt{2}/2),$$

$$f_{\min} = -1 \text{ at } (\sqrt{2}, -\sqrt{2}/2), (-\sqrt{2}, \sqrt{2}/2).$$

Step 3: Verification Let $u = x$ and $v = 2y$. Then $u^2 + v^2 = 4$ and $f = uv/2$. Since

$$|uv| \leq \frac{u^2 + v^2}{2} = 2,$$

we have $|f| \leq 1$, with equality precisely when $|u| = |v|$. This reproduces all four points.

Question 3

Find the points on the parabola

$$y = x^2$$

closest to $A = (0, 2)$.

Tasks

1. Formulate the squared-distance objective and the constraint.
2. Solve the Lagrange system, including all branches.
3. Compare candidates and justify that the reported minimum is absolute despite the noncompact constraint.

Question 3 – Solution

Strategy. Minimize squared distance, which has the same minimizers as distance but simpler derivatives.

Step 1: Set up Let

$$f(x, y) = x^2 + (y - 2)^2, \quad g(x, y) = y - x^2.$$

The system $\nabla f = \lambda \nabla g$ is

$$2x = -2\lambda x, \quad 2(y - 2) = \lambda, \quad y = x^2.$$

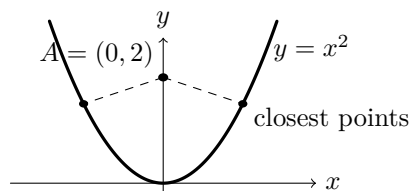
Step 2: Solve branches If $x = 0$, then $y = 0$, producing the candidate $(0, 0)$ with squared distance 4.

If $x \neq 0$, the first equation gives $\lambda = -1$. The second gives $2(y - 2) = -1$, so $y = 3/2$. Hence

$$x = \pm \sqrt{\frac{3}{2}}.$$

At either point the squared distance is

$$\frac{3}{2} + \left(-\frac{1}{2}\right)^2 = \boxed{\frac{7}{4}}.$$



Step 3: Conclusion Comparing candidates gives the two closest points

$$\boxed{\left(\pm \sqrt{\frac{3}{2}}, \frac{3}{2}\right)},$$

each at distance $\boxed{\sqrt{7}/2}$ from A . Along the parabola, $f(x, x^2) = x^2 + (x^2 - 2)^2 \rightarrow \infty$ as $|x| \rightarrow \infty$. Therefore no minimizing sequence can escape to infinity, and the smaller candidate value is the absolute minimum.

Question 4

A rectangular box has positive side lengths x, y, z and fixed sum of its three side lengths

$$x + y + z = 12.$$

Find the dimensions giving the largest volume $V = xyz$.

Tasks

1. Solve the three-variable Lagrange multiplier system.
2. Find the maximum volume.
3. Verify global maximality by checking the boundary of the closed simplex and using an inequality.

Question 4 – Solution

Strategy. The multiplier equations force equal positive side lengths; then compare with the zero-volume boundary and verify by AM–GM.

Step 1: Lagrange system For $g = x + y + z$,

$$\nabla V = \langle yz, xz, xy \rangle = \lambda \langle 1, 1, 1 \rangle.$$

Because $x, y, z > 0$, equality of the three components gives

$$yz = xz = xy,$$

and division by positive factors yields $x = y = z$. The constraint then gives

$$\boxed{x = y = z = 4}.$$

Step 2: Volume

$$\boxed{V_{\max} = 4 \cdot 4 \cdot 4 = 64}.$$

Step 3: Global verification Allowing $x, y, z \geq 0$ closes the feasible simplex. Its boundary has at least one zero side, so $V = 0$ there. In the positive interior, AM–GM gives

$$\frac{x + y + z}{3} \geq (xyz)^{1/3}.$$

Since the left side is 4,

$$xyz \leq 64,$$

with equality only when $x = y = z$. Thus the Lagrange candidate is the unique positive absolute maximizer.

Question 5

The sphere

$$x^2 + y^2 + z^2 = 9$$

meets the plane

$$x + y + z = 3.$$

Find the highest and lowest points of their intersection curve.

Tasks

1. Use two Lagrange multipliers to extremize the height $f(x, y, z) = z$.
2. Solve the complete system and identify both points.
3. Verify the result geometrically or by direct substitution.

Question 5 – Solution

Strategy. At a constrained extreme with two regular constraints, ∇f lies in the span of both constraint gradients.

Step 1: Two-multiplier system Let

$$g = x^2 + y^2 + z^2, \quad h = x + y + z.$$

Then

$$\langle 0, 0, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle + \mu \langle 1, 1, 1 \rangle.$$

The first two component equations imply $2\lambda(x - y) = 0$. If $\lambda = 0$, then the first equation gives $\mu = 0$, contradicting the third. Hence $x = y = a$.

Step 2: Enforce both constraints The plane gives $z = 3 - 2a$. Substitution into the sphere gives

$$2a^2 + (3 - 2a)^2 = 9,$$

or

$$6a(a - 2) = 0.$$

Thus $a = 0$, $z = 3$, or $a = 2$, $z = -1$. Therefore

$$\boxed{\text{highest point } (0, 0, 3)}, \quad \boxed{\text{lowest point } (2, 2, -1)}.$$

Step 3: Verification Both points satisfy the sphere and plane equations. Their heights are 3 and -1 . The intersection is a closed circle, hence compact; the two candidates exhaust the multiplier equations, so these are the absolute height extrema.

Question 6

Consider maximizing and minimizing

$$f(x, y) = x$$

subject to the constraint

$$g(x, y) = x^2 + y^2 = 0.$$

Tasks

1. Determine the feasible set and its constrained extrema directly.
2. Show that no multiplier λ satisfies $\nabla f = \lambda \nabla g$ at the feasible point.
3. Explain which constraint-qualification hypothesis fails and why this does not contradict the method.

Question 6 – Solution

Strategy. Inspect the feasible set before applying the multiplier equation; the constraint has a singular gradient at its only point.

Step 1: Direct analysis The equation $x^2 + y^2 = 0$ over the reals forces

$$x = 0, \quad y = 0.$$

Thus the feasible set is the singleton $\{(0, 0)\}$. Its only function value is 0, so

$$\boxed{f_{\min} = f_{\max} = 0 \text{ at } (0, 0)}.$$

Step 2: Multiplier failure

$$\nabla f(0, 0) = \langle 1, 0 \rangle, \quad \nabla g(0, 0) = \langle 0, 0 \rangle.$$

The equation

$$\langle 1, 0 \rangle = \lambda \langle 0, 0 \rangle$$

has no solution for any real λ .

Step 3: Interpretation The standard Lagrange multiplier theorem assumes $\nabla g \neq \mathbf{0}$ at the constrained extremum. That regularity condition fails here, so the theorem makes no promise. The example shows that singular feasible points must be checked separately; it does not invalidate the multiplier method at regular constraint points.

Question 7

Find the constrained extrema of

$$f(x, y) = x^2 + y^2$$

subject to

$$x^2 + xy + y^2 = 3.$$

Tasks

1. Solve the Lagrange equations without assuming x or y is nonzero.
2. Determine all maximizing and minimizing points and values.
3. Verify the result by rewriting the constraint in rotated directions $y = x$ and $y = -x$.

Question 7 – Solution

Strategy. Eliminate the multiplier by combining the two component equations; the resulting factorization identifies the principal directions.

Step 1: Multiplier equations

$$\langle 2x, 2y \rangle = \lambda \langle 2x + y, x + 2y \rangle, \quad x^2 + xy + y^2 = 3.$$

Multiply the first component equation by y , the second by x , and subtract. The left sides cancel, leaving

$$\lambda(y^2 - x^2) = 0.$$

The constraint rules out $(0, 0)$, and $\lambda = 0$ would force $x = y = 0$. Hence

$$y = \pm x.$$

Step 2: Candidate values If $y = x$, the constraint is $3x^2 = 3$, so $(x, y) = (1, 1)$ or $(-1, -1)$ and $f = 2$.

If $y = -x$, the constraint is $x^2 = 3$, giving $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ with $f = 6$. Therefore

$$f_{\min} = 2 \text{ at } (1, 1), (-1, -1),$$

$$f_{\max} = 6 \text{ at } (\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}).$$

Step 3: Verification The constraint ellipse is compact. Along its $y = x$ principal direction its radius squared is 2, while along $y = -x$ its radius squared is 6. These are exactly the values of $x^2 + y^2$, confirming the classifications and completeness.

Question 8

For the family

$$f_a(x, y) = x^2 + ay^2,$$

the point $P = (1, 2)$ is a constrained extremum on the line $x + y = 3$.

Tasks

1. Determine the parameter a using the Lagrange condition.
2. Classify the constrained extremum.
3. Verify the classification by substituting the constraint into the resulting function.

Question 8 – Solution

Strategy. Parallel gradients at the specified regular point determine the parameter; a one-variable restriction then establishes the classification.

Step 1: Recover a For $g = x + y$,

$$\nabla f_a(1, 2) = \langle 2, 4a \rangle, \quad \nabla g = \langle 1, 1 \rangle.$$

The equation $\langle 2, 4a \rangle = \lambda \langle 1, 1 \rangle$ gives $\lambda = 2$ and $4a = 2$. Hence

$$\boxed{a = \frac{1}{2}}.$$

Step 2: Restrict to the line Set $y = 3 - x$. Then

$$f_{1/2}(x, 3 - x) = x^2 + \frac{1}{2}(3 - x)^2.$$

Completing the square,

$$x^2 + \frac{1}{2}(3 - x)^2 = \frac{3}{2}(x - 1)^2 + 3.$$

Therefore the constrained value is minimized uniquely at $x = 1$, $y = 2$.

Step 3: Result and verification

$$\boxed{P = (1, 2) \text{ is a strict constrained minimum}}, \quad \boxed{f_{1/2}(1, 2) = 3}.$$

The restricted quadratic tends to infinity as $|x| \rightarrow \infty$, so the minimum is also absolute along the entire constraint line; no constrained maximum exists.

Question 9

A firm must choose positive inputs x, y satisfying

$$xy = 100.$$

Its cost is

$$C(x, y) = 3x + 2y.$$

Tasks

1. Use Lagrange multipliers to find the cost-minimizing inputs.
2. Compute the minimum cost and interpret the multiplier balance.
3. Verify global minimality by substitution or an inequality.

Question 9 – Solution

Strategy. The multiplier equations balance marginal costs against the constraint normal; then a one-variable check rules out other behavior.

Step 1: Multiplier equations With $g = xy$,

$$\langle 3, 2 \rangle = \lambda \langle y, x \rangle, \quad xy = 100.$$

Thus $3 = \lambda y$ and $2 = \lambda x$. Eliminating λ gives

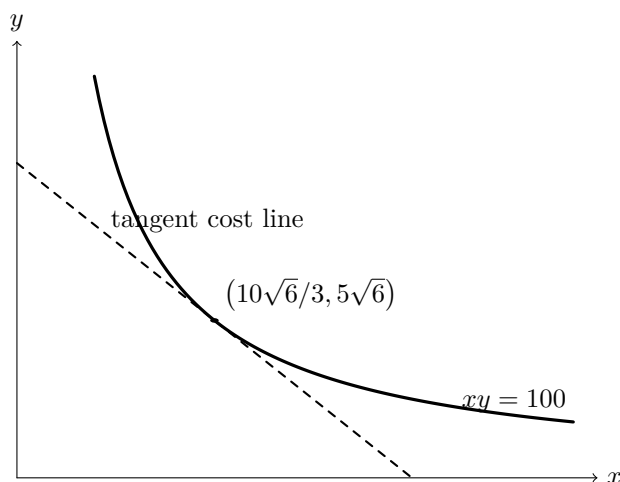
$$3x = 2y, \quad y = \frac{3}{2}x.$$

The positive constraint then yields

$$x^2 = \frac{200}{3},$$

so

$$x = \frac{10\sqrt{6}}{3}, \quad y = 5\sqrt{6}.$$



Step 2: Cost and interpretation

$$C_{\min} = 3 \left(\frac{10\sqrt{6}}{3} \right) + 2(5\sqrt{6}) = 20\sqrt{6}.$$

The relation $3/y = 2/x = \lambda$ says the cost-gradient components are proportional to the production-constraint gradient components.

Step 3: Global verification Using $y = 100/x$ for $x > 0$,

$$C(x) = 3x + \frac{200}{x} \longrightarrow \infty$$

as $x \rightarrow 0^+$ or $x \rightarrow \infty$. Its only critical point is the one found above, and $C''(x) = 400/x^3 > 0$. Hence it is the unique absolute constrained minimum.

Question 10

Find the absolute extrema of

$$f(x, y, z) = xyz$$

on the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6.$$

Tasks

1. Solve the Lagrange equations for candidates with nonzero product.
2. Treat zero-coordinate candidates and compare all possible values.
3. List every maximizing and minimizing point and verify the sign pattern.

Question 10 – Solution

Strategy. Multiplying each component equation by its corresponding coordinate reveals fixed ratios among x^2, y^2, z^2 .

Step 1: Nonzero-product candidates The multiplier system is

$$yz = 2\lambda x, \quad xz = 4\lambda y, \quad xy = 6\lambda z.$$

When $xyz \neq 0$, multiply these equations by x, y, z , respectively:

$$xyz = 2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2.$$

Thus

$$2x^2 = 4y^2 = 6z^2 = K.$$

Substitution into the constraint gives

$$\frac{K}{2} + 2\frac{K}{4} + 3\frac{K}{6} = \frac{3K}{2} = 6,$$

so $K = 4$. Therefore

$$x^2 = 2, \quad y^2 = 1, \quad z^2 = \frac{2}{3}.$$

Step 2: Values and zero cases At these eight sign choices,

$$|xyz| = \sqrt{2} \cdot 1 \cdot \sqrt{\frac{2}{3}} = \frac{2}{\sqrt{3}}.$$

Any feasible candidate with a zero coordinate has $f = 0$, which lies strictly between the positive and negative nonzero values.

Step 3: All extrema The ellipsoid is compact, so comparison is decisive:

$$f_{\max} = \frac{2}{\sqrt{3}}$$

at the four points $(\pm\sqrt{2}, \pm 1, \pm\sqrt{2/3})$ having an even number of minus signs, and

$$f_{\min} = -\frac{2}{\sqrt{3}}$$

at the four such points having an odd number of minus signs. The parity rule follows because the product is positive for an even number of negative factors and negative for an odd number.