

Question 1

Evaluate

$$\lim_{(x,y) \rightarrow (2,-1)} (x^2y + 3xy^2 - 2y + 5).$$

Tasks

1. State why direct substitution is valid.
2. Evaluate the limit exactly.
3. Verify the arithmetic term by term.

Question 1 – Solution

Strategy. A polynomial in two variables is continuous everywhere, so its limit equals its value at the target point.

Step 1: Continuity Each monomial and constant is continuous, and finite sums of continuous functions are continuous. Direct substitution is therefore valid.

Step 2: Substitute

$$\begin{aligned}x^2y + 3xy^2 - 2y + 5 &= (2)^2(-1) + 3(2)(-1)^2 - 2(-1) + 5 \\&= -4 + 6 + 2 + 5 = 9.\end{aligned}$$

Thus

$$\boxed{\lim_{(x,y) \rightarrow (2,-1)} (x^2y + 3xy^2 - 2y + 5) = 9}.$$

Verification. The four contributions are -4 , 6 , 2 , and 5 ; their sum is 9 . No denominator or restricted operation can fail at $(2, -1)$.

Question 2

Evaluate

$$\lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + xy - 3x - y + 2}{x - 1}.$$

Tasks

1. Explain why direct substitution is inconclusive.
2. Factor and simplify on the punctured domain.
3. Evaluate the limit and distinguish it from the function value.

Question 2 – Solution

Strategy. Factor the numerator to expose a common factor with the vanishing denominator.

Step 1: Indeterminate form At $(1, 2)$, numerator and denominator are both 0, so substitution gives $0/0$, not a limit value.

Step 2: Factor Treat the numerator as a polynomial in y :

$$x^2 - 3x + 2 + y(x - 1) = (x - 1)(x - 2) + y(x - 1) = (x - 1)(x + y - 2).$$

Whenever $x \neq 1$,

$$\frac{(x - 1)(x + y - 2)}{x - 1} = x + y - 2.$$

Step 3: Limit The simplified function is continuous, so

$$\lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + xy - 3x - y + 2}{x - 1} = 1.$$

The original quotient is undefined on the entire line $x = 1$, including the target, but points of its domain approach $(1, 2)$ and the missing line does not change the limit.

Question 3

Determine whether

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$$

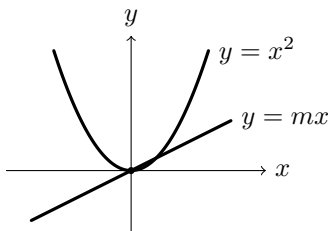
exists.

Tasks

1. Test every straight-line path $y = mx$.
2. Test a curved path suggested by the denominator.
3. Explain why the first test alone cannot prove existence.

Question 3 – Solution

Strategy. Compare straight lines with the balancing path $y = kx^2$, where x^4 and y^2 have the same order.



Step 1: Lines Along $y = mx$,

$$\frac{x^2(mx)}{x^4 + m^2x^2} = \frac{mx}{x^2 + m^2} \rightarrow 0$$

for $m \neq 0$; the path $y = 0$ also gives 0.

Step 2: Parabolas Along $y = kx^2$,

$$\frac{x^2(kx^2)}{x^4 + k^2x^4} = \frac{k}{1 + k^2}.$$

For example, $k = 1$ gives $1/2$, whereas $k = 0$ gives 0.

Conclusion. Since two paths produce different values,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4 + y^2} \text{ does not exist.}$$

Agreement along all lines is necessary but not sufficient because nonlinearly approaching paths remain possible.

Question 4

Evaluate

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2}.$$

Tasks

1. Establish a useful bound without selecting paths.
2. Apply the squeeze theorem.
3. Give an independent polar-coordinate check.

Question 4 – Solution

Strategy. Bound the numerator using $x^2 \leq x^2 + y^2$ and then compare with a quantity tending to zero.

Step 1: Bound For $(x, y) \neq (0, 0)$,

$$0 \leq \frac{x^2 y^2}{x^2 + y^2} \leq \frac{(x^2 + y^2) y^2}{x^2 + y^2} = y^2.$$

As $(x, y) \rightarrow (0, 0)$, $y^2 \rightarrow 0$.

Step 2: Squeeze Both bounding functions tend to 0, hence

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2} = 0}.$$

Step 3: Polar check With $x = r \cos \theta$ and $y = r \sin \theta$,

$$\frac{x^2 y^2}{x^2 + y^2} = r^2 \cos^2 \theta \sin^2 \theta.$$

Because $0 \leq \cos^2 \theta \sin^2 \theta \leq 1/4$, its absolute value is at most $r^2/4 \rightarrow 0$, uniformly in direction.

Question 5

For

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq (0, 0),$$

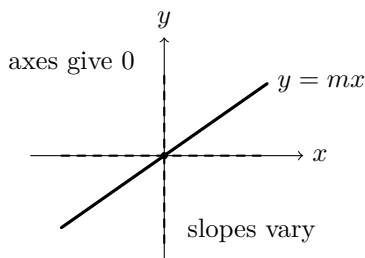
compare its two iterated limits with its joint limit at the origin.

Tasks

1. Compute both iterated limits.
2. Test the joint limit along $y = mx$.
3. Explain why the conclusions differ.

Question 5 – Solution

Strategy. Iterated limits approach along coordinate-aligned stages; the joint limit must agree along every path.



Step 1: Iterated limits For fixed $y \neq 0$, $\lim_{x \rightarrow 0} xy/(x^2 + y^2) = 0$, so

$$\lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) = 0.$$

Similarly, for fixed $x \neq 0$, the inner y -limit is 0, giving

$$\lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = 0.$$

Step 2: Joint paths Along $y = mx$,

$$f(x, mx) = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}.$$

This is 0 for $m = 0$ but $1/2$ for $m = 1$.

Conclusion. Both iterated limits equal $\boxed{0}$, yet the joint limit $\boxed{\text{does not exist}}$. Iterated limits inspect only two staged approaches and cannot control arbitrary simultaneous approaches.

Question 6

Let

$$D = \{(x, y) : x \geq 0, 0 \leq y \leq \sqrt{3}x\}, \quad f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

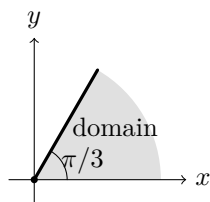
Find the limit of f as $(x, y) \rightarrow (0, 0)$ within D , or prove it does not exist.

Tasks

1. Describe the allowed directions.
2. Compare two valid boundary paths.
3. State the relative-limit conclusion.

Question 6 – Solution

Strategy. A relative limit uses only paths in the domain, but all such paths must still agree.



Step 1: Directions The wedge is $0 \leq \theta \leq \pi/3$. In polar form,

$$f = \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2} = \cos 2\theta.$$

Step 2: Boundary paths Along $y = 0$ ($\theta = 0$), $f = 1$. Along $y = \sqrt{3}x$ ($\theta = \pi/3$),

$$f = \cos(2\pi/3) = -\frac{1}{2}.$$

Both paths stay in D and approach the origin.

Conclusion. Because the two permitted approaches disagree,

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ (x,y) \in D}} f(x,y) \text{ does not exist.}$$

Question 7

Evaluate

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right).$$

Tasks

1. Explain why the oscillatory factor has no limit.
2. Bound the full expression.
3. Prove the limit despite the oscillation.

Question 7 – Solution

Strategy. The sine factor is bounded even though it oscillates; the radial factor forces the product to zero.

Step 1: Absolute bound Since $|\sin u| \leq 1$,

$$\left| (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) \right| \leq x^2 + y^2.$$

Step 2: Squeeze As $(x, y) \rightarrow (0, 0)$, the right side tends to 0. Therefore

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) = 0.$$

Step 3: Why oscillation is harmless Along radii for which $1/(x^2 + y^2)$ hits alternating sine peaks, the sine values do not settle. Nevertheless, the product's magnitude never exceeds the shrinking radius squared, so every path is controlled by the same bound.

Question 8

Prove directly from the ε - δ definition that

$$\lim_{(x,y) \rightarrow (1,-2)} (3x - 4y) = 11.$$

Tasks

1. Rewrite the output error.
2. Bound it by the Euclidean input distance.
3. Give an explicit valid choice of δ .

Question 8 – Solution

Strategy. Use Cauchy–Schwarz to compare the linear output change with the distance to $(1, -2)$.

Step 1: Error Let $h = x - 1$ and $k = y + 2$. Then

$$|(3x - 4y) - 11| = |3h - 4k|.$$

Step 2: Cauchy–Schwarz

$$|3h - 4k| \leq \sqrt{3^2 + (-4)^2} \sqrt{h^2 + k^2} = 5\sqrt{(x - 1)^2 + (y + 2)^2}.$$

Step 3: Choose δ Given $\varepsilon > 0$, set

$$\boxed{\delta = \varepsilon/5}.$$

If $0 < \sqrt{(x - 1)^2 + (y + 2)^2} < \delta$, then

$$|(3x - 4y) - 11| < 5\delta = \varepsilon.$$

This is exactly the two-variable limit definition, so the claimed limit is proved.

Question 9

Suppose (a, b) lies strictly inside the disk $x^2 + y^2 < 25$. Prove

$$\lim_{(x,y) \rightarrow (a,b)} \sqrt{25 - x^2 - y^2} = \sqrt{25 - a^2 - b^2},$$

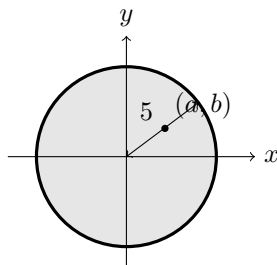
and give a concrete neighborhood that remains inside the domain.

Tasks

1. Find a positive safety margin to the boundary.
2. Choose a disk around (a, b) contained in the domain.
3. Justify the limit by composition.

Question 9 – Solution

Strategy. Use the triangle inequality to keep nearby points inside the radius-5 disk, then apply continuity.



Step 1: Margin Let $s = \sqrt{a^2 + b^2} < 5$. The radial margin is $5 - s > 0$.

Step 2: Neighborhood Choose

$$\delta_0 = \frac{5 - s}{2}.$$

If $\sqrt{(x - a)^2 + (y - b)^2} < \delta_0$, then by the triangle inequality

$$\sqrt{x^2 + y^2} \leq s + \sqrt{(x - a)^2 + (y - b)^2} < s + \frac{5 - s}{2} < 5.$$

Thus this entire neighborhood lies in the square-root domain.

Step 3: Composition The polynomial $25 - x^2 - y^2$ is continuous and remains positive there; $\sqrt{\cdot}$ is continuous on positive inputs. Their composition is continuous at (a, b) , proving the boxed limit stated in the question.

Question 10

Define

$$F(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ c, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Determine $\lim_{(x,y) \rightarrow (0,0)} F(x, y)$ using polar coordinates.
2. Find the unique value of c making F continuous at the origin.
3. Verify the estimate uniformly in angle.

Question 10 – Solution

Strategy. Recognize the cubic numerator's polar form and isolate one factor of radial distance.

Step 1: Convert With $x = r \cos \theta$ and $y = r \sin \theta$,

$$x^3 - 3xy^2 = r^3(\cos^3 \theta - 3 \cos \theta \sin^2 \theta) = r^3 \cos 3\theta.$$

Since $x^2 + y^2 = r^2$, for $r > 0$,

$$F(x, y) = r \cos 3\theta.$$

Step 2: Uniform estimate For every angle,

$$|F(x, y)| = r |\cos 3\theta| \leq r = \sqrt{x^2 + y^2} \longrightarrow 0.$$

Hence

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} F(x, y) = 0}.$$

Step 3: Continuity value Continuity requires the assigned value to equal the limit, so the unique choice is

$$\boxed{c = 0}.$$

The bound is independent of θ , verifying that no directional or curved approach can change the result.

Question 1

Let

$$f(x, y) = 4x^3y^2 - 5x^2y + 7y^3 - 6.$$

Tasks

1. Find f_x and f_y .
2. Evaluate both at $(1, -2)$.
3. Verify that each differentiation treats the other variable as constant.

Question 1 – Solution

Strategy. Differentiate term by term, holding the non-differentiated variable fixed.

Step 1: Differentiate with respect to x

$$f_x = 12x^2y^2 - 10xy.$$

The terms $7y^3$ and -6 are constant with respect to x .

Step 2: Differentiate with respect to y

$$f_y = 8x^3y - 5x^2 + 21y^2.$$

Step 3: Evaluate

$$f_x(1, -2) = 12(1)(4) - 10(1)(-2) = \boxed{68},$$

$$f_y(1, -2) = 8(1)(-2) - 5(1) + 21(4) = \boxed{63}.$$

Verification. The x -derivative lowers only powers of x ; the y -derivative lowers only powers of y . Substitution gives $48 + 20 = 68$ and $-16 - 5 + 84 = 63$.

Question 2

For

$$g(x, y) = \frac{x^2 + 3y}{2x - y},$$

find its first partial derivatives on its domain.

Tasks

1. State the domain restriction.
2. Compute g_x and g_y using the quotient rule.
3. Simplify and check the denominators.

Question 2 – Solution

Strategy. Apply the one-variable quotient rule separately in each variable and preserve the original domain.

Step 1: Domain The function and its partial derivatives are defined where

$$\boxed{2x - y \neq 0}.$$

Step 2: x -partial Holding y fixed,

$$\begin{aligned} g_x &= \frac{(2x)(2x - y) - 2(x^2 + 3y)}{(2x - y)^2} \\ &= \boxed{\frac{2x^2 - 2xy - 6y}{(2x - y)^2}}. \end{aligned}$$

Step 3: y -partial Holding x fixed,

$$\begin{aligned} g_y &= \frac{3(2x - y) - (x^2 + 3y)(-1)}{(2x - y)^2} \\ &= \boxed{\frac{x^2 + 6x}{(2x - y)^2}}. \end{aligned}$$

Verification. Both results have the squared quotient-rule denominator and remain restricted to $2x - y \neq 0$; expanding each unsimplified numerator reproduces the displayed form.

Question 3

Let

$$h(x, y) = e^{xy} \sin(x + y).$$

Tasks

1. Compute h_x and h_y .
2. Evaluate both at $(0, \pi/2)$.
3. Identify the product-rule contribution that vanishes at the point.

Question 3 – Solution

Strategy. Use the product rule, differentiating each composite factor with only one variable active.

Step 1: x -partial

$$h_x = ye^{xy} \sin(x+y) + e^{xy} \cos(x+y) = e^{xy} [y \sin(x+y) + \cos(x+y)].$$

Step 2: y -partial

$$h_y = xe^{xy} \sin(x+y) + e^{xy} \cos(x+y) = e^{xy} [x \sin(x+y) + \cos(x+y)].$$

Step 3: Evaluate At $(0, \pi/2)$, $e^{xy} = 1$, $\sin(x+y) = 1$, and $\cos(x+y) = 0$. Therefore

$$\boxed{h_x(0, \pi/2) = \pi/2}, \quad \boxed{h_y(0, \pi/2) = 0}.$$

The cosine contribution vanishes in both partials; the $xe^{xy} \sin(x+y)$ contribution in h_y also vanishes because $x = 0$.

Question 4

On the domain $x > 0$, define

$$p(x, y) = x^y.$$

Tasks

1. Find p_x by treating y as constant.
2. Find p_y using an exponential representation.
3. Evaluate both partial derivatives at $(e, 2)$ and verify their forms.

Question 4 – Solution

Strategy. Use the ordinary power rule for x and write $x^y = e^{y \ln x}$ when differentiating with respect to the exponent.

Step 1: x -partial With y fixed,

$$\boxed{p_x(x, y) = yx^{y-1}}.$$

Step 2: y -partial Since $p = e^{y \ln x}$ and $\ln x$ is constant with respect to y ,

$$\boxed{p_y(x, y) = x^y \ln x}.$$

Step 3: Evaluate

$$p_x(e, 2) = 2e, \quad p_y(e, 2) = e^2 \ln e = e^2.$$

Thus

$$\boxed{p_x(e, 2) = 2e, \quad p_y(e, 2) = e^2}.$$

Verification. For fixed $y = 2$, $p = x^2$ has derivative $2x$. For fixed $x = e$, $p = e^y$ has derivative e^y , confirming both values.

Question 5

Let

$$F(x, y, z) = x^2yz + \ln(x + z) - ye^z, \quad x + z > 0.$$

Tasks

1. Compute F_x , F_y , and F_z .
2. Evaluate all three at $(1, 2, 0)$.
3. Check which terms are constant in each differentiation.

Question 5 – Solution

Strategy. Differentiate with one variable active at a time and keep the logarithm's domain restriction.

Step 1: Partial derivatives

$$F_x = 2xyz + \frac{1}{x+z},$$

$$F_y = x^2z - e^z,$$

$$F_z = x^2y + \frac{1}{x+z} - ye^z.$$

Step 2: Evaluate At $(1, 2, 0)$,

$$F_x = 0 + 1 = \boxed{1}, \quad F_y = 0 - 1 = \boxed{-1},$$

$$F_z = 2 + 1 - 2 = \boxed{1}.$$

Verification. In F_x , $-ye^z$ is constant; in F_y , $\ln(x+z)$ is constant; in F_z , every term varies with z . The point satisfies $x+z=1>0$.

Question 6

The equation

$$x^2 + y^2 + z^2 + xyz = 4$$

defines z locally as a function of x and y near $P = (1, 1, 1)$.

Tasks

1. Find z_x and z_y by partial implicit differentiation.
2. Evaluate them at P .
3. State the nonzero factor that permits solving for the derivatives.

Question 6 – Solution

Strategy. Differentiate the equation while treating $z = z(x, y)$, then collect the terms containing the requested derivative.

Step 1: Differentiate in x

$$2x + 2zz_x + yz + xyz_x = 0,$$

so

$$z_x = -\frac{2x + yz}{2z + xy}.$$

Step 2: Differentiate in y

$$2y + 2zz_y + xz + xyz_y = 0,$$

so

$$z_y = -\frac{2y + xz}{2z + xy}.$$

Step 3: Evaluate At $(1, 1, 1)$, the common denominator is $2z + xy = 3 \neq 0$. Hence

$$z_x(1, 1) = -1, \quad z_y(1, 1) = -1.$$

Substitution into the differentiated equations gives $2 + 2(-1) + 1 + (-1) = 0$ in each case.

Question 7

For the surface

$$z = f(x, y) = x^2 + xy + y^2,$$

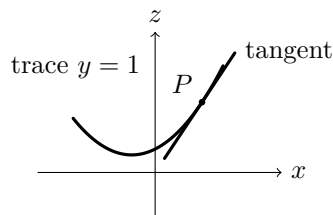
hold $y = 1$ fixed and examine the point $P = (1, 1, 3)$.

Tasks

1. Find $f_x(1, 1)$.
2. Obtain the tangent line to the trace $y = 1$ at P .
3. Verify the slope by differentiating the trace directly.

Question 7 – Solution

Strategy. Restrict the surface to the vertical plane $y = 1$; the resulting one-variable curve has slope $f_x(1, 1)$.



Step 1: Partial derivative

$$f_x = 2x + y, \quad \boxed{f_x(1, 1) = 3}.$$

Step 2: Trace and tangent At $y = 1$,

$$z = x^2 + x + 1.$$

The tangent line in the plane $y = 1$ passes through $(x, z) = (1, 3)$ with slope 3:

$$\boxed{y = 1, \quad z - 3 = 3(x - 1)}.$$

Step 3: Direct check Differentiating the trace gives $dz/dx = 2x + 1$, which equals 3 at $x = 1$, matching the partial derivative.

Question 8

Determine whether constants a , b , and c exist such that

$$q(x, y) = ax^2 + bxy + cy^2$$

satisfies

$$q_x(1, 1) = 7, \quad q_y(1, 1) = 8, \quad q(1, 1) = 6.$$

Tasks

1. Translate the data into a linear system.
2. Solve for a, b, c or prove inconsistency.
3. Verify your conclusion against all three conditions.

Question 8 – Solution

Strategy. Compute the partial derivatives symbolically, evaluate at $(1, 1)$, and solve the resulting three equations.

Step 1: Equations

$$q_x = 2ax + by, \quad q_y = bx + 2cy.$$

Thus

$$2a + b = 7, \quad b + 2c = 8, \quad a + b + c = 6.$$

Step 2: Solve From the first two equations,

$$a = \frac{7-b}{2}, \quad c = \frac{8-b}{2}.$$

Substitution into $a + b + c = 6$ gives

$$\frac{7-b}{2} + b + \frac{8-b}{2} = \frac{15}{2},$$

which cannot equal 6. Therefore the supplied data are inconsistent.

Conclusion.

No constants a, b, c satisfy all three conditions.

Verification. In fact, adding the two derivative equations gives $2(a + b + c) = 15$, forcing $q(1, 1) = 15/2$, not 6.

Question 9

Define

$$u(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $u_x(0, 0)$ from the definition.
2. Compute $u_y(0, 0)$ from the definition.
3. Compute formulas for u_x, u_y away from the origin and explain why they do not determine the values at the origin.

Question 9 – Solution

Strategy. At the exceptional point, use one-variable difference quotients along the coordinate axes.

Step 1: x -partial at the origin

$$u_x(0,0) = \lim_{h \rightarrow 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \boxed{0}.$$

Step 2: y -partial at the origin

$$u_y(0,0) = \lim_{k \rightarrow 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \rightarrow 0} \frac{0}{k} = \boxed{0}.$$

Step 3: Away from the origin Quotient differentiation gives

$$\boxed{u_x = \frac{2xy^3}{(x^2 + y^2)^2}}, \quad \boxed{u_y = \frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2}} \quad ((x,y) \neq (0,0)).$$

These formulas were derived where the denominator is nonzero, so substituting $(0,0)$ would be invalid. The definition supplies the missing partial values.

Question 10

Define

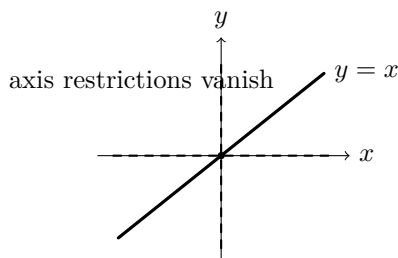
$$v(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Show that both first partial derivatives exist at the origin.
2. Test continuity along $y = x$.
3. Explain what this example proves about partial derivatives and continuity.

Question 10 – Solution

Strategy. Partial derivatives inspect axis restrictions only; compare those restrictions with a diagonal approach.



Step 1: Axis difference quotients Since $v(h, 0) = 0$ and $v(0, k) = 0$,

$$v_x(0, 0) = \lim_{h \rightarrow 0} \frac{v(h, 0) - v(0, 0)}{h} = 0,$$

$$v_y(0, 0) = \lim_{k \rightarrow 0} \frac{v(0, k) - v(0, 0)}{k} = 0.$$

Thus

$$\boxed{v_x(0, 0) = v_y(0, 0) = 0}.$$

Step 2: Diagonal path Along $y = x$ with $x \neq 0$,

$$v(x, x) = \frac{x^2}{2x^2} = \frac{1}{2}.$$

Therefore $\lim_{x \rightarrow 0} v(x, x) = 1/2 \neq v(0, 0) = 0$.

Conclusion. The function is not continuous at the origin even though both first partial derivatives exist there. Hence

$$\boxed{\text{existence of } v_x, v_y \text{ at a point does not imply continuity there.}}$$

Question 1

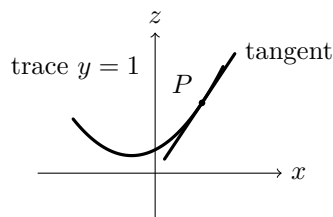
For $f(x, y) = x^2 + 3xy - y^2$ at $P = (1, 2, 3)$:

Tasks

1. Find f_x, f_y at $(1, 2)$.
2. Interpret them as trace slopes.
3. Write both trace tangent lines.

Question 1 – Solution

Strategy. Hold one input fixed at a time.



Step 1: Rates $f_x = 2x + 3y$, $f_y = 3x - 2y$, hence $\boxed{f_x(1, 2) = 8}$ and $\boxed{f_y(1, 2) = -1}$.

Step 2: Meaning With $y = 2$, height rises 8 units per x unit; with $x = 1$, it falls 1 unit per y unit.

Step 3: Lines $\boxed{y = 2, z - 3 = 8(x - 1)}$ and $\boxed{x = 1, z - 3 = -(y - 2)}$.

Question 2

Temperature is $T(x, y) = 80 - 2x^2 - y^2$ degrees Celsius, with distance in centimeters.

Tasks

1. Find $T_x(3, 4), T_y(3, 4)$ with units.
2. Interpret signs and magnitudes.
3. Estimate separate changes for 0.05 cm moves in $+x$ and $+y$.

Question 2 – Solution

Strategy. Read each partial as temperature change per centimeter with the other coordinate fixed.

Step 1: Rates $T_x = -4x$, $T_y = -2y$, so $T_x = -12^\circ\text{C}/\text{cm}$ and $T_y = -8^\circ\text{C}/\text{cm}$ at $(3, 4)$.

Step 2: Meaning Both directions cool; the x trace is locally steeper.

Step 3: Estimates $\Delta T_x \approx -12(.05) = -0.60^\circ\text{C}$ and $\Delta T_y \approx -8(.05) = -0.40^\circ\text{C}$.

Question 3

Revenue is $R(p, a) = 1200p - 30p^2 + 50a - 2a^2 + 4pa$, where p is dollars and a is hundreds of advertising dollars.

Tasks

1. Compute $R_p(10, 5), R_a(10, 5)$.
2. State units.
3. Interpret both values.

Question 3 – Solution

Strategy. Attach output units per input unit.

Step 1: Compute $R_p = 1200 - 60p + 4a$ and $R_a = 50 - 4a + 4p$, so $\boxed{R_p(10, 5) = 620}$ and $\boxed{R_a(10, 5) = 70}$.

Step 2: Units These are revenue dollars per price dollar and revenue dollars per hundred advertising dollars.

Step 3: Meaning Locally, a \$1 price rise predicts \$620 more revenue; an added \$100 advertising predicts \$70 more, with the other input fixed.

Question 4

Pressure is $P(V, T) = 8T/V$ kPa, with V in liters and T in kelvins.

Tasks

1. Find $P_V(4, 300)$, $P_T(4, 300)$.
2. Supply units.
3. Interpret each sign.

Question 4 – Solution

Strategy. Vary one physical input while fixing the other.

Step 1: Volume $P_V = -8T/V^2$, so $P_V(4, 300) = -150 \text{ kPa/L}$.

Step 2: Temperature $P_T = 8/V$, so $P_T(4, 300) = 2 \text{ kPa/K}$.

Step 3: Meaning Added volume lowers pressure locally; added temperature raises it. The statements are instantaneous and hold the other variable fixed.

Question 5

At a terrain point, $f_x = -0.25$ and $f_y = 0.60$, with all distances in meters.

Tasks

1. Interpret both as rise/run.
2. Find signed trace angles.
3. Identify the steeper trace.

Question 5 – Solution

Strategy. A trace slope is the tangent of its signed inclination.

Step 1: Rise/run A one-meter positive x move predicts -0.25 m elevation change; a positive y move predicts 0.60 m, separately.

Step 2: Angles $\alpha_x = \arctan(-.25) \approx -14.0^\circ$ and $\alpha_y = \arctan(.60) \approx 31.0^\circ$.

Step 3: Compare Since $.60 > .25$, the y trace is steeper; its positive sign indicates ascent.

Question 6

Near $(2, 3)$, the columns give x and the rows give y :

	1.9	2.0	2.1
2.9	7.62	7.80	7.98
3.0	7.71	7.90	8.09
3.1	7.80	8.00	8.20

Tasks

1. Estimate $f_x(2, 3)$ centrally.
2. Estimate $f_y(2, 3)$ centrally.
3. Interpret both.

Question 6 – Solution

Strategy. Use the center row and column.

Step 1: x rate $f_x \approx (8.09 - 7.71)/.2 = \boxed{1.90}$.

Step 2: y rate $f_y \approx (8.00 - 7.80)/.2 = \boxed{1.00}$.

Step 3: Meaning Output rises about 1.90 per x unit with y fixed and 1.00 per y unit with x fixed. Symmetric differences reduce one-sided bias.

Question 7

Adjacent contours differ by 6, with labels increasing in both positive coordinate directions. Along $+x$ they are 2 units apart; along $+y$ they are 3 units apart.

Tasks

1. Estimate both partial derivatives.
2. Relate spacing to magnitude.
3. Explain how decreasing labels change signs.

Question 7 – Solution

Strategy. Divide contour-value change by coordinate distance.

Step 1: Estimates $f_x \approx 6/2 = 3$ and $f_y \approx 6/3 = 2$.

Step 2: Spacing Equal output change over less distance gives larger slope, so the x trace is steeper.

Step 3: Sign If labels decrease in a positive direction, the corresponding partial is negative. Spacing controls magnitude; label order controls sign.

Question 8

Cylinder volume is $V(r, h) = \pi r^2 h$.

Tasks

1. Interpret V_r, V_h geometrically.
2. Evaluate at $(3, 10)$ cm.
3. Estimate separate effects of 0.1 cm increases.

Question 8 – Solution

Strategy. Connect derivatives to familiar areas.

Step 1: Geometry $V_r = 2\pi rh$ is circumference times height; $V_h = \pi r^2$ is base area.

Step 2: Values $V_r = 60\pi \text{ cm}^2$, $V_h = 9\pi \text{ cm}^2$.

Step 3: Estimates $\Delta V_r \approx 6\pi \text{ cm}^3$ and $\Delta V_h \approx 0.9\pi \text{ cm}^3$. The radial change has the larger first-order effect.

Question 9

At one point a concentration field has $C_x = 4$, $C_y = 0$, $C_t = -1.5$ in appropriate mg/L per meter or minute units.

Tasks

1. Interpret each.
2. Explain what $C_y = 0$ does not imply.
3. Predict separate changes for .2 m in $+x$, .2 m in $+y$, and 2 minutes.

Question 9 – Solution

Strategy. Preserve held-fixed conditions and units.

Step 1: Meaning Concentration rises 4 mg/L per x meter, has zero instantaneous y slope, and falls 1.5 mg/L per minute.

Step 2: Caution $C_y = 0$ at one point does not mean independence from y nearby.

Step 3: Predictions $\Delta C_x \approx .8 \text{ mg/L}$, $\Delta C_y \approx 0$, and $\Delta C_t \approx -3.0 \text{ mg/L}$.

Question 10

At the same input, models satisfy $A_x = 5$, $A_y = -2$ and $B_x = -1$, $B_y = -2$.

Tasks

1. Compare positive- x responses.
2. Compare positive- y responses.
3. State what cannot be concluded from these data.

Question 10 – Solution

Strategy. Partial derivatives give local coordinate-specific rates.

Step 1: x A locally rises at 5 per unit; B falls at 1 per unit. Their signs oppose and A has greater absolute sensitivity.

Step 2: y Both locally fall at 2 per positive y unit.

Step 3: Limits We cannot infer $A > B$, global monotonicity, or exact finite changes without more information.

The data justify local rate comparisons only.

Question 1

For

$$f(x, y) = x^4y^2 - 3x^2y^3 + 2xy,$$

Tasks

1. Find $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.
2. Evaluate them at $(1, -1)$.
3. Verify the mixed-partial equality algebraically.

Question 1 – Solution

Strategy. Compute both first partials before differentiating again.

Step 1: First partials

$$f_x = 4x^3y^2 - 6xy^3 + 2y, \quad f_y = 2x^4y - 9x^2y^2 + 2x.$$

Step 2: Second partials

$$\boxed{f_{xx} = 12x^2y^2 - 6y^3}, \quad \boxed{f_{xy} = 8x^3y - 18xy^2 + 2},$$
$$\boxed{f_{yx} = 8x^3y - 18xy^2 + 2}, \quad \boxed{f_{yy} = 2x^4 - 18x^2y}.$$

Step 3: Evaluate

$$\boxed{f_{xx}(1, -1) = 18}, \quad \boxed{f_{xy}(1, -1) = f_{yx}(1, -1) = -24}, \quad \boxed{f_{yy}(1, -1) = 20}.$$

The two mixed formulas coincide term by term, as expected for a polynomial.

Question 2

Let $g(x, y) = e^{xy} \cos y$.

Tasks

1. Compute g_{xx} and g_{xy} .
2. Compute g_{yx} independently from g_y .
3. Evaluate them at $(0, \pi)$ and verify agreement.

Question 2 – Solution

Strategy. Keep product-rule terms grouped by their exponential factor.

Step 1: Differentiate from g_x Since $g_x = ye^{xy} \cos y$,

$$\boxed{g_{xx} = y^2 e^{xy} \cos y},$$

$$\boxed{g_{xy} = e^{xy} [(1 + xy) \cos y - y \sin y]}.$$

Step 2: Reverse order

$$g_y = e^{xy} (x \cos y - \sin y),$$

$$g_{yx} = ye^{xy} (x \cos y - \sin y) + e^{xy} \cos y = \boxed{e^{xy} [(1 + xy) \cos y - y \sin y]}.$$

Step 3: Evaluate At $(0, \pi)$, $e^{xy} = 1$, $\cos \pi = -1$, and $\sin \pi = 0$:

$$\boxed{g_{xx} = -\pi^2}, \quad \boxed{g_{xy} = g_{yx} = -1}.$$

Question 3

For $u(x, y) = \ln(x^2 + y^2)$ on its natural domain:

Tasks

1. Find all second partial derivatives.
2. Show that $u_{xx} + u_{yy} = 0$.
3. Explain why the result cannot be extended to the origin by substitution.

Question 3 – Solution

Strategy. Differentiate the rational first partials and preserve the punctured-plane domain.

Step 1: First partials

$$u_x = \frac{2x}{x^2 + y^2}, \quad u_y = \frac{2y}{x^2 + y^2}.$$

Step 2: Second partials

$$\boxed{u_{xx} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}}, \quad \boxed{u_{yy} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}},$$
$$\boxed{u_{xy} = u_{yx} = -\frac{4xy}{(x^2 + y^2)^2}}.$$

Step 3: Cancellation

$$u_{xx} + u_{yy} = \frac{2(y^2 - x^2 + x^2 - y^2)}{(x^2 + y^2)^2} = \boxed{0}.$$

The origin is excluded from u , and every denominator above vanishes there; substitution cannot create a derivative at a point outside the domain.

Question 4

Find the third-order derivative f_{xxy} for

$$f(x, y) = x^2 e^{xy}.$$

Tasks

1. Compute it in the encoded order.
2. Compute f_{yxx} independently.
3. Explain the agreement and evaluate at $(1, 0)$.

Question 4 – Solution

Strategy. Follow each subscript successively and retain all product-rule terms.

Step 1: Route x, x, y

$$\begin{aligned}f_x &= e^{xy}(2x + x^2y), \\f_{xx} &= e^{xy}(2 + 4xy + x^2y^2), \\f_{xxy} &= e^{xy}(6x + 6x^2y + x^3y^2).\end{aligned}$$

Step 2: Route y, x, x

$$\begin{aligned}f_y &= x^3e^{xy}, \quad f_{yx} = e^{xy}(3x^2 + x^3y), \\f_{yxx} &= e^{xy}(6x + 6x^2y + x^3y^2).\end{aligned}$$

Step 3: Verify Polynomial and exponential factors are smooth, so the derivative order may be permuted. At $(1, 0)$ both expressions equal $\boxed{6}$.

Question 5

A proposed Hessian is

$$H_f(x, y) = \begin{pmatrix} 6x - 2y & -2x + 4y \\ -2x + 4y & 4x + 6y \end{pmatrix}.$$

Tasks

1. Check its compatibility conditions.
2. Recover one possible $f(x, y)$.
3. Describe every function with this Hessian.

Question 5 – Solution

Strategy. Integrate f_{xx} and use the other entries to determine the unknown functions of integration.

Step 1: Compatibility The off-diagonal entries agree, and

$$\partial_y f_{xx} = -2 = \partial_x f_{xy}, \quad \partial_y f_{xy} = 4 = \partial_x f_{yy}.$$

Step 2: Reconstruct

$$f_x = 3x^2 - 2xy + A(y).$$

Since $f_{xy} = -2x + A'(y) = -2x + 4y$, $A(y) = 2y^2 + C$. Integrating in x ,

$$f = x^3 - x^2y + 2xy^2 + Cx + B(y).$$

The condition $f_{yy} = 4x + B''(y) = 4x + 6y$ gives $B = y^3 + Dy + E$.

Result

$$\boxed{f = x^3 - x^2y + 2xy^2 + y^3 + Cx + Dy + E}.$$

Conversely, direct differentiation shows every choice of C, D, E has exactly the proposed Hessian.

Question 6

Assume the mixed partials of f are continuous near (a, b) . Define

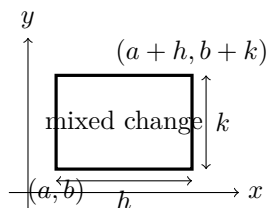
$$D(h, k) = \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}.$$

Tasks

1. Interpret the numerator as a rectangular mixed change.
2. Apply the mean value theorem in each variable.
3. Find $\lim_{(h,k) \rightarrow (0,0)} D(h, k)$.

Question 6 – Solution

Strategy. Apply one-variable mean value arguments successively across the small rectangle.



Step 1: Difference structure Let

$$G(y) = f(a + h, y) - f(a, y).$$

The numerator is $G(b + k) - G(b)$.

Step 2: Mean value steps For some η between b and $b + k$,

$$G(b + k) - G(b) = kG'(\eta) = k[f_y(a + h, \eta) - f_y(a, \eta)].$$

For some ξ between a and $a + h$, the bracket equals $hf_{yx}(\xi, \eta)$. Hence

$$D(h, k) = f_{yx}(\xi, \eta).$$

Step 3: Limit As $(h, k) \rightarrow (0, 0)$, $(\xi, \eta) \rightarrow (a, b)$. Continuity and equality of mixed partials give

$$\boxed{\lim D(h, k) = f_{yx}(a, b) = f_{xy}(a, b)}.$$

Question 7

For $w(x, y, z) = x^2yz^3 + \sin(xz)$:

Tasks

1. Find w_{xyz} .
2. Find w_{zzx} .
3. Evaluate both at $(1, 2, 0)$ and discuss derivative order.

Question 7 – Solution

Strategy. Follow the subscripts and keep polynomial and trigonometric contributions separate.

Step 1: w_{xyz}

$$w_x = 2xyz^3 + z \cos(xz), \quad w_{xy} = 2xz^3, \quad \boxed{w_{xyz} = 6xz^2}.$$

Step 2: w_{zzx}

$$\begin{aligned} w_z &= 3x^2yz^2 + x \cos(xz), \\ w_{zz} &= 6x^2yz - x^2 \sin(xz), \\ \boxed{w_{zzx} &= 12xyz - 2x \sin(xz) - x^2 z \cos(xz)}. \end{aligned}$$

Step 3: Evaluate Both happen to equal $\boxed{0}$ at $(1, 2, 0)$. Smoothness permits reordering derivatives that contain the same multiset of variables; xyz and zzx are different derivative types and need not agree generally.

Question 8

Let $f(x, y) = x^2y^2/(x^2 + y^2)$ away from $(0, 0)$ and $f(0, 0) = 0$.

Tasks

1. Find $f_x(0, y)$ and $f_y(x, 0)$ from definitions.
2. Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.
3. Explain why axis data suffice here.

Question 8 – Solution

Strategy. Restrict first derivatives to an axis, then differentiate that restriction at zero.

Step 1: Axis first partials For $y \neq 0$,

$$f_x(0, y) = \lim_{h \rightarrow 0} \frac{h^2 y^2 / (h^2 + y^2)}{h} = \lim_{h \rightarrow 0} \frac{h y^2}{h^2 + y^2} = 0.$$

At $y = 0$ it is also zero. By symmetry, $f_y(x, 0) = 0$ for every x .

Step 2: Mixed values

$$f_{xy}(0, 0) = \lim_{k \rightarrow 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = 0,$$

$$f_{yx}(0, 0) = \lim_{h \rightarrow 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = 0.$$

Thus $\boxed{f_{xy}(0, 0) = f_{yx}(0, 0) = 0}$.

Verification. Each definition uses only the relevant axis restriction, so off-axis values do not enter these two particular quotients.

Question 9

Suppose u satisfies $u_{tt} = 9u_{xx}$ and

$$u(x, t) = A \cos(2x) \cos(\omega t).$$

Tasks

1. Compute u_{tt} and u_{xx} .
2. Determine all real ω for nonzero A .
3. Verify the result and analyze the $A = 0$ edge case.

Question 9 – Solution

Strategy. Substitute the separated trial form and compare its common factors.

Step 1: Derivatives

$$u_{tt} = -A\omega^2 \cos(2x) \cos(\omega t), \quad u_{xx} = -4A \cos(2x) \cos(\omega t).$$

Step 2: Match For $A \neq 0$, equality for every (x, t) requires

$$-\omega^2 = 9(-4), \quad \omega^2 = 36,$$

so $\boxed{\omega = \pm 6}$.

Step 3: Verify Directly, $u_{tt} = -36u = 9(-4u) = 9u_{xx}$. Because $\cos(-6t) = \cos(6t)$, the signs produce the same function. If $A = 0$, the zero function satisfies the equation for every ω .

Question 10

Find all real constants a, b, c for which

$$u(x, y) = e^{ax+by} \cos(cx)$$

satisfies $u_{xx} + u_{yy} = 0$ everywhere.

Tasks

1. Compute the pure second partials.
2. Separate sine and cosine coefficients.
3. Solve all conditions, including $c = 0$.

Question 10 – Solution

Strategy. Equality for every x forces independent trigonometric coefficients to vanish.

Step 1: Differentiate

$$\begin{aligned}u_{xx} &= e^{ax+by}[(a^2 - c^2)\cos(cx) - 2ac\sin(cx)], \\u_{yy} &= b^2 e^{ax+by}\cos(cx).\end{aligned}$$

Thus the sum is

$$e^{ax+by}[(a^2 + b^2 - c^2)\cos(cx) - 2ac\sin(cx)].$$

Step 2: Nonzero c If $c \neq 0$, independence of sine and cosine gives

$$a^2 + b^2 - c^2 = 0, \quad ac = 0.$$

Thus $a = 0$ and $b = \pm c$.

Step 3: Edge case If $c = 0$, the condition is $a^2 + b^2 = 0$, hence $a = b = 0$. This is already included below:

$$\boxed{(a, b, c) = (0, c, c) \text{ or } (0, -c, c), \quad c \in \mathbb{R}}.$$

Substitution makes both coefficients vanish, verifying sufficiency.

Question 1

For

$$z = x^2 e^y + \ln(x + y), \quad x + y > 0,$$

Tasks

1. Find the total differential dz .
2. Evaluate it at $(1, 1)$ for $dx = -0.02$, $dy = 0.03$.
3. Identify the contribution from each input change.

Question 1 – Solution

Strategy. Use $dz = f_x dx + f_y dy$ and evaluate coefficients before applying the increments.

Step 1: Partial derivatives

$$f_x = 2xe^y + \frac{1}{x+y}, \quad f_y = x^2e^y + \frac{1}{x+y}.$$

Thus

$$dz = \left(2xe^y + \frac{1}{x+y} \right) dx + \left(x^2e^y + \frac{1}{x+y} \right) dy.$$

Step 2: Evaluate At $(1, 1)$,

$$dz = (2e + \tfrac{1}{2})(-0.02) + (e + \tfrac{1}{2})(0.03).$$

Combining terms,

$$dz = -0.04e - 0.01 + 0.03e + 0.015 = \boxed{0.005 - 0.01e \approx -0.02218}.$$

Step 3: Contributions The x change contributes $-(0.04e + 0.01)$; the y change contributes $0.03e + 0.015$. Their signs oppose, and the x contribution has larger magnitude.

Question 2

Use differentials to estimate

$$(4.02)^2\sqrt{8.97}.$$

Tasks

1. Choose a two-variable function and nearby convenient base point.
2. Compute the differential estimate.
3. State the estimated value and identify the neglected order of error.

Question 2 – Solution

Strategy. Let $f(x, y) = x^2\sqrt{y}$ and linearize its change from $(4, 9)$.

Step 1: Base data

$$f(4, 9) = 16(3) = 48, \quad dx = 0.02, \quad dy = -0.03.$$

$$f_x = 2x\sqrt{y}, \quad f_y = \frac{x^2}{2\sqrt{y}}.$$

At $(4, 9)$, $f_x = 24$ and $f_y = 8/3$.

Step 2: Differential

$$df = 24(0.02) + \frac{8}{3}(-0.03) = 0.48 - 0.08 = 0.40.$$

Step 3: Estimate

$$\boxed{(4.02)^2\sqrt{8.97} \approx 48 + 0.40 = 48.40}.$$

The differential keeps terms linear in dx, dy and neglects quadratic and higher-order increment effects.

Question 3

For $f(x, y) = x^2 + y^2$, start at $(3, 4)$ and use $dx = 0.1$, $dy = -0.2$.

Tasks

1. Compute df .
2. Compute the exact change Δf .
3. Derive the discrepancy $\Delta f - df$ and explain its sign.

Question 3 – Solution

Strategy. Expand the exact finite change so the first-order and second-order parts separate visibly.

Step 1: Differential

$$df = 2x \, dx + 2y \, dy = 6(0.1) + 8(-0.2) = \boxed{-1.0}.$$

Step 2: Exact change

$$\Delta f = (3.1)^2 + (3.8)^2 - (3^2 + 4^2) = 9.61 + 14.44 - 25 = \boxed{-0.95}.$$

Step 3: Discrepancy Algebraically,

$$\Delta f = 2x \, dx + 2y \, dy + (dx)^2 + (dy)^2 = df + (dx)^2 + (dy)^2.$$

Therefore

$$\boxed{\Delta f - df = 0.01 + 0.04 = 0.05}.$$

It is nonnegative because the omitted terms are squares; the upward curvature makes the exact change exceed the linear prediction.

Question 4

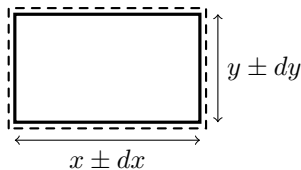
A rectangular plate has measured sides $x = 30.0$ cm and $y = 20.0$ cm, each accurate to within 0.1 cm. Its area is $A = xy$.

Tasks

1. Use differentials to estimate the maximum absolute area error.
2. Estimate the maximum relative and percentage errors.
3. Compare with the exact worst-case outward error.

Question 4 – Solution

Strategy. For a worst-case differential bound, replace signed increments by their maximum absolute values.



Step 1: Absolute error

$$dA = y \, dx + x \, dy.$$

Thus

$$|dA| \leq 20(0.1) + 30(0.1) = \boxed{5.0 \text{ cm}^2}.$$

Step 2: Relative error Since $A = 600 \text{ cm}^2$,

$$\frac{|dA|}{A} \approx \frac{5}{600} = \boxed{\frac{1}{120} \approx 0.00833},$$

or $\boxed{0.833\%}$.

Step 3: Exact comparison The largest outward area is

$$(30.1)(20.1) = 605.01,$$

so the exact positive error is 5.01 cm^2 . The differential misses only the product $dx \, dy = 0.01 \text{ cm}^2$.

Question 5

A cone has volume $V(r, h) = \frac{1}{3}\pi r^2 h$. At $r = 5$ cm, $h = 12$ cm, suppose $dr = 0.04$ cm and $dh = -0.10$ cm.

Tasks

1. Find dV exactly in terms of π .
2. Determine which measurement change dominates the estimate.
3. Compute the relative differential dV/V in two ways.

Question 5 – Solution

Strategy. Compute component contributions, then confirm the relative formula by logarithmic structure.

Step 1: Differential

$$dV = \frac{2\pi r h}{3} dr + \frac{\pi r^2}{3} dh.$$

At the given dimensions,

$$dV = 40\pi(0.04) + \frac{25\pi}{3}(-0.10) = 1.6\pi - \frac{5}{6}\pi = \boxed{\frac{23\pi}{30} \text{ cm}^3}.$$

Step 2: Compare The radial contribution is 1.6π ; the height contribution is $-5\pi/6$. The radial contribution dominates, so the net predicted change is positive.

Step 3: Relative form

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h} = 2\frac{0.04}{5} - \frac{0.10}{12} = \boxed{\frac{23}{3000}}.$$

Also $V = 100\pi$, and $(23\pi/30)/(100\pi) = 23/3000$, verifying the result.

Question 6

Points constrained to the sphere

$$x^2 + y^2 + z^2 = 49$$

undergo small changes near $(2, 3, 6)$.

Tasks

1. Derive the differential constraint relating dx, dy, dz .
2. Estimate dz when $dx = 0.03$ and $dy = -0.02$.
3. Explain geometrically why the first-order change satisfies this relation.

Question 6 – Solution

Strategy. Differentiate the constant constraint; its total differential must vanish along allowed infinitesimal changes.

Step 1: Constraint

$$2x \, dx + 2y \, dy + 2z \, dz = 0,$$

so, because $z = 6 \neq 0$,

$$\boxed{dz = -\frac{x \, dx + y \, dy}{z}}.$$

Step 2: Estimate

$$dz = -\frac{2(0.03) + 3(-0.02)}{6} = -\frac{0.06 - 0.06}{6} = \boxed{0}.$$

Step 3: Geometry The radius vector $\langle x, y, z \rangle$ is normal to the sphere. The relation

$$\langle x, y, z \rangle \cdot \langle dx, dy, dz \rangle = 0$$

says the first-order displacement is perpendicular to that normal, hence tangent to the sphere. Here the chosen horizontal change is already orthogonal to the radius projection, so no first-order vertical correction is needed.

Question 7

Determine whether

$$\omega = (2xy + 3) dx + (x^2 + 4y) dy$$

is the differential of a scalar function $f(x, y)$.

Tasks

1. Test the necessary compatibility condition.
2. Recover every possible f if the test passes.
3. Verify by redifferentiating.

Question 7 – Solution

Strategy. If $df = M dx + N dy$, then $M_y = N_x$ on a simply connected domain.

Step 1: Compatibility Here

$$M = 2xy + 3, \quad N = x^2 + 4y.$$

Then

$$M_y = 2x = N_x,$$

so the form is compatible throughout $\mathbb{R}^2$.

Step 2: Recover f Integrate $f_x = M$ in x :

$$f = x^2y + 3x + C(y).$$

Then

$$f_y = x^2 + C'(y) = x^2 + 4y,$$

so $C'(y) = 4y$ and $C(y) = 2y^2 + C_0$.

Result and verification

$$\boxed{f(x, y) = x^2y + 3x + 2y^2 + C_0}.$$

Indeed, differentiating gives $df = (2xy + 3) dx + (x^2 + 4y) dy$. The additive constant is invisible to the differential.

Question 8

A quantity is

$$Q = \frac{x^3 \sqrt{y}}{z^2}, \quad x, y, z > 0.$$

Tasks

1. Derive a relative-differential formula for dQ/Q .
2. Estimate the signed percentage change if x rises 1%, y falls 4%, and z rises 0.5%.
3. Identify the dominant contribution.

Question 8 – Solution

Strategy. Divide the total differential by Q , or equivalently differentiate the logarithmic power structure.

Step 1: Relative differential

$$\ln Q = 3 \ln x + \frac{1}{2} \ln y - 2 \ln z.$$

Differentiating gives

$$\boxed{\frac{dQ}{Q} = 3 \frac{dx}{x} + \frac{1}{2} \frac{dy}{y} - 2 \frac{dz}{z}}.$$

Step 2: Percentage estimate

$$\frac{dQ}{Q} \approx 3(0.01) + \frac{1}{2}(-0.04) - 2(0.005) = 0.03 - 0.02 - 0.01 = \boxed{0}.$$

Thus the predicted first-order percentage change is $\boxed{0\%}$.

Step 3: Contributions The x increase contributes $+3\%$, the y decrease -2% , and the z increase -1% . The x term is individually largest, but the three first-order effects cancel exactly.

Question 9

A differential field is proposed as

$$df = (y^2 + 2x) dx + (2xy + e^y) dy, \quad f(0, 0) = 5.$$

Tasks

1. Reconstruct f .
2. Use df to estimate $f(0.02, -0.01)$ from the origin.
3. Compare the estimate with the exact value through second order.

Question 9 – Solution

Strategy. Reconstruct the function first, then separate its linear and nonlinear increment terms.

Step 1: Reconstruct Integrating $f_x = y^2 + 2x$ gives

$$f = xy^2 + x^2 + C(y).$$

Since $f_y = 2xy + C'(y) = 2xy + e^y$, $C = e^y + C_0$. The initial value gives $1 + C_0 = 5$, so

$$\boxed{f = xy^2 + x^2 + e^y + 4}.$$

Step 2: Differential estimate At $(0, 0)$,

$$f_x = 0, \quad f_y = 1.$$

For $dx = 0.02$, $dy = -0.01$,

$$f(0.02, -0.01) \approx 5 + df = 5 - 0.01 = \boxed{4.99}.$$

Step 3: Exact comparison Exactly,

$$f = 0.02(0.01)^2 + (0.02)^2 + e^{-0.01} + 4 \approx 4.99045183.$$

The estimate error is about $\boxed{0.00045183}$, while the second-order correction is $dx^2 + dy^2/2 = 0.00045$. Thus the second-order estimate is 4.99045, consistent with the exact value.

Question 10

Define

$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $f_x(0, 0)$ and $f_y(0, 0)$.
2. Test whether a differential $df = f_x dx + f_y dy$ gives a valid first-order approximation at the origin.
3. Identify a path that proves or disproves differentiability.

Question 10 – Solution

Strategy. Existing partial derivatives propose a linear map, but differentiability requires the remainder divided by distance to vanish along every path.

Step 1: Partial

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0)}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1,$$

$$f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k)}{k} = 0.$$

The candidate differential is therefore $L(dx, dy) = dx$.

Step 2: Remainder test Differentiability would require

$$\frac{f(x,y) - x}{\sqrt{x^2 + y^2}} \rightarrow 0.$$

Along $y = x$,

$$f(x,x) = \frac{x^3}{2x^2} = \frac{x}{2},$$

so

$$\frac{f(x,x) - x}{\sqrt{2x^2}} = \frac{-x/2}{\sqrt{2}|x|} = -\frac{\text{sgn}(x)}{2\sqrt{2}},$$

which does not tend to zero.

Conclusion Although both partials exist, f is not differentiable at the origin, so

$df = dx$ is not a valid total first-order approximation there.

Question 1

Let $z = x^2y + \sin y$, where $x = t^2 - 1$ and $y = e^t$.

Tasks

1. Find dz/dt using the multivariable chain rule.
2. Evaluate it at $t = 0$.
3. Verify by differentiating the explicit composite function.

Question 1 – Solution

Strategy. Combine the two routes $t \rightarrow x \rightarrow z$ and $t \rightarrow y \rightarrow z$.

Step 1: Ingredients

$$z_x = 2xy, \quad z_y = x^2 + \cos y, \quad x' = 2t, \quad y' = e^t.$$

Therefore

$$\frac{dz}{dt} = 2xy(2t) + (x^2 + \cos y)e^t.$$

Step 2: Evaluate At $t = 0$, $x = -1$ and $y = 1$, giving

$$\left. \frac{dz}{dt} \right|_{t=0} = 1 + \cos 1.$$

Step 3: Direct check The composite is

$$z(t) = (t^2 - 1)^2 e^t + \sin(e^t).$$

Its derivative is $4t(t^2 - 1)e^t + (t^2 - 1)^2 e^t + \cos(e^t)e^t$, which at zero is $1 + \cos 1$.

Question 2

Suppose $F(x, y) = \ln(x^2 + y^2)$ and a curve is $x = 3 \cos t$, $y = 3 \sin t$.

Tasks

1. Compute dF/dt by the chain rule.
2. Explain the cancellation geometrically.
3. Verify using the explicit composite.

Question 2 – Solution

Strategy. Evaluate the two chain-rule contributions before simplifying.

Step 1: Chain rule

$$F_x = \frac{2x}{x^2 + y^2}, \quad F_y = \frac{2y}{x^2 + y^2},$$
$$x' = -3 \sin t, \quad y' = 3 \cos t.$$

Thus

$$\frac{dF}{dt} = \frac{2x(-3 \sin t) + 2y(3 \cos t)}{x^2 + y^2} = 0,$$

after substituting $x = 3 \cos t$, $y = 3 \sin t$.

Step 2: Meaning The curve remains on the circle $x^2 + y^2 = 9$, and F depends only on that squared radius. Hence $\boxed{dF/dt = 0}$.

Step 3: Direct check $F(3 \cos t, 3 \sin t) = \ln 9$, whose derivative is zero.

Question 3

Let

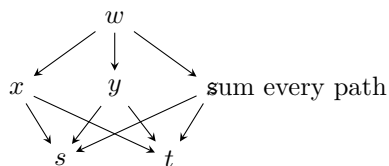
$$w = x^2 + yz, \quad x = s + t, \quad y = st, \quad z = s^2 - t^2.$$

Tasks

1. Draw or use the dependency structure to find w_s, w_t .
2. Evaluate both at $(s, t) = (1, 2)$.
3. Verify one result by explicit expansion.

Question 3 – Solution

Strategy. Sum the contribution along every dependency path.



Step 1: Derivatives

$$w_x = 2x, \quad w_y = z, \quad w_z = y.$$

$$x_s = 1, \quad y_s = t, \quad z_s = 2s; \quad x_t = 1, \quad y_t = s, \quad z_t = -2t.$$

Therefore

$$\boxed{w_s = 2x + zt + 2sy}, \quad \boxed{w_t = 2x + zs - 2ty}.$$

Step 2: Evaluate At $(1, 2)$, $(x, y, z) = (3, 2, -3)$:

$$\boxed{w_s = 6 - 6 + 4 = 4}, \quad \boxed{w_t = 6 - 3 - 8 = -5}.$$

Step 3: Check Expanding gives $w = (s+t)^2 + st(s^2 - t^2)$. Differentiating in s yields $2(s+t) + t(3s^2 - t^2)$, which equals 4 at $(1, 2)$.

Question 4

A surface satisfies $x^2 + y^2 + z^2 = 14$. Along a path on the surface, $x = t$, $y = t^2$, and $z > 0$ near $t = 1$.

Tasks

1. Find dz/dt implicitly without solving for $z(t)$ first.
2. Evaluate at $t = 1$.
3. Verify using the positive explicit branch.

Question 4 – Solution

Strategy. Differentiate the constraint along the path; its total rate must remain zero.

Step 1: Differentiate

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} = 0,$$

so

$$\boxed{\frac{dz}{dt} = -\frac{xx' + yy'}{z}}.$$

Step 2: Evaluate At $t = 1$, $x = 1$, $y = 1$, $z = \sqrt{14 - 1 - 1} = 2\sqrt{3}$, while $x' = 1$, $y' = 2$. Hence

$$\boxed{\left. \frac{dz}{dt} \right|_1 = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}}.$$

Step 3: Check The positive branch is $z(t) = \sqrt{14 - t^2 - t^4}$. Differentiation gives $z' = (-2t - 4t^3)/(2z)$, which produces the same value at $t = 1$.

Question 5

Let $u(x, y)$ be differentiable and set $x = r \cos \theta$, $y = r \sin \theta$.

Tasks

1. Derive formulas for u_r and u_θ .
2. Apply them to $u = x^2 - y^2$.
3. Verify against the polar form of u .

Question 5 – Solution

Strategy. Treat r and θ as independent inputs to the Cartesian coordinate map.

Step 1: General formulas

$$\boxed{u_r = u_x \cos \theta + u_y \sin \theta},$$

$$\boxed{u_\theta = -ru_x \sin \theta + ru_y \cos \theta}.$$

Step 2: Apply For $u = x^2 - y^2$, $u_x = 2x$, $u_y = -2y$. Substituting $x = r \cos \theta$, $y = r \sin \theta$ gives

$$u_r = 2r(\cos^2 \theta - \sin^2 \theta) = \boxed{2r \cos 2\theta},$$

$$u_\theta = -4r^2 \sin \theta \cos \theta = \boxed{-2r^2 \sin 2\theta}.$$

Step 3: Verify Directly, $u = r^2 \cos 2\theta$; its two ordinary partial derivatives match both formulas.

Question 6

Let

$$F(x, y) = \begin{pmatrix} x^2 - y \\ xy \end{pmatrix}, \quad G(s, t) = \begin{pmatrix} e^s \cos t \\ e^s \sin t \end{pmatrix}.$$

Tasks

1. Find the Jacobian matrices J_F and J_G .
2. Use matrix chain rule to find $J_{F \circ G}(0, \pi/2)$.
3. Check one entry directly.

Question 6 – Solution

Strategy. Evaluate J_F at $G(s, t)$ before multiplying by J_G .

Step 1: Jacobians

$$J_F = \begin{pmatrix} 2x & -1 \\ y & x \end{pmatrix}, \quad J_G = \begin{pmatrix} e^s \cos t & -e^s \sin t \\ e^s \sin t & e^s \cos t \end{pmatrix}.$$

At $(0, \pi/2)$, $G = (0, 1)$, so

$$J_F(0, 1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Step 2: Multiply

$$J_{F \circ G}(0, \pi/2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The composite Jacobian is $-I_2$.

Step 3: Check The first component is $e^{2s} \cos^2 t - e^s \sin t$; its s -derivative at the point is $0 - 1 = -1$, matching the upper-left entry.

Question 7

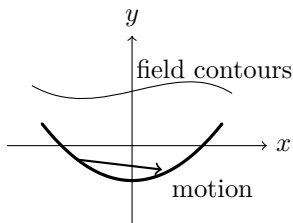
A temperature field is $T(x, y, t) = x^2 + 2xy + 3t$. A particle follows $x = t^2$, $y = 1 - t$.

Tasks

1. Find the rate of temperature change experienced by the particle.
2. Evaluate it at $t = 1$.
3. Separate spatial-motion and explicit-time contributions.

Question 7 – Solution

Strategy. The experienced rate includes motion through space and direct evolution of the field.



Step 1: Extended chain rule

$$\frac{dT}{dt} = T_x x' + T_y y' + T_t.$$

Here

$$T_x = 2x + 2y, \quad T_y = 2x, \quad T_t = 3, \quad x' = 2t, \quad y' = -1.$$

Thus

$$\boxed{\frac{dT}{dt} = (2x + 2y)(2t) - 2x + 3}.$$

Step 2: Evaluate At $t = 1$, $(x, y) = (1, 0)$:

$$\boxed{\frac{dT}{dt} = 2(2) - 2 + 3 = 5}.$$

Step 3: Contributions Motion in x contributes 4, motion in y contributes -2 , and explicit time dependence contributes 3. Their sum is 5 temperature units per time unit.

Question 8

Suppose $z = f(x, y)$ has $f_x(2, 0) = 3$ and $f_y(2, 0) = -4$. The inputs obey

$$x = u^2 + v, \quad y = u - v^2.$$

At $(u, v) = (1, 1)$:

Tasks

1. Find z_u and z_v .
2. Find all input changes (du, dv) producing zero first-order change in z .
3. Verify the cancellation using dx, dy .

Question 8 – Solution

Strategy. First propagate u, v changes to x, y , then combine them with the supplied partials.

Step 1: Chain derivatives At $(1, 1)$, $(x, y) = (2, 0)$ and

$$x_u = 2, \ x_v = 1, \ y_u = 1, \ y_v = -2.$$

Therefore

$$\boxed{z_u = 3(2) + (-4)(1) = 2}, \quad \boxed{z_v = 3(1) + (-4)(-2) = 11}.$$

Step 2: Zero-change inputs The total differential is

$$dz = 2 \, du + 11 \, dv.$$

Thus every first-order neutral change satisfies

$$\boxed{2 \, du + 11 \, dv = 0},$$

equivalently $(du, dv) = \lambda(11, -2)$.

Step 3: Verify For $(du, dv) = (11, -2)$, $dx = 2(11) - 2 = 20$ and $dy = 11 - 2(-2) = 15$. Then $dz = 3(20) - 4(15) = 0$.

Question 9

Variables x, y are defined implicitly by

$$x + y = s, \quad xy = t.$$

Near $(x, y) = (1, 2)$, regard x, y as functions of (s, t) .

Tasks

1. Find x_s, y_s, x_t, y_t by differentiating the system.
2. Evaluate them at $(1, 2)$.
3. Explain why the calculation fails when $x = y$.

Question 9 – Solution

Strategy. Differentiate both constraints and solve the resulting 2×2 linear systems.

Step 1: With respect to s

$$x_s + y_s = 1, \quad yx_s + xy_s = 0.$$

Solving gives

$$\boxed{x_s = \frac{x}{x-y}}, \quad \boxed{y_s = -\frac{y}{x-y}}.$$

Step 2: With respect to t

$$x_t + y_t = 0, \quad yx_t + xy_t = 1,$$

so

$$\boxed{x_t = \frac{1}{y-x}}, \quad \boxed{y_t = \frac{1}{x-y}}.$$

At $(1, 2)$:

$$\boxed{x_s = -1, \ y_s = 2, \ x_t = 1, \ y_t = -1}.$$

Step 3: Edge case The determinant of the coefficient matrix is $x - y$. When $x = y$, the system cannot locally distinguish the two roots; division by $x - y$ correctly signals failure of this local inverse description.

Question 10

Let $z = f(x(t), y(t))$, where f has continuous second partials.

Tasks

1. Derive a formula for d^2z/dt^2 .
2. Organize it into curvature-of- f and acceleration-of-path terms.
3. Apply it to $f = x^2 + xy + y^2$, $x = \cos t$, $y = \sin t$ at $t = 0$.

Question 10 – Solution

Strategy. Differentiate $z' = f_x x' + f_y y'$ and apply the chain and product rules to every factor.

Step 1: General formula

$$z'' = f_{xx}(x')^2 + 2f_{xy}x'y' + f_{yy}(y')^2 + f_x x'' + f_y y''.$$

The first three terms measure the function's second-order response along the velocity; the last two measure the effect of path acceleration on the first partials.

Step 2: Data For $f = x^2 + xy + y^2$,

$$f_x = 2x + y, \quad f_y = x + 2y, \quad f_{xx} = 2, \quad f_{xy} = 1, \quad f_{yy} = 2.$$

At $t = 0$, $(x, y) = (1, 0)$, $(x', y') = (0, 1)$, and $(x'', y'') = (-1, 0)$.

Step 3: Evaluate

$$z'' = 2(0)^2 + 2(1)(0)(1) + 2(1)^2 + (2)(-1) + (1)(0) = \boxed{0}.$$

Directly, $z = \cos^2 t + \sin t \cos t + \sin^2 t = 1 + \frac{1}{2} \sin 2t$, whose second derivative is $-\sin 2t$, also zero at $t = 0$.

Question 1

For

$$f(x, y) = x^2y + e^{x-y},$$

find the directional derivative at $P = (1, 0)$ in the direction $v = \langle 3, 4 \rangle$.

Tasks

1. Normalize the direction.
2. Compute the gradient at P .
3. Evaluate and interpret the directional derivative.

Question 1 – Solution

Strategy. A directional derivative uses a unit vector: $D_u f = \nabla f \cdot u$.

Step 1: Normalize

$$\|v\| = 5, \quad u = \boxed{\langle 3/5, 4/5 \rangle}.$$

Step 2: Gradient

$$f_x = 2xy + e^{x-y}, \quad f_y = x^2 - e^{x-y}.$$

At $(1, 0)$,

$$\nabla f(1, 0) = \langle e, 1 - e \rangle.$$

Step 3: Dot product

$$D_u f(1, 0) = e \frac{3}{5} + (1 - e) \frac{4}{5} = \boxed{\frac{4 - e}{5}}.$$

The positive value means f initially increases by approximately $(4 - e)/5$ output units per unit distance traveled in the specified direction.

Question 2

Let $g(x, y) = \ln(x^2 + y^2)$. Find its rate of change at $P = (1, -2)$ toward $Q = (4, 2)$.

Tasks

1. Construct the unit vector from P to Q .
2. Compute the directional derivative exactly.
3. Decide whether g initially increases or decreases.

Question 2 – Solution

Strategy. Subtract endpoint coordinates in the correct order before normalizing.

Step 1: Direction

$$\overrightarrow{PQ} = \langle 3, 4 \rangle, \quad u = \langle 3/5, 4/5 \rangle.$$

Step 2: Gradient

$$\nabla g = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle,$$

so

$$\nabla g(1, -2) = \langle 2/5, -4/5 \rangle.$$

Therefore

$$D_u g(P) = \frac{2}{5} \frac{3}{5} - \frac{4}{5} \frac{4}{5} = \boxed{-\frac{2}{5}}.$$

Step 3: Interpret The negative sign means g decreases initially toward Q . This is consistent with the initial motion having a component toward smaller distance from the origin.

Question 3

At a point P , a differentiable function has $\nabla f(P) = \langle -2, 5 \rangle$.

Tasks

1. Find the maximum and minimum directional derivatives and their directions.
2. Find all unit directions giving zero directional derivative.
3. Find the directional derivative at angle $\pi/6$ from the positive x -axis.

Question 3 – Solution

Strategy. Use $D_u f = \|\nabla f\| \cos \alpha$, where α is the angle from the gradient to the unit direction.

Step 1: Extremes Since $\|\nabla f\| = \sqrt{29}$,

$$D_{\max} = \sqrt{29} \text{ in direction } \frac{1}{\sqrt{29}} \langle -2, 5 \rangle,$$

$$D_{\min} = -\sqrt{29} \text{ in direction } \frac{1}{\sqrt{29}} \langle 2, -5 \rangle.$$

Step 2: Zero directions Unit vectors perpendicular to $\langle -2, 5 \rangle$ are

$$u = \pm \frac{1}{\sqrt{29}} \langle 5, 2 \rangle.$$

Step 3: Prescribed angle For $u = \langle \sqrt{3}/2, 1/2 \rangle$,

$$D_u f = -2 \frac{\sqrt{3}}{2} + 5 \frac{1}{2} = \frac{5}{2} - \sqrt{3}.$$

Question 4

At P , $\nabla f(P) = \langle 6, 8 \rangle$. Determine every unit direction u for which $D_u f(P) = 5$.

Tasks

1. Convert the condition to geometry.
2. Find both unit vectors exactly.
3. Verify normalization and the required rate.

Question 4 – Solution

Strategy. Decompose the desired unit vector into components parallel and perpendicular to the gradient.

Step 1: Unit basis Let

$$e = \frac{1}{10} \langle 6, 8 \rangle = \langle 3/5, 4/5 \rangle, \quad n = \langle -4/5, 3/5 \rangle.$$

These form an orthonormal basis. Write $u = ae + bn$ with $a^2 + b^2 = 1$.

Step 2: Rate condition

$$\nabla f \cdot u = 10a = 5,$$

so $a = 1/2$ and $b = \pm\sqrt{3}/2$. Thus

$$u = \frac{1}{2} \langle 3/5, 4/5 \rangle \pm \frac{\sqrt{3}}{2} \langle -4/5, 3/5 \rangle.$$

Equivalently,

$$u = \left\langle \frac{3 \mp 4\sqrt{3}}{10}, \frac{4 \pm 3\sqrt{3}}{10} \right\rangle.$$

Step 3: Verify Orthonormality gives $\|u\|^2 = 1/4 + 3/4 = 1$, and the perpendicular component contributes zero to the dot product, leaving $10(1/2) = 5$.

Question 5

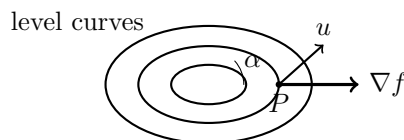
Let f be continuously differentiable near P . At P , its level curve is smooth and $\nabla f(P) \neq 0$. A unit direction u makes angle α with $\nabla f(P)$.

Tasks

1. Derive $D_u f(P)$ in terms of α .
2. Explain why tangent motion along the level curve gives zero rate.
3. Determine the sign of the rate for acute, right, and obtuse α .

Question 5 – Solution

Strategy. Combine the dot-product formula with the gradient's normal relationship to level curves.



Step 1: Angular formula

$$D_u f(P) = \nabla f(P) \cdot u = \|\nabla f(P)\| \cos \alpha.$$

Step 2: Tangent direction Along the level curve, f is constant, so its rate is zero. Geometrically the tangent is perpendicular to ∇f , giving $\alpha = \pi/2$ and $\cos \alpha = 0$.

Step 3: Signs The rate is positive for acute α , zero for a right angle, and negative for obtuse α . Its magnitude is greatest when u is parallel or antiparallel to the gradient.

Question 6

For the scalar field

$$F(x, y, z) = xe^{yz} + z^2,$$

find the directional derivative at $P = (2, 0, 1)$ toward $Q = (-1, 2, 3)$.

Tasks

1. Find the three-dimensional unit direction.
2. Compute $\nabla F(P)$.
3. Evaluate the rate and verify its units conceptually.

Question 6 – Solution

Strategy. The same gradient-dot-unit-vector formula applies in three dimensions.

Step 1: Direction

$$\overrightarrow{PQ} = \langle -3, 2, 2 \rangle, \quad \|\overrightarrow{PQ}\| = \sqrt{17}, \quad u = \frac{1}{\sqrt{17}} \langle -3, 2, 2 \rangle.$$

Step 2: Gradient

$$F_x = e^{yz}, \quad F_y = xze^{yz}, \quad F_z = xye^{yz} + 2z.$$

At $(2, 0, 1)$,

$$\nabla F(P) = \langle 1, 2, 2 \rangle.$$

Step 3: Rate

$$D_u F(P) = \frac{-3 + 4 + 4}{\sqrt{17}} = \boxed{\frac{5}{\sqrt{17}}}.$$

Because u is dimensionless and unit length, the result measures change in F per unit spatial distance.

Question 7

Define

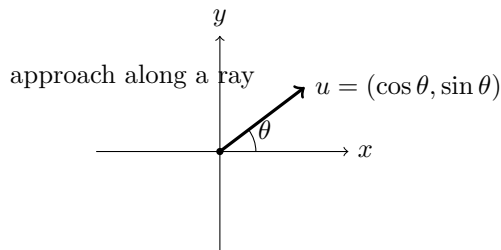
$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $D_u f(0, 0)$ from the definition for every unit $u = \langle \cos \theta, \sin \theta \rangle$.
2. Compare with $\nabla f(0, 0) \cdot u$.
3. Explain what this proves about directional derivatives and differentiability.

Question 7 – Solution

Strategy. Approach the origin along the ray hu and keep the signed parameter h .



Step 1: Definition For $h \neq 0$,

$$f(h \cos \theta, h \sin \theta) = \frac{h^3 \cos^3 \theta}{h^2} = h \cos^3 \theta.$$

Therefore

$$D_u f(0, 0) = \lim_{h \rightarrow 0} \frac{h \cos^3 \theta}{h} = \cos^3 \theta.$$

Step 2: Gradient comparison Axis definitions give $f_x(0, 0) = 1$ and $f_y(0, 0) = 0$, so

$$\nabla f(0, 0) \cdot u = \cos \theta,$$

which differs from $\cos^3 \theta$ for most angles.

Conclusion Every directional derivative exists, but its dependence on u is not linear. Thus f is not differentiable at the origin, and the gradient formula is not valid there.

Question 8

At $P = (1, 2)$, an unknown differentiable function satisfies

$$D_{u_1} f(P) = 4, \quad u_1 = \langle 1, 0 \rangle,$$

$$D_{u_2} f(P) = 1, \quad u_2 = \langle 3/5, 4/5 \rangle.$$

Tasks

1. Reconstruct $\nabla f(P)$.
2. Find the steepest-ascent direction and rate.
3. Predict the derivative in direction $\langle 0, -1 \rangle$ and verify consistency.

Question 8 – Solution

Strategy. Treat directional measurements as linear equations for the two gradient components.

Step 1: Reconstruct Write $\nabla f(P) = \langle a, b \rangle$. The first measurement gives $a = 4$. The second gives

$$\frac{3}{5}a + \frac{4}{5}b = 1.$$

Thus $12/5 + 4b/5 = 1$, so $b = -7/4$ and

$$\boxed{\nabla f(P) = \langle 4, -7/4 \rangle}.$$

Step 2: Maximum

$$\|\nabla f(P)\| = \sqrt{16 + \frac{49}{16}} = \frac{\sqrt{305}}{4}.$$

The steepest-ascent direction is

$$\boxed{\frac{1}{\sqrt{305}} \langle 16, -7 \rangle},$$

and its rate is $\boxed{\sqrt{305}/4}$.

Step 3: New direction For $\langle 0, -1 \rangle$,

$$D_{\langle 0, -1 \rangle} f = \langle 4, -7/4 \rangle \cdot \langle 0, -1 \rangle = \boxed{7/4}.$$

Substitution into both original dot products reproduces 4 and 1.

Question 9

Let $f(x, y, z) = x^2 + y^2 + z^2$. At $P = (1, 2, 2)$, restrict directions to be tangent to the plane $x + y + z = 5$.

Tasks

1. Find the unit tangent direction giving the greatest increase of f .
2. Find that maximum constrained rate.
3. Identify the exceptional case in which no preferred tangent direction exists.

Question 9 – Solution

Strategy. Project the gradient onto the plane's tangent subspace, then normalize the projection.

Step 1: Data $\nabla f(P) = \langle 2, 4, 4 \rangle$. A unit normal to the plane is

$$n = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle.$$

The tangential projection is

$$p = \nabla f - (\nabla f \cdot n)n = \langle 2, 4, 4 \rangle - \frac{10}{3} \langle 1, 1, 1 \rangle = \frac{1}{3} \langle -4, 2, 2 \rangle.$$

Step 2: Direction and rate Since $\|p\| = 2\sqrt{6}/3$,

$$\boxed{u_{\max} = \frac{1}{\sqrt{6}} \langle -2, 1, 1 \rangle}, \quad \boxed{D_{\max, \text{tangent}} f = \frac{2\sqrt{6}}{3}}.$$

Step 3: Verify $u_{\max} \cdot \langle 1, 1, 1 \rangle = 0$, so it is tangent, and $\nabla f \cdot u_{\max} = 2\sqrt{6}/3$. If the projected gradient were zero, every tangent directional derivative would be zero and no unique maximizing tangent direction would exist.

Question 10

At a point P , a differentiable function has unknown gradient. Its directional derivative equals 3 in every unit direction making angle $\pi/3$ with a fixed unit vector a .

Tasks

1. Determine what this implies about the component of $\nabla f(P)$ along a .
2. In two dimensions, use the two such directions to prove the perpendicular component is zero.
3. Determine $\nabla f(P)$ and the maximum directional derivative.

Question 10 – Solution

Strategy. Resolve the gradient into components parallel and perpendicular to a and compare symmetric directions.

Step 1: Decompose Let b be a unit vector perpendicular to a , and write

$$\nabla f = Aa + Bb.$$

The two unit directions at angles $\pm\pi/3$ from a are

$$u_{\pm} = \frac{1}{2}a \pm \frac{\sqrt{3}}{2}b.$$

Step 2: Use both measurements

$$3 = \nabla f \cdot u_{\pm} = \frac{A}{2} \pm \frac{\sqrt{3}B}{2}.$$

Adding gives $6 = A$, while subtracting gives $B = 0$.

Step 3: Result Therefore

$$\boxed{\nabla f(P) = 6a}.$$

The greatest directional derivative is the gradient magnitude,

$$\boxed{D_{\max}f(P) = 6},$$

attained uniquely in direction a . Checking either prescribed direction gives $6 \cos(\pi/3) = 3$.