

Question 1

A point $P = (x, y, z)$ is known only through its octant and its distances to the coordinate axes.

Given

$$x < 0, \quad y > 0, \quad z < 0,$$

$$d(P, x\text{-axis}) = \sqrt{41},$$

$$d(P, y\text{-axis}) = \sqrt{34},$$

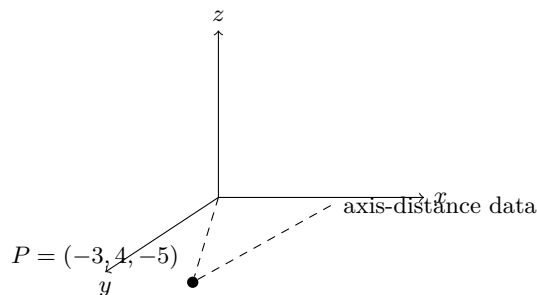
$$d(P, z\text{-axis}) = 5.$$

Tasks

1. Determine the coordinates of P .
2. Verify your result using all three axis distances.
3. Explain why only two of the three distance measurements would not determine P uniquely.

Question 1 – Solution

Strategy Squared distance to an axis is the sum of the squares of the other two coordinates. Solve first for x^2, y^2, z^2 , then use the octant to determine signs.



Equations The data give

$$y^2 + z^2 = 41, \quad x^2 + z^2 = 34, \quad x^2 + y^2 = 25.$$

Adding the last two equations and subtracting the first yields $2x^2 = 18$, so $x^2 = 9$. Similarly,

$$2y^2 = 41 + 25 - 34 = 32, \quad 2z^2 = 41 + 34 - 25 = 50.$$

Thus $|x| = 3$, $|y| = 4$, and $|z| = 5$. The stated octant fixes the signs:

$$\boxed{P = (-3, 4, -5)}.$$

Verification The resulting axis distances are

$$\sqrt{4^2 + (-5)^2} = \sqrt{41}, \quad \sqrt{(-3)^2 + (-5)^2} = \sqrt{34}, \quad \sqrt{(-3)^2 + 4^2} = 5.$$

With only two distances, one squared coordinate would remain free, so the third is essential.

Question 2

A sphere passes through four specified points in three-dimensional space.

Given

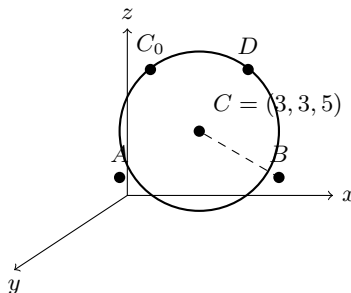
$$\begin{aligned}A &= (1, 0, 2), & B &= (5, 0, 2), \\C &= (1, 6, 2), & D &= (1, 0, 8).\end{aligned}$$

Tasks

1. Find the center and radius of the sphere.
2. Write its Cartesian equation.
3. Explain why the four given points determine a unique sphere.
4. Verify that all four points satisfy your equation.

Question 2 – Solution

Strategy A sphere's center is equidistant from all four points. Perpendicular-bisector conditions can be obtained by equating squared distances.



Center Let the center be $K = (h, k, \ell)$. From $d(A, K)^2 = d(B, K)^2$,

$$(h - 1)^2 + k^2 + (\ell - 2)^2 = (h - 5)^2 + k^2 + (\ell - 2)^2,$$

so $h = 3$. Comparing A with C gives $k = 3$, and comparing A with D gives $\ell = 5$. Hence the center is $(3, 3, 5)$ and

$$r^2 = (3 - 1)^2 + (3 - 0)^2 + (5 - 2)^2 = 22.$$

Therefore

$$\boxed{(x - 3)^2 + (y - 3)^2 + (z - 5)^2 = 22}.$$

Uniqueness and verification The three difference vectors from A point independently in the x -, y -, and z -directions, so their perpendicular-bisector planes meet in exactly one point. Substitution shows that each given point produces squared distance 22.

Question 3

Consider the fixed points

$$A = (-2, 0, 0), \quad B = (4, 0, 0),$$

and let $P = (x, y, z)$ satisfy the distance condition

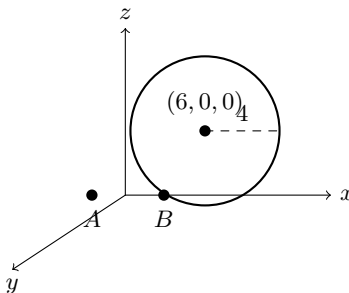
$$d(P, A) = 2d(P, B).$$

Tasks

1. Derive a Cartesian equation for the locus of all possible points P .
2. Identify the surface by stating its center and radius.
3. Determine which of A and B lies inside the surface.
4. Explain geometrically what the factor 2 requires of points on the locus.

Question 3 – Solution

Strategy Square the distance relation, expand, and complete the square. The resulting surface is an Apollonius sphere.



Derivation The condition is

$$(x + 2)^2 + y^2 + z^2 = 4((x - 4)^2 + y^2 + z^2).$$

After collecting terms and dividing by 3,

$$x^2 - 12x + y^2 + z^2 + 20 = 0.$$

Completing the square gives

$$\boxed{(x - 6)^2 + y^2 + z^2 = 16}.$$

Thus the locus is the sphere centered at $(6, 0, 0)$ with radius 4.

Interpretation Point B is 2 units from the center and lies inside; A is 8 units from the center and lies outside. A point must remain closer to B because its distance to A is required to be twice as large.

Verification At the axial endpoints $(2, 0, 0)$ and $(10, 0, 0)$, the distance pairs are $(4, 2)$ and $(12, 6)$, respectively, confirming the ratio.

Question 4

Let S be the set of points whose distance to the xy -plane is no greater than their distance to the z -axis:

$$S = \{P : d(P, xy\text{-plane}) \leq d(P, z\text{-axis})\}.$$

Tasks

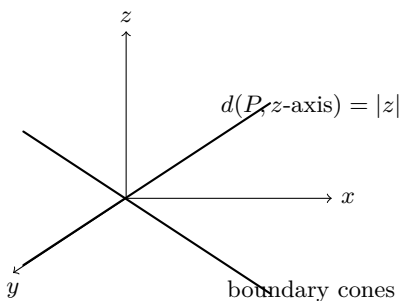
1. Derive a Cartesian inequality describing S .
2. Describe the boundary surface of S .
3. Determine whether each test point belongs to S :

$$P_1 = (1, 2, 3), \quad P_2 = (2, -2, 1).$$

4. Describe the reflectional and rotational symmetries of S .

Question 4 – Solution

Strategy Translate each geometric distance into coordinates: distance to the xy -plane is $|z|$, while distance to the z -axis is the radial distance $\sqrt{x^2 + y^2}$.



Inequality and boundary We require

$$|z| \leq \sqrt{x^2 + y^2} \iff \boxed{z^2 \leq x^2 + y^2}.$$

The boundary $z^2 = x^2 + y^2$ is a double cone with vertex at the origin. The set S is the region on or outside that cone in the radial direction.

Membership For $(1, 2, 3)$, $z^2 = 9 > 1^2 + 2^2 = 5$, so it is not in S . For $(2, -2, 1)$, $z^2 = 1 \leq 8$, so it is in S .

Symmetry and verification The inequality uses only squares, so changing any coordinate's sign preserves membership; rotations about the z -axis also preserve $x^2 + y^2$. On the xy -plane, every point satisfies it, while nonzero points on the z -axis do not, matching the distance interpretation.

Question 5

A rectangular laboratory is aligned with the coordinate axes and has opposite corners

$$O = (0, 0, 0), \quad V = (8, 6, 4).$$

A wireless transmitter will be installed at a point P inside the laboratory. Define

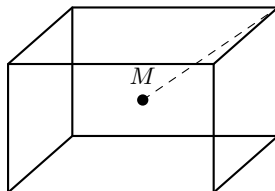
$$R(P) = \max\{d(P, C) : C \text{ is one of the eight corners}\}.$$

Design objective

1. Find the location of P that minimizes $R(P)$.
2. Find the minimum possible value of $R(P)$.
3. Prove that no other transmitter location can produce a smaller greatest distance.

Question 5 – Solution

Strategy Opposite corners force a lower bound: for any transmitter location, at least one of its distances to an opposite pair must be at least half the pair's separation. The box center attains that bound for every pair.



Lower bound The space diagonal has length

$$d(O, V) = \sqrt{8^2 + 6^2 + 4^2} = \sqrt{116} = 2\sqrt{29}.$$

For any location P , the triangle inequality gives

$$d(O, V) \leq d(O, P) + d(P, V).$$

Therefore at least one of $d(O, P)$ and $d(P, V)$ is at least $\sqrt{29}$.

Attaining the bound The midpoint of every space diagonal is

$$M = (4, 3, 2).$$

Each corner differs from M by coordinates $\pm 4, \pm 3, \pm 2$, so every corner is exactly

$$\sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$$

units away. Hence the optimal location is $\boxed{(4, 3, 2)}$, and the minimized greatest distance is $\boxed{\sqrt{29}}$.

Verification The lower bound and attained value agree, which proves global optimality rather than merely checking a plausible point.

Question 6

Two points determine a directed line segment:

$$A = (-4, 1, 7), \quad B = (6, -3, -1).$$

Tasks

1. Find the point P that divides $\overline{AB}$ internally in the ratio

$$AP : PB = 2 : 3.$$

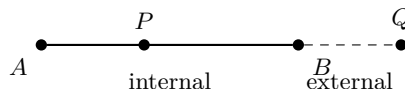
2. Find the point Q that divides the same line externally in the ratio

$$AQ : QB = 2 : 3.$$

3. Verify both ratios using three-dimensional distances, and distinguish clearly between internal and external division.

Question 6 – Solution

Strategy Parameterize the line by $X(t) = A + t(B - A)$. Internal division has $0 < t < 1$; external division requires t outside that interval.



Internal point Since $B - A = (10, -4, -8)$ and $AP/AB = 2/(2 + 3) = 2/5$,

$$P = A + \frac{2}{5}(B - A) = (-4, 1, 7) + (4, -8/5, -16/5) = \boxed{(0, -3/5, 19/5)}.$$

External point For $Q = A + t(B - A)$ beyond A , take $t < 0$. The ratio is

$$\frac{AQ}{QB} = \frac{-t}{1-t} = \frac{2}{3},$$

which gives $t = -2$. Thus

$$Q = A - 2(B - A) = \boxed{(-24, 9, 23)}.$$

Verification Because $P - A = (2/5)(B - A)$ and $B - P = (3/5)(B - A)$, their lengths have ratio $2 : 3$. Also $Q - A = -2(B - A)$ and $B - Q = 3(B - A)$, again giving $AQ : QB = 2 : 3$.

Question 7

Find every sphere satisfying all of the following conditions:

1. The sphere lies entirely in the octant

$$x \geq 0, \quad y \leq 0, \quad z \geq 0.$$

2. It is tangent to each of the three coordinate planes.

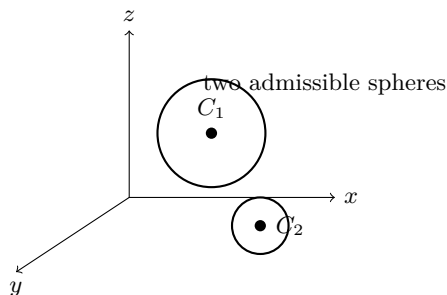
3. It passes through the point

$$P = (1, -2, 3).$$

For each sphere, state its center, radius, and Cartesian equation. Then give a geometric explanation for why two different spheres satisfy the same conditions.

Question 7 – Solution

Strategy Tangency to all three coordinate planes makes the center's coordinate magnitudes equal to the radius. The specified octant fixes their signs.



Center condition If the radius is $r > 0$, then the center must be

$$C = (r, -r, r).$$

Since P lies on the sphere,

$$(1 - r)^2 + (-2 + r)^2 + (3 - r)^2 = r^2.$$

Expanding gives $2r^2 - 12r + 14 = 0$, or

$$r^2 - 6r + 7 = 0.$$

Hence $r = 3 \pm \sqrt{2}$, both positive.

Results The two spheres are

$$(x - r)^2 + (y + r)^2 + (z - r)^2 = r^2, \quad r = 3 \pm \sqrt{2}.$$

Why two? Along the line of possible centers $(r, -r, r)$, the distance to P first decreases and then increases, so the equation “distance to center equals r ” can be met at two scales. Direct substitution verifies both roots, and each sphere has extreme coordinates touching—but not crossing—the three coordinate planes.

Question 8

A transformation T is performed in this order:

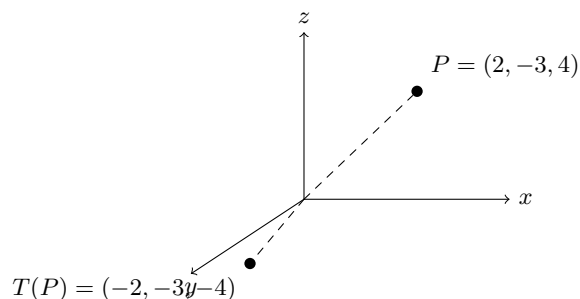
1. reflect across the yz -plane;
2. then reflect the result across the xy -plane.

Tasks

1. Find a coordinate formula for $T(x, y, z)$.
2. Compute $T(2, -3, 4)$.
3. Identify the single rigid motion equivalent to this composition.
4. Prove directly from the three-dimensional distance formula that T preserves the distance between every pair of points.

Question 8 – Solution

Strategy Track which coordinate changes sign under each reflection. Then compare squared distances before and after the transformation.



Coordinate rule Reflection across the yz -plane changes x to $-x$; reflection across the xy -plane changes z to $-z$. Therefore

$$\boxed{T(x, y, z) = (-x, y, -z)}, \quad T(2, -3, 4) = \boxed{(-2, -3, -4)}.$$

Geometric interpretation The y -coordinate is fixed while both perpendicular coordinates reverse. This is a rotation through 180° about the y -axis.

Distance preservation For $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$,

$$d(TP, TQ)^2 = (-x_1 + x_2)^2 + (y_1 - y_2)^2 + (-z_1 + z_2)^2 = d(P, Q)^2.$$

Both distances are nonnegative, so they are equal.

Verification Applying T twice restores every coordinate, as expected for a half-turn.

Question 9

A rectangular box has edges parallel to the coordinate axes and opposite vertices

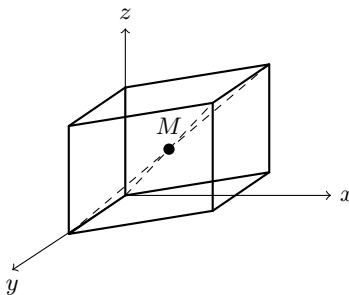
$$A = (1, -2, 3), \quad G = (7, 4, 13).$$

Tasks

1. List all eight vertices systematically.
2. Show that the four space diagonals have a common midpoint, and find it.
3. Find the center and radius of the sphere passing through all eight vertices.
4. Verify that every vertex lies on this sphere without substituting the vertices one at a time.

Question 9 – Solution

Strategy Each vertex independently chooses one of the two available values for each coordinate. Opposite vertices choose complementary values.



Vertices The coordinate choices are $x \in \{1, 7\}$, $y \in \{-2, 4\}$, $z \in \{3, 13\}$. Thus the vertices are

$$(1, -2, 3), (1, -2, 13), (1, 4, 3), (1, 4, 13),$$

$$(7, -2, 3), (7, -2, 13), (7, 4, 3), (7, 4, 13).$$

Common midpoint Every opposite pair averages the low and high value in each coordinate, giving

$$M = \left(\frac{1+7}{2}, \frac{-2+4}{2}, \frac{3+13}{2} \right) = \boxed{(4, 1, 8)}.$$

Hence all four space diagonals bisect one another at M .

Circumsphere From M to any vertex, the coordinate changes have magnitudes 3, 3, 5. Therefore

$$\boxed{r = \sqrt{3^2 + 3^2 + 5^2} = \sqrt{43}}.$$

Verification Every vertex has one of the sign patterns $(\pm 3, \pm 3, \pm 5)$ relative to M , so all eight lie on the sphere.

Question 10

A proposed sphere is required to satisfy all three conditions below:

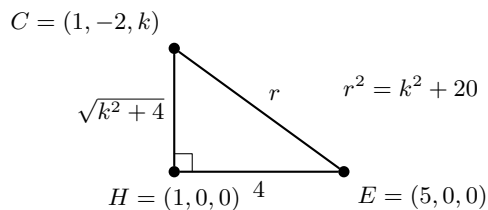
1. It intersects the x -axis at $A = (-3, 0, 0)$ and $B = (5, 0, 0)$.
2. Its center lies in the plane $y = -2$.
3. It is tangent to the xy -plane.

Consistency test

Determine whether such a sphere exists. If it does, find every possible center, radius, and equation. If it does not, prove precisely which requirements are incompatible.

Question 10 – Solution

Strategy The center lies on the perpendicular-bisector plane of the two axial intersection points. Tangency to the xy -plane relates the radius to the center's height.



Center coordinates The midpoint of the two intersections has x -coordinate 1, so write the center as

$$C = (1, -2, k).$$

The squared radius, using $(5, 0, 0)$, is

$$r^2 = (5 - 1)^2 + (0 + 2)^2 + k^2 = 20 + k^2.$$

Tangency to the xy -plane requires $r = |k|$, hence $r^2 = k^2$. Combining these equations would give $k^2 = 20 + k^2$, an impossibility.

Conclusion Therefore no sphere satisfies all the stated conditions.

Consistency review The two axial intersection points already force every possible center onto $x = 1$. Requiring $y = -2$ means the center has a nonzero horizontal offset from the x -axis. Consequently the radius needed to reach either axial point is strictly greater than $|k|$, while tangency to the xy -plane requires it to equal $|k|$. The contradiction is geometric, not an algebraic accident.

Question 1

A line passes through the points

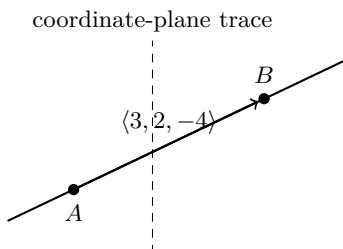
$$A = (-2, 1, 3), \quad B = (4, 5, -5).$$

Tasks

1. Write the line in vector, parametric, and symmetric forms using a primitive integer direction vector.
2. Find the line's intersection with each coordinate plane.
3. Verify that the three plane-intersection points lie on your line.

Question 1 – Solution

Strategy Subtract the given points to obtain a direction, reduce it, and set one coordinate at a time equal to zero to find the coordinate-plane traces.



Line equations Since $B - A = \langle 6, 4, -8 \rangle = 2 \langle 3, 2, -4 \rangle$,

$$\vec{r} = \langle -2, 1, 3 \rangle + t \langle 3, 2, -4 \rangle,$$

$$x = -2 + 3t, \quad y = 1 + 2t, \quad z = 3 - 4t,$$

$$\frac{x + 2}{3} = \frac{y - 1}{2} = \frac{z - 3}{-4}.$$

Coordinate-plane traces Setting $z = 0$, $x = 0$, and $y = 0$ gives, respectively,

$$L \cap xy = \left(\frac{1}{4}, \frac{5}{2}, 0 \right), \quad L \cap yz = \left(0, \frac{7}{3}, \frac{1}{3} \right), \quad L \cap xz = \left(-\frac{7}{2}, 0, 5 \right).$$

Verification These correspond to $t = 3/4$, $2/3$, and $-1/2$ in the same parametric equations, so all three points lie on the line.

Question 2

Consider the line

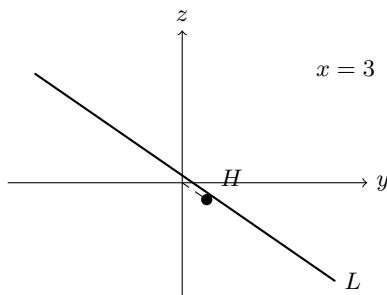
$$L : \quad x = 3, \quad \frac{y+2}{4} = \frac{z-1}{-3}.$$

Tasks

1. Write vector and parametric equations for L .
2. Explain geometrically what the equation $x = 3$ says about the line.
3. Find the point on L closest to the origin and determine that minimum distance.

Question 2 – Solution

Strategy Use the common symmetric ratio as a parameter. For the closest point, minimize the squared distance to avoid unnecessary square roots.



Representations Let the common ratio equal t . Then

$$\vec{r} = \langle 3, -2, 1 \rangle + t \langle 0, 4, -3 \rangle, \quad x = 3, \quad y = -2 + 4t, \quad z = 1 - 3t.$$

The zero x -component of the direction means the entire line lies in the plane $x = 3$ and is parallel to the yz -plane.

Closest point The squared distance from the origin is

$$D^2(t) = 9 + (-2 + 4t)^2 + (1 - 3t)^2 = 25t^2 - 22t + 14.$$

Its minimum occurs at $t = 22/50 = 11/25$. Hence

$$H = \left(3, -\frac{6}{25}, -\frac{8}{25} \right), \quad d(O, L) = \frac{\sqrt{229}}{5}.$$

Verification The vector H from the origin has dot product 0 with $\langle 0, 4, -3 \rangle$, confirming that OH is perpendicular to L .

Question 3

Points

$$A = (1, -2, 3), \quad B = (5, 4, -1), \quad P = (4, u, v)$$

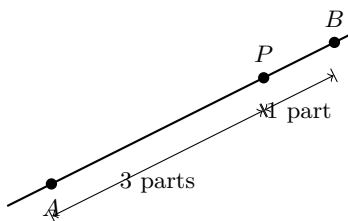
are collinear, and P lies between A and B .

Tasks

1. Determine u and v .
2. Write a parametric equation of the common line.
3. Find the ratio $AP : PB$ and verify it using distances.

Question 3 – Solution

Strategy Parameterize the segment from A to B . The known x -coordinate of P determines the parameter, after which the other coordinates and division ratio follow.



Parameter and coordinates Since $B - A = \langle 4, 6, -4 \rangle$,

$$\vec{r}(t) = \langle 1, -2, 3 \rangle + t \langle 4, 6, -4 \rangle.$$

At P , $1 + 4t = 4$, so $t = 3/4$. Therefore

$$u = -2 + 6\left(\frac{3}{4}\right) = \frac{5}{2}, \quad v = 3 - 4\left(\frac{3}{4}\right) = 0,$$

and

$$\boxed{P = \left(4, \frac{5}{2}, 0\right)}.$$

Division ratio The parameter places P three-fourths of the way from A to B , so

$$\boxed{AP : PB = 3 : 1}.$$

Verification We have $P - A = \frac{3}{4}(B - A)$ and $B - P = \frac{1}{4}(B - A)$; taking lengths preserves the ratio $3 : 1$.

Question 4

For a real parameter k , consider

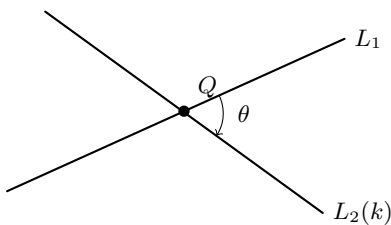
$$\begin{aligned}L_1 : \vec{r} &= \langle 1, 2, 0 \rangle + s \langle 2, -1, 3 \rangle, \\L_2(k) : \vec{r} &= \langle 4, -2, 7 \rangle + t \langle 1, k, -1 \rangle.\end{aligned}$$

Tasks

1. Find the value of k for which the lines intersect.
2. Find the intersection point.
3. For that value of k , find the acute angle between the lines.

Question 4 – Solution

Strategy The x - and z -coordinate equations determine s and t without k ; the y -coordinate then determines the required parameter.



Intersection Equating x - and z -coordinates gives

$$1 + 2s = 4 + t, \quad 3s = 7 - t.$$

The unique solution is $s = 2$, $t = 1$. The y -coordinates then require

$$2 - s = -2 + kt \implies 0 = -2 + k,$$

so $k = 2$. Both lines meet at $Q = (5, 0, 6)$.

Angle The directions are $\vec{d}_1 = \langle 2, -1, 3 \rangle$ and $\vec{d}_2 = \langle 1, 2, -1 \rangle$. Thus

$$\cos \theta = \frac{|\vec{d}_1 \cdot \vec{d}_2|}{\|\vec{d}_1\| \|\vec{d}_2\|} = \frac{3}{\sqrt{14}\sqrt{6}} = \frac{3}{2\sqrt{21}}.$$

Hence $\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right) \approx 70.9^\circ$.

Verification Substitution of $s = 2$, $t = 1$, and $k = 2$ produces $(5, 0, 6)$ in both equations.

Question 5

Let

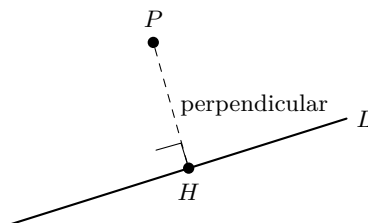
$$P = (4, 3, -1), \quad L : \vec{r} = \langle 1, -1, 2 \rangle + t \langle 2, 1, 2 \rangle.$$

Tasks

1. Find the point H on L closest to P .
2. Write an equation of the line through P that intersects L perpendicularly at H .
3. Find the distance from P to L .

Question 5 – Solution

Strategy At the closest point, the displacement from the line to P is perpendicular to the line's direction vector.



Perpendicular foot Write $A = (1, -1, 2)$ and $\vec{d} = \langle 2, 1, 2 \rangle$. For $H = A + t\vec{d}$,

$$(P - H) \cdot \vec{d} = 0 \implies (P - A - t\vec{d}) \cdot \vec{d} = 0.$$

Since $(P - A) \cdot \vec{d} = 4$ and $\vec{d} \cdot \vec{d} = 9$, $t = 4/9$. Therefore

$$H = \left(\frac{17}{9}, -\frac{5}{9}, \frac{26}{9} \right).$$

Perpendicular line and distance A direction from P to H is $\langle -19, -32, 35 \rangle$ after multiplying by 9. Thus

$$\vec{r} = \langle 4, 3, -1 \rangle + u \langle -19, -32, 35 \rangle.$$

Also,

$$d(P, L) = \|P - H\| = \frac{\sqrt{2610}}{9} = \frac{\sqrt{290}}{3}.$$

Verification $\langle -19, -32, 35 \rangle \cdot \langle 2, 1, 2 \rangle = -38 - 32 + 70 = 0$, and H satisfies the original line equation.

Question 6

Consider the two lines

$$L_1 : \vec{r} = s \langle 1, 2, 0 \rangle ,$$

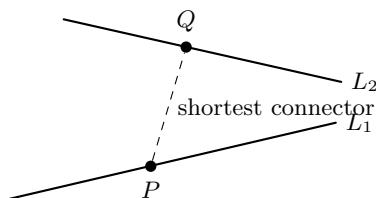
$$L_2 : \vec{r} = \langle 1, 0, 3 \rangle + t \langle 2, -1, 1 \rangle .$$

Tasks

1. Show that the lines are skew.
2. Find the point on each line at which their shortest connecting segment begins and ends.
3. Determine the distance between the lines.

Question 6 – Solution

Strategy For closest points $P \in L_1$ and $Q \in L_2$, the connector $Q - P$ must be perpendicular to both line directions.



Classification The directions $\langle 1, 2, 0 \rangle$ and $\langle 2, -1, 1 \rangle$ are not parallel. Solving the first two intersection equations gives $s = 1/5$, $t = -2/5$, but the third coordinates are then 0 and $13/5$, so the lines do not intersect. They are skew.

Closest points Let

$$P = (s, 2s, 0), \quad Q = (1 + 2t, -t, 3 + t).$$

The conditions $(Q - P) \cdot \langle 1, 2, 0 \rangle = 0$ and $(Q - P) \cdot \langle 2, -1, 1 \rangle = 0$ yield

$$s = \frac{1}{5}, \quad t = -\frac{5}{6}.$$

Thus

$$\boxed{P = \left(\frac{1}{5}, \frac{2}{5}, 0\right)}, \quad \boxed{Q = \left(-\frac{2}{3}, \frac{5}{6}, \frac{13}{6}\right)}.$$

Distance and verification Since $Q - P = \frac{13}{30} \langle -2, 1, 5 \rangle$,

$$\boxed{d(L_1, L_2) = \frac{13\sqrt{30}}{30} = \frac{13}{\sqrt{30}}}.$$

The vector $\langle -2, 1, 5 \rangle$ has zero dot product with both line directions.

Question 7

A line through

$$P = (-1, 2, 0)$$

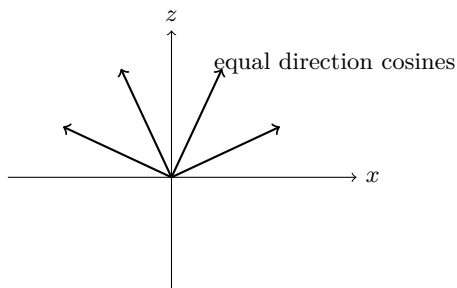
makes equal acute angles with the x -, y -, and z -axes (treated as unoriented lines). Here the first octant includes its boundary planes.

Tasks

1. Find all geometrically distinct lines through P having this property.
2. Determine which of these lines contain at least one point in the first octant.
3. State the parameter intervals corresponding to the portions lying in the first octant.

Question 7 – Solution

Strategy Equal inclination means the three direction cosines have equal absolute value. Direction vectors that differ only by an overall nonzero factor define the same line.



All lines Up to reversal, there are four sign patterns. The distinct lines are

$$\vec{r} = \langle -1, 2, 0 \rangle + t\vec{d}$$

where

$$\vec{d} \in \{ \langle 1, 1, 1 \rangle, \langle 1, 1, -1 \rangle, \langle 1, -1, 1 \rangle, \langle 1, -1, -1 \rangle \}.$$

Each direction has direction cosines with absolute value $1/\sqrt{3}$.

First-octant test For $\langle 1, 1, 1 \rangle$, the coordinates are $(-1 + t, 2 + t, t)$, all nonnegative when $t \geq 1$. For $\langle 1, -1, 1 \rangle$, they are $(-1 + t, 2 - t, t)$, all nonnegative when $1 \leq t \leq 2$.

For $\langle 1, 1, -1 \rangle$ and $\langle 1, -1, -1 \rangle$, the requirements $x \geq 0$ and $z \geq 0$ force incompatible signs of t . Thus exactly the first and third listed lines enter the first octant.

Verification The stated intervals satisfy all three coordinate inequalities, including the boundary planes.

Question 8

Two lines intersect at $Q = (1, -2, 3)$ and have direction vectors

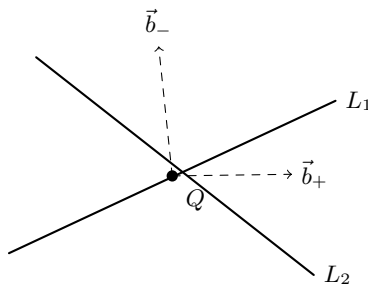
$$\vec{d}_1 = \langle 3, 0, 4 \rangle, \quad \vec{d}_2 = \langle 0, 4, 3 \rangle.$$

Tasks

1. Find equations of both angle-bisector lines through Q .
2. Determine which bisector corresponds to the acute angle between the given lines.
3. Prove that the two bisector lines are perpendicular.

Question 8 – Solution

Strategy Angle-bisector directions are the sum and difference of unit direction vectors. Here the given vectors already have equal length.



Bisector directions Since $\|\vec{d}_1\| = \|\vec{d}_2\| = 5$, the sum and difference give

$$\vec{b}_+ = \langle 3, 4, 7 \rangle, \quad \vec{b}_- = \langle 3, -4, 1 \rangle.$$

Therefore the bisectors are

$$\boxed{\vec{r} = \langle 1, -2, 3 \rangle + t \langle 3, 4, 7 \rangle}, \quad \boxed{\vec{r} = \langle 1, -2, 3 \rangle + u \langle 3, -4, 1 \rangle}.$$

Acute-angle bisector Because $\vec{d}_1 \cdot \vec{d}_2 = 12 > 0$, the directions form an acute angle. Their sum $\vec{b}_+$ lies between them and bisects that acute angle; $\vec{b}_-$ bisects the supplementary angle.

Perpendicularity Directly,

$$\vec{b}_+ \cdot \vec{b}_- = 3(3) + 4(-4) + 7(1) = 9 - 16 + 7 = 0.$$

Hence the two angle-bisector lines are perpendicular.

Verification Dotting $\vec{b}_+$ with each original direction gives the same normalized cosine, confirming equal angles.

Question 9

A line intersects the xy -plane at

$$A = (2, -1, 0)$$

and the yz -plane at

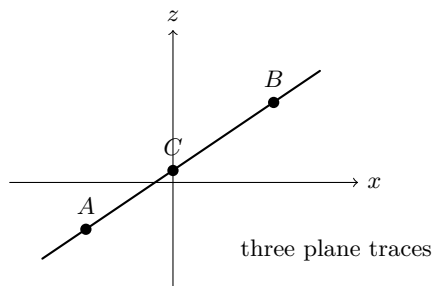
$$B = (0, 3, 4).$$

Tasks

1. Write vector, parametric, and symmetric equations of the line.
2. Find its intersection with the xz -plane.
3. Determine the ratio in which the xz -plane intersection divides $\overline{AB}$.

Question 9 – Solution

Strategy The two traces determine the line. Parameterize from A toward B , then impose $y = 0$ for the xz -plane.



Line equations A direction is $B - A = \langle -2, 4, 4 \rangle = 2 \langle -1, 2, 2 \rangle$. Thus

$$\vec{r} = \langle 2, -1, 0 \rangle + t \langle -1, 2, 2 \rangle,$$

$$x = 2 - t, \quad y = -1 + 2t, \quad z = 2t,$$

and

$$\frac{x - 2}{-1} = \frac{y + 1}{2} = \frac{z}{2}.$$

xz -plane trace Set $y = 0$. Then $-1 + 2t = 0$, so $t = 1/2$ and

$$C = \left(\frac{3}{2}, 0, 1 \right).$$

Division ratio In this reduced parameterization, B occurs at $t = 2$. Thus C lies one-half unit from A in parameter and three-halves units from B , giving

$$AC : CB = 1 : 3.$$

Verification C has $y = 0$, and substituting its coordinates into the symmetric equation gives the common value $1/2$.

Question 10

Two particles move for $t \geq 0$ according to

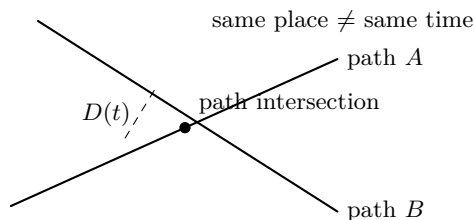
$$\begin{aligned}\vec{r}_A(t) &= \langle 0, 1, 2 \rangle + t \langle 2, -1, 1 \rangle, \\ \vec{r}_B(t) &= \langle 6, -2, 5 \rangle + t \langle -1, 1, 0 \rangle.\end{aligned}$$

Tasks

1. Determine whether their geometric paths intersect.
2. Determine whether the particles collide.
3. Find the time and value of their minimum separation for $t \geq 0$.

Question 10 – Solution

Strategy Path intersection allows different parameters, whereas collision uses the same time. Minimize the squared norm of the relative-position vector.



Paths versus collision Solving $\vec{r}_A(s) = \vec{r}_B(u)$ gives $s = 3$, $u = 0$, so the paths intersect at $\boxed{(6, -2, 5)}$. The particles reach that point at different times and therefore do not collide. Equating positions at the same t would require all components of the relative vector below to vanish, which never occurs.

Separation At a common time t ,

$$\vec{r}_A(t) - \vec{r}_B(t) = \langle -6 + 3t, 3 - 2t, -3 + t \rangle.$$

Hence

$$D^2(t) = (-6 + 3t)^2 + (3 - 2t)^2 + (-3 + t)^2 = 14t^2 - 54t + 54.$$

The vertex occurs at

$$t = \frac{54}{28} = \frac{27}{14} > 0.$$

Substitution gives $D_{\min}^2 = 27/14$, so

$$\boxed{D_{\min} = \sqrt{\frac{27}{14}} = \frac{3\sqrt{42}}{14}}.$$

Verification The quadratic opens upward, and the minimizing time lies in the allowed domain. The nonzero minimum confirms independently that no collision occurs.

Question 1

A plane passes through

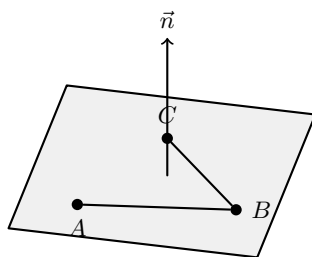
$$A = (1, 0, 2), \quad B = (3, 1, -1), \quad C = (-1, 4, 0).$$

Tasks

1. Derive a Cartesian equation of the plane from two in-plane vectors.
2. Write a unit normal vector.
3. Find all three coordinate-axis intercepts and verify the three given points.

Question 1 – Solution

Strategy Form $\overrightarrow{AB}$ and $\overrightarrow{AC}$; their cross product is normal to the plane.



Step 1: In-plane vectors Subtract coordinates component by component:

$$\begin{aligned}\overrightarrow{AB} &= B - A = \langle 3 - 1, 1 - 0, -1 - 2 \rangle = \langle 2, 1, -3 \rangle, \\ \overrightarrow{AC} &= C - A = \langle -1 - 1, 4 - 0, 0 - 2 \rangle = \langle -2, 4, -2 \rangle.\end{aligned}$$

Step 2: Normal vector Their cross product is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1(-2) - (-3)(4), (-3)(-2) - 2(-2), 2(4) - 1(-2) \rangle = \langle 10, 10, 10 \rangle.$$

Divide by 10 to use the primitive normal $\vec{n} = \langle 1, 1, 1 \rangle$. The point-normal equation through A is

$$(x - 1) + (y - 0) + (z - 2) = 0.$$

Therefore

$$\boxed{x + y + z = 3}, \quad \boxed{\hat{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle}.$$

Step 3: Intercepts On the x -axis, $y = z = 0$, so $x = 3$. The other axes work similarly:

$$\boxed{(3, 0, 0), (0, 3, 0), (0, 0, 3)}.$$

Verification Substitution of A , B , and C gives 3 each time. The two in-plane vectors dot to zero with $\langle 1, 1, 1 \rangle$, independently confirming the orientation.

Question 2

A plane contains the line

$$L : \vec{r} = \langle 1, -2, 0 \rangle + t \langle 2, 1, -1 \rangle$$

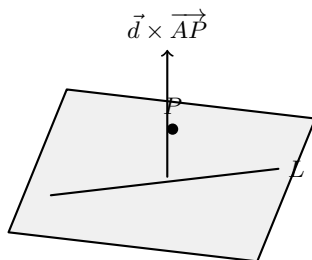
and the point $P = (0, 1, 4)$.

Tasks

1. Find a Cartesian equation of the plane.
2. Explain why the data determine a unique plane.
3. Verify that the entire line and the point lie in the plane.

Question 2 – Solution

Strategy Combine the line direction with a vector from a point on the line to P . Their cross product is a plane normal.



Step 1: Two directions in the plane Take $A = (1, -2, 0)$ from the line. Its direction is $\vec{d} = \langle 2, 1, -1 \rangle$, while

$$\overrightarrow{AP} = P - A = \langle 0 - 1, 1 - (-2), 4 - 0 \rangle = \langle -1, 3, 4 \rangle.$$

Step 2: Find a normal Compute

$$\vec{d} \times \overrightarrow{AP} = \langle 1(4) - (-1)(3), (-1)(-1) - 2(4), 2(3) - 1(-1) \rangle = \langle 7, -7, 7 \rangle.$$

Thus $\vec{n} = \langle 1, -1, 1 \rangle$. Using point A ,

$$(x - 1) - (y + 2) + (z - 0) = 0,$$

so the plane is

$$\boxed{x - y + z = 3}.$$

Uniqueness The vectors $\vec{d}$ and $\overrightarrow{AP}$ are not parallel, so they span exactly one plane.

Verification Point P gives $0 - 1 + 4 = 3$. A general point on L is $(1 + 2t, -2 + t, -t)$, and

$$(1 + 2t) - (-2 + t) - t = 3$$

for every t , proving that the whole line is contained in the plane.

Question 3

For a real parameter k , let

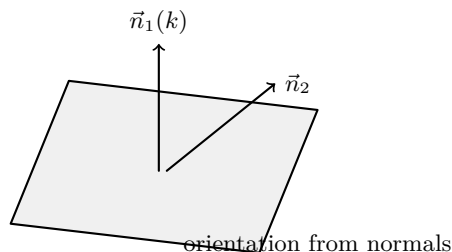
$$\Pi_k : kx + 2y - z = 5, \quad \Sigma : 2x - y + 2z = 7.$$

Tasks

1. Find k if the planes are perpendicular.
2. Determine whether any k makes the planes parallel.
3. Find every k for which the acute angle between the planes is 60° .

Question 3 – Solution

Strategy Plane orientation is controlled by the normals $\vec{n}_k = \langle k, 2, -1 \rangle$ and $\vec{m} = \langle 2, -1, 2 \rangle$.



Step 1: Perpendicularity Calculate

$$\vec{n}_k \cdot \vec{m} = k(2) + 2(-1) + (-1)(2) = 2k - 4.$$

Perpendicular normals require $2k - 4 = 0$, hence $\boxed{k = 2}$.

Step 2: Parallelism Parallelism would require $\langle k, 2, -1 \rangle = \lambda \langle 2, -1, 2 \rangle$. From $2 = -\lambda$, $\lambda = -2$; but the third coordinate would require $-1 = 2(-2) = -4$, a contradiction. Hence $\boxed{\text{no real } k \text{ makes them parallel}}$.

Step 3: The 60° condition The angle formula gives

$$\frac{|2k - 4|}{3\sqrt{k^2 + 5}} = \frac{1}{2}.$$

Cross-multiply and square:

$$2|2k - 4| = 3\sqrt{k^2 + 5} \implies 16(k - 2)^2 = 9(k^2 + 5).$$

Expanding gives $16k^2 - 64k + 64 = 9k^2 + 45$, so

$$7k^2 - 64k + 19 = 0.$$

The quadratic formula yields

$$k = \frac{64 \pm \sqrt{64^2 - 4(7)(19)}}{14} = \frac{64 \pm 18\sqrt{11}}{14},$$

and therefore

$$\boxed{k = \frac{32 \pm 9\sqrt{11}}{7}}.$$

Verification Neither root makes the original denominator zero, and substitution into the unsquared equation gives the positive value $1/2$, so no extraneous root remains.

Question 4

Consider the planes

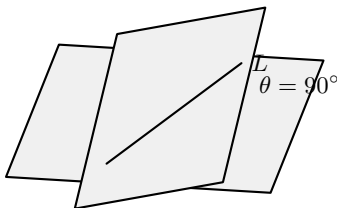
$$\Pi_1 : x + y - z = 2, \quad \Pi_2 : 2x - y + z = 1.$$

Tasks

1. Find a vector and parametric equation for their line of intersection.
2. Find the dihedral angle between the planes.
3. Verify the line equation in both plane equations.

Question 4 – Solution

Strategy The intersection direction is perpendicular to both normals. Solve the two plane equations for one point and one free parameter.



Step 1: Solve simultaneously Add the equations:

$$(x + y - z) + (2x - y + z) = 2 + 1 \implies 3x = 3 \implies x = 1.$$

Substitute $x = 1$ into $x + y - z = 2$:

$$1 + y - z = 2 \implies y = 1 + z.$$

Let the free coordinate be $z = t$. Then $y = 1 + t$, so

$$\boxed{\vec{r} = \langle 1, 1, 0 \rangle + t \langle 0, 1, 1 \rangle}.$$

Step 2: Confirm the direction The normals are $\vec{n}_1 = \langle 1, 1, -1 \rangle$ and $\vec{n}_2 = \langle 2, -1, 1 \rangle$. Their cross product is $\langle 0, -3, -3 \rangle$, parallel to $\langle 0, 1, 1 \rangle$.

Step 3: Angle Compute $\vec{n}_1 \cdot \vec{n}_2 = 2 - 1 - 1 = 0$. Therefore

$$\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|} = 0, \quad \boxed{\theta = 90^\circ}.$$

Verification Substitution of $(1, 1 + t, t)$ gives 2 in Π_1 and 1 in Π_2 for every t .

Question 5

Let $P = (4, -1, 5)$ and

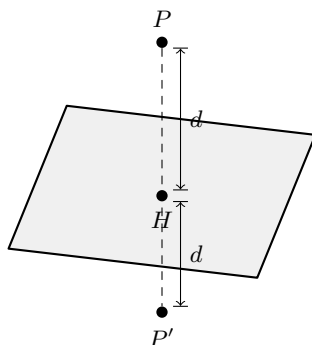
$$\Pi : 2x - y + 2z = 3.$$

Tasks

1. Find the perpendicular projection H of P onto Π .
2. Find the distance from P to Π .
3. Reflect P across Π and find the image point P' .
4. Verify the midpoint and perpendicularity conditions.

Question 5 – Solution

Strategy Move from P along the normal $\vec{n} = \langle 2, -1, 2 \rangle$. The signed equation value determines the required multiple.



Step 1: Signed displacement Write the plane as $F(x, y, z) = 2x - y + 2z - 3 = 0$. At P ,

$$2(4) - (-1) + 2(5) - 3 = 16, \quad \|\vec{n}\|^2 = 9.$$

Step 2: Projection A point on the perpendicular through P is $P - s\vec{n}$. To reach the plane,

$$F(P - s\vec{n}) = F(P) - s\|\vec{n}\|^2 = 16 - 9s = 0,$$

so $s = 16/9$. Hence

$$H = P - \frac{16}{9}\vec{n} = \left(4 - \frac{32}{9}, -1 + \frac{16}{9}, 5 - \frac{32}{9}\right) = \boxed{\left(\frac{4}{9}, \frac{7}{9}, \frac{13}{9}\right)}.$$

Step 3: Distance Since $P - H = (16/9)\vec{n}$,

$$d(P, \Pi) = \frac{16}{9}\|\vec{n}\| = \frac{16}{9}(3), \quad \boxed{d(P, \Pi) = \frac{16}{3}}.$$

Step 4: Reflection Since H is the midpoint of PP' ,

$$P' = 2H - P = P - \frac{32}{9}\vec{n} = \boxed{\left(-\frac{28}{9}, \frac{23}{9}, -\frac{19}{9}\right)}.$$

Verification Substituting H into the plane gives 3. Also $P - H = (16/9)\vec{n}$, so PH is perpendicular to the plane, and direct averaging of P and P' gives H .

Question 6

Among all points on the plane

$$\Pi : x + 2y + 2z = 9,$$

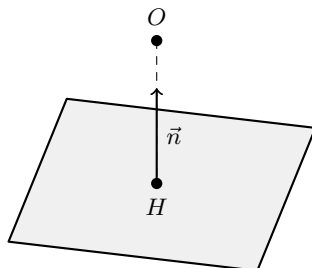
find the point closest to the origin.

Tasks

1. Determine the closest point and minimum distance.
2. Prove global minimality without using calculus.
3. Find the sphere centered at the origin tangent to Π .

Question 6 – Solution

Strategy The shortest segment from a point to a plane follows the plane's normal. Use Cauchy–Schwarz for a global proof.



Step 1: Follow the normal The normal is $\vec{n} = \langle 1, 2, 2 \rangle$ and

$$\|\vec{n}\| = \sqrt{1^2 + 2^2 + 2^2} = 3.$$

A point on the normal line through the origin has form $(t, 2t, 2t)$. Imposing the plane equation gives

$$t + 2(2t) + 2(2t) = 9 \implies 9t = 9 \implies t = 1.$$

Hence

$$\boxed{H = (1, 2, 2)}, \quad \boxed{d(O, \Pi) = \|H\| = 3}.$$

Step 2: Global proof For any $X = (x, y, z)$ on Π , $\vec{n} \cdot X = 9$. Cauchy–Schwarz gives

$$9 = |\vec{n} \cdot X| \leq \|\vec{n}\| \|X\| = 3\|X\|,$$

so $\|X\| \geq 3$. Equality occurs precisely when X is parallel to $\vec{n}$, as H is.

Step 3: Tangent sphere A sphere centered at the origin and reaching H has radius 3, so its equation is

$$\boxed{x^2 + y^2 + z^2 = 9}.$$

Its radius to H is normal to Π , so the plane is tangent there.

Question 7

Let

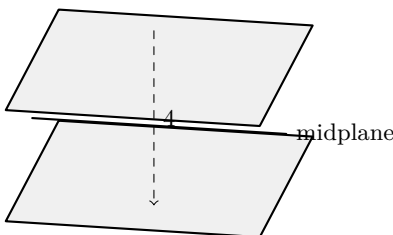
$$\Pi_1 : 2x - 2y + z = 7, \quad \Pi_2 : 2x - 2y + z = -5.$$

Tasks

1. Find the distance between the parallel planes.
2. Derive the plane exactly halfway between them.
3. Find endpoints of one shortest segment joining the planes and verify its midpoint lies on the halfway plane.

Question 7 – Solution

Strategy Put the equations over the same normal. Their constant difference divided by the normal length is the separation.



Step 1: Distance Both planes have normal $\vec{n} = \langle 2, -2, 1 \rangle$, whose length is

$$\|\vec{n}\| = \sqrt{2^2 + (-2)^2 + 1^2} = 3.$$

Because their left sides are identical, the distance is the absolute difference of constants divided by $\|\vec{n}\|$:

$$d(\Pi_1, \Pi_2) = \frac{|7 - (-5)|}{3} = 4.$$

Step 2: Halfway plane A parallel plane has form $2x - 2y + z = c$. Equal distance from c to 7 and -5 requires

$$7 - c = c - (-5) \implies 2c = 2 \implies c = 1.$$

Thus

$$2x - 2y + z = 1.$$

Step 3: Shortest connector Choose $A = (3, 0, 1)$; indeed $2(3) - 2(0) + 1 = 7$. The unit normal is $\vec{n}/3$, so moving four units toward Π_2 gives

$$B = A - \frac{4}{3} \langle 2, -2, 1 \rangle = \left(\frac{1}{3}, \frac{8}{3}, -\frac{1}{3} \right).$$

Substitution gives $2(1/3) - 2(8/3) - 1/3 = -5$, so $B \in \Pi_2$. Its midpoint with A is

$$M = \left(\frac{5}{3}, \frac{4}{3}, \frac{1}{3} \right).$$

Verification $2(5/3) - 2(4/3) + 1/3 = 1$. Also $B - A$ is parallel to the common normal, so AB is perpendicular to both planes and is therefore a shortest connector.

Question 8

All planes containing the intersection of

$$\Pi_1 : x + y + z = 1 \quad \text{and} \quad \Pi_2 : x - y + 2z = 3$$

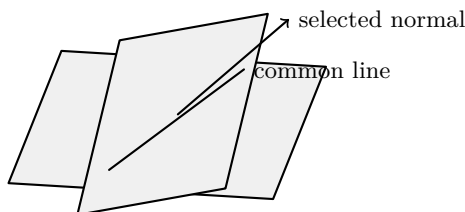
form a pencil of planes.

Tasks

1. Write a one-parameter equation for this pencil.
2. Find the member perpendicular to $\Sigma : x + 2y - z = 4$.
3. Verify that the selected plane contains the entire common line of Π_1 and Π_2 .

Question 8 – Solution

Strategy Every combination $F_1 + \lambda F_2 = 0$ vanishes wherever both original plane expressions vanish. Choose λ using normal-vector perpendicularity.



Step 1: Build the pencil Move all terms to the left:

$$F_1 = x + y + z - 1, \quad F_2 = x - y + 2z - 3.$$

At every point on the common line, $F_1 = F_2 = 0$. Therefore every equation

$$F_1 + \lambda F_2 = 0 \quad (\lambda \in \mathbb{R}), \quad \text{together with } F_2 = 0$$

contains that line. The member $F_2 = 0$ is also needed for the full pencil; its normal has dot product -3 with $\langle 1, 2, -1 \rangle$, so it is not the requested perpendicular member. Collecting coefficients shows its normal is

$$\vec{n}_\lambda = \langle 1 + \lambda, 1 - \lambda, 1 + 2\lambda \rangle.$$

Step 2: Impose perpendicularity The normal of Σ is $\vec{m} = \langle 1, 2, -1 \rangle$. Perpendicular planes have perpendicular normals, so

$$\vec{n}_\lambda \cdot \langle 1, 2, -1 \rangle = 2 - 3\lambda = 0,$$

because $(1 + \lambda) + 2(1 - \lambda) - (1 + 2\lambda) = 2 - 3\lambda$. Thus $\lambda = 2/3$. Multiplying by 3 gives

$$3F_1 + 2F_2 = 0,$$

and expanding yields

$$5x + y + 7z = 9.$$

Verification Any point common to Π_1 and Π_2 has $F_1 = F_2 = 0$, so it satisfies $3F_1 + 2F_2 = 0$. Its normal $\langle 5, 1, 7 \rangle$ dots with $\langle 1, 2, -1 \rangle$ to 0.

Question 9

Find the locus of points equidistant from the intersecting planes

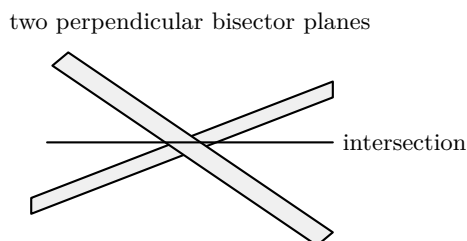
$$\Pi_1 : x - 2y + 2z - 3 = 0, \quad \Pi_2 : 2x + 2y - z + 6 = 0.$$

Tasks

1. Derive equations for both components of the locus.
2. Explain why they are angle-bisector planes.
3. Prove that the two resulting planes are perpendicular.

Question 9 – Solution

Strategy Use the point-to-plane distance formula with absolute values. Equal normal lengths simplify the two sign cases.



Step 1: Distance equation The normal lengths are

$$\sqrt{1^2 + (-2)^2 + 2^2} = 3, \quad \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

Thus equality of point-to-plane distances becomes

$$|x - 2y + 2z - 3| = |2x + 2y - z + 6|.$$

Step 2: Resolve the absolute values Let the two expressions be F_1 and F_2 . The equation $|F_1| = |F_2|$ gives two cases.

If $F_1 = F_2$, then

$$x - 2y + 2z - 3 = 2x + 2y - z + 6 \implies x + 4y - 3z + 9 = 0.$$

If $F_1 = -F_2$, then

$$x - 2y + 2z - 3 = -2x - 2y + z - 6 \implies 3x + z + 3 = 0.$$

Therefore the locus consists of

$$\boxed{x + 4y - 3z + 9 = 0}, \quad \boxed{3x + z + 3 = 0}.$$

Interpretation On either plane, the signed perpendicular displacements from Π_1 and Π_2 have equal magnitude. These are precisely the internal and external angle-bisector planes, and both contain the original planes' intersection line.

Perpendicularity Normals may be chosen as $\langle 1, 4, -3 \rangle$ and $\langle 3, 0, 1 \rangle$. Their dot product is $3 - 3 = 0$, so the bisector planes are perpendicular.

Verification Substituting any point from either boxed plane into the absolute-value equation recovers equal distances.

Question 10

A plane cuts the positive coordinate axes at A , B , and C . The centroid of the intercept triangle ABC is

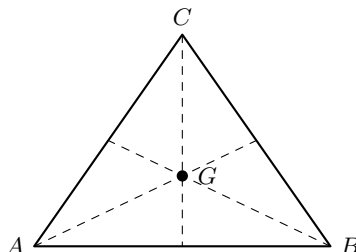
$$G = (2, 3, 4).$$

Tasks

1. Determine A , B , and C .
2. Find the plane in intercept and Cartesian forms.
3. Find the area of triangle ABC .
4. Find the volume of the tetrahedron bounded by the plane and the coordinate planes.

Question 10 – Solution

Strategy Write $A = (a, 0, 0)$, $B = (0, b, 0)$, and $C = (0, 0, c)$. Their centroid directly determines the three intercepts.



Step 1: Recover the intercepts Write $A = (a, 0, 0)$, $B = (0, b, 0)$, and $C = (0, 0, c)$. Then

$$G = \frac{A + B + C}{3} = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right),$$

so $a/3 = 2$, $b/3 = 3$, and $c/3 = 4$. Hence

$$A = (6, 0, 0), \quad B = (0, 9, 0), \quad C = (0, 0, 12).$$

Step 2: Plane equation A plane with intercepts a, b, c has equation $x/a + y/b + z/c = 1$. Therefore

$$\frac{x}{6} + \frac{y}{9} + \frac{z}{12} = 1, \quad 6x + 4y + 3z = 36.$$

Step 3: Triangle area Compute

$$\begin{aligned} \overrightarrow{AB} &= \langle -6, 9, 0 \rangle, & \overrightarrow{AC} &= \langle -6, 0, 12 \rangle, \\ \overrightarrow{AB} \times \overrightarrow{AC} &= \langle 9(12), 0(-6) - (-6)(12), (-6)(0) - 9(-6) \rangle = \langle 108, 72, 54 \rangle. \end{aligned}$$

Its length is

$$\sqrt{108^2 + 72^2 + 54^2} = \sqrt{19764} = 18\sqrt{61}.$$

Hence $\text{Area}(ABC) = 9\sqrt{61}.$

Step 4: Tetrahedron volume The coordinate-axis edges from the origin are mutually perpendicular with lengths 6, 9, 12. A tri-rectangular tetrahedron has one-sixth the corresponding box volume, so

$$V = \frac{1}{6}(6)(9)(12) = 108.$$

Verification Substitution of each intercept into $6x + 4y + 3z = 36$ succeeds, and their coordinate average is $(2, 3, 4)$.

Question 1

Classify and sketch the surface

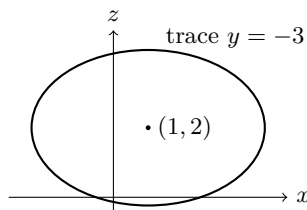
$$4x^2 + y^2 + 9z^2 - 8x + 6y - 36z + 4 = 0.$$

Tasks

1. Convert to standard form and classify it.
2. Find its center and semiaxes.
3. Derive its principal traces.
4. Find all coordinate-axis intercepts.

Question 1 – Solution

Strategy Group like variables and complete each square.



Step 1: Complete squares Write

$$4(x^2 - 2x) + (y^2 + 6y) + 9(z^2 - 4z) + 4 = 0.$$

Using $x^2 - 2x = (x - 1)^2 - 1$, $y^2 + 6y = (y + 3)^2 - 9$, and $z^2 - 4z = (z - 2)^2 - 4$ gives

$$4(x - 1)^2 + (y + 3)^2 + 9(z - 2)^2 = 45.$$

Divide by 45:

$$\frac{(x - 1)^2}{45/4} + \frac{(y + 3)^2}{45} + \frac{(z - 2)^2}{5} = 1.$$

Step 2: Geometry This is an ellipsoid centered at $(1, -3, 2)$ with semiaxes $3\sqrt{5}/2$, $3\sqrt{5}$, and $\sqrt{5}$. Its six axis endpoints relative to the center are obtained by adding and subtracting those lengths in the corresponding coordinate.

Step 3: Principal traces Through the center, the traces are

$$\begin{aligned} x = 1 : \frac{(y + 3)^2}{45} + \frac{(z - 2)^2}{5} &= 1, & y = -3 : \frac{(x - 1)^2}{45/4} + \frac{(z - 2)^2}{5} &= 1, \\ z = 2 : \frac{(x - 1)^2}{45/4} + \frac{(y + 3)^2}{45} &= 1. \end{aligned}$$

Step 4: Axis intercepts Set the other two coordinates to zero in the original equation. This gives

$$(1, 0, 0), \quad (0, -3 \pm \sqrt{5}, 0), \quad (0, 0, 2 \pm 4\sqrt{2}/3).$$

Substitution verifies each intercept and trace.

Question 2

Analyze

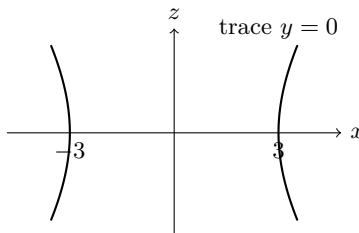
$$\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{16} = 1.$$

Tasks

1. Identify the surface and its axis.
2. Derive all three coordinate-plane traces.
3. Find the waist dimensions.
4. Explain why the surface is connected.

Question 2 – Solution

Strategy Read the signs and inspect traces one coordinate at a time.



Step 1: Classification Two positive squared terms and one negative term equal 1, so this is a hyperboloid of one sheet. The negative term is the z -term, hence the axis is the z -axis.

Step 2: Horizontal traces Setting $z = c$ gives

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 + \frac{c^2}{16}.$$

Every right side is positive, so every horizontal plane cuts an ellipse. At $c = 0$ the waist ellipse has semiaxes 3 and 2.

Step 3: Vertical traces At $y = 0$,

$$\frac{x^2}{9} - \frac{z^2}{16} = 1,$$

and at $x = 0$,

$$\frac{y^2}{4} - \frac{z^2}{16} = 1.$$

Both are hyperbolas opening perpendicular to the z -axis.

Conclusion Because a nonempty ellipse exists for every real $z = c$ and varies continuously with c , the surface is connected.

Question 3

For real k , classify

$$x^2 + y^2 - kz^2 = 1.$$

Tasks

1. Classify the surface for $k < 0$, $k = 0$, and $k > 0$.
2. Describe boundedness in each case.
3. Compare horizontal and vertical traces across the transition.

Question 3 – Solution

Strategy The sign of the z^2 coefficient controls the canonical type.

Case 1: $k < 0$ Write $k = -a$ with $a > 0$. Then $x^2 + y^2 + az^2 = 1$, an ellipsoid of revolution, bounded in every direction.

Case 2: $k = 0$ The equation becomes $x^2 + y^2 = 1$ with unrestricted z , so it is a circular cylinder of radius 1 parallel to the z -axis, unbounded in z .

Case 3: $k > 0$ The equation has two positive terms and one negative term, so it is a one-sheet hyperboloid about the z -axis. At $z = c$, the circle has radius $\sqrt{1 + kc^2}$; the surface is unbounded in all three coordinates.

Verification In every case the xy -trace is the unit circle. Vertical traces change from ellipses, to parallel lines, to hyperbolas, confirming the transition.

Question 4

Analyze the hyperbolic paraboloid

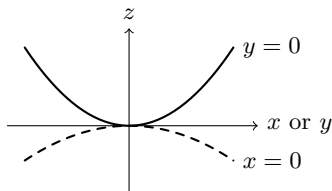
$$z = \frac{x^2}{4} - \frac{y^2}{9}.$$

Tasks

1. Derive the principal vertical traces.
2. Classify horizontal traces for $z = c$.
3. Find every line through the origin contained in the surface.
4. Justify the term “saddle.”

Question 4 – Solution

Strategy Fix one coordinate at a time, then factor the level-zero equation.



Step 1: Vertical traces If $y = 0$, then $z = x^2/4$, an upward parabola. If $x = 0$, then $z = -y^2/9$, a downward parabola. Their opposite concavities create the saddle.

Step 2: Horizontal traces At $z = c$,

$$\frac{x^2}{4} - \frac{y^2}{9} = c.$$

For $c > 0$ the hyperbola opens in the x -direction; for $c < 0$ it opens in the y -direction.

Step 3: Lines through the origin A contained line $(x, y, z) = t(a, b, c)$ must satisfy $tc = t^2(a^2/4 - b^2/9)$ for all t , so $c = 0$ and $a^2/4 = b^2/9$. Factoring gives

$$\left(\frac{x}{2} - \frac{y}{3}\right) \left(\frac{x}{2} + \frac{y}{3}\right) = 0.$$

Thus the two lines are $\boxed{z = 0, y = \pm 3x/2}$. Direct substitution confirms every point on each line lies on the surface.

Question 5

A centered quadric has horizontal traces only for $|z| \geq 3$, where

$$x^2 + y^2 = 4 \left(\frac{z^2}{9} - 1 \right).$$

Tasks

1. Reconstruct and classify the quadric.
2. Find its vertices and axis.
3. Derive both principal vertical traces.
4. Explain the empty central region.

Question 5 – Solution

Strategy Rearrange the trace family and interpret both its signs and its existence condition.

Step 1: Reconstruct Move terms and divide by 4:

$$x^2 + y^2 = \frac{4}{9}z^2 - 4 \implies \boxed{\frac{z^2}{9} - \frac{x^2}{4} - \frac{y^2}{4} = 1}.$$

Step 2: Geometry One positive square and two negative squares equal 1, so this is a two-sheet hyperboloid with z -axis. Setting $x = y = 0$ gives vertices $\boxed{(0, 0, \pm 3)}$.

Step 3: Vertical traces Setting $y = 0$ and $x = 0$ gives

$$\boxed{\frac{z^2}{9} - \frac{x^2}{4} = 1}, \quad \boxed{\frac{z^2}{9} - \frac{y^2}{4} = 1}.$$

Step 4: Gap Since $x^2 + y^2 \geq 0$, the given trace requires $z^2/9 - 1 \geq 0$, hence $|z| \geq 3$. No surface points lie in the central slab $-3 < z < 3$.

Question 6

Find and describe the intersection of

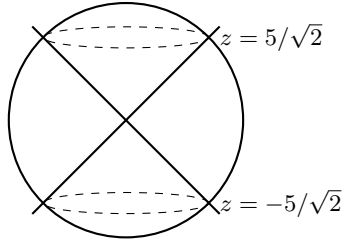
$$x^2 + y^2 + z^2 = 25 \quad \text{and} \quad z^2 = x^2 + y^2.$$

Tasks

1. Determine every component of the intersection.
2. Find each component's exact height and radius.
3. Verify the result in both original equations.

Question 6 – Solution

Strategy Substitute the cone relation into the sphere, then retain both signs of z .



Step 1: Solve for height Since $x^2 + y^2 = z^2$, the sphere gives

$$z^2 + z^2 = 25 \implies 2z^2 = 25 \implies z = \pm \frac{5}{\sqrt{2}}.$$

Step 2: Solve for radius On either level,

$$x^2 + y^2 = z^2 = \frac{25}{2}.$$

Thus the intersection consists of two circles:

$$\boxed{z = \pm \frac{5}{\sqrt{2}}, \quad x^2 + y^2 = \frac{25}{2}}.$$

Each has radius $\boxed{5/\sqrt{2}}$. Both signs are essential because the cone is double. Substitution verifies both circles in both original equations.

Question 7

The ellipsoid

$$\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

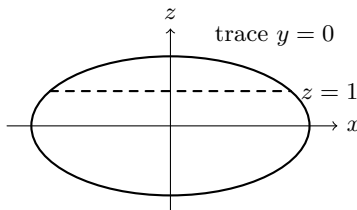
is cut by the plane $z = 1$.

Tasks

1. Find the section in standard form.
2. Determine its semiaxes and area.
3. Compare it quantitatively with the central section $z = 0$.

Question 7 – Solution

Strategy Substitute the fixed height and rewrite the remaining equation in standard ellipse form.



Step 1: Substitute With $z = 1$,

$$\frac{x^2}{16} + \frac{y^2}{9} + \frac{1}{4} = 1 \implies \frac{x^2}{16} + \frac{y^2}{9} = \frac{3}{4}.$$

Divide by $3/4$:

$$\boxed{\frac{x^2}{12} + \frac{y^2}{27/4} = 1}.$$

The semiaxes are $\boxed{2\sqrt{3}}$ and $\boxed{3\sqrt{3}/2}$.

Step 2: Area Therefore

$$A = \pi ab = \pi(2\sqrt{3})(3\sqrt{3}/2) = \boxed{9\pi}.$$

Comparison At $z = 0$, the semiaxes are 4 and 3, so the central area is 12π . The section at $z = 1$ has area $3/4$ as large, matching the right side $3/4$ before normalization.

Question 8

An elliptic paraboloid has vertex $(1, -2, 3)$, axis parallel to the z -axis, opens downward, and its trace in $z = -1$ is

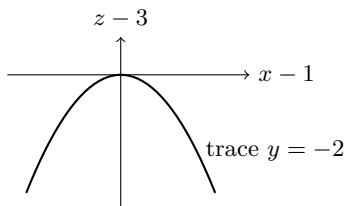
$$\frac{(x-1)^2}{8} + \frac{(y+2)^2}{4} = 1.$$

Tasks

1. Reconstruct the surface equation.
2. Find the trace at $z = 2$ in standard form.
3. State that trace's semiaxes and verify the supplied trace.

Question 8 – Solution

Strategy Use translated form $z - 3 = -A(x - 1)^2 - B(y + 2)^2$ and fit the known level.



Step 1: Use the known trace At $z = -1$, the vertical drop is 4. The given ellipse is equivalent to

$$\frac{1}{2}(x - 1)^2 + (y + 2)^2 = 4.$$

Hence the surface equation is

$$z - 3 = -\frac{1}{2}(x - 1)^2 - (y + 2)^2.$$

Step 2: New trace Set $z = 2$:

$$1 = \frac{1}{2}(x - 1)^2 + (y + 2)^2,$$

or

$$\frac{(x - 1)^2}{2} + (y + 2)^2 = 1.$$

Its semiaxes are $\sqrt{2}$ and 1. Substituting $z = -1$ back into the derived surface recovers the supplied ellipse.

Question 9

Classify the real point sets

$$x^2 + y^2 + z^2 - 2xy - 2xz + 2yz = c$$

Tasks

1. Factor the quadratic expression.
2. Classify the real point set for $c < 0$, $c = 0$, and $c > 0$.
3. Explain the degeneration geometrically.

Question 9 – Solution

Strategy Test whether the quadratic expression is a perfect square.

Step 1: Factor Expanding

$$(x - y - z)^2 = x^2 + y^2 + z^2 - 2xy - 2xz + 2yz.$$

Thus the equation is simply

$$(x - y - z)^2 = c.$$

Case analysis If $c < 0$, no real square can equal c , so the set is empty. If $c = 0$, then $x - y - z = 0$, a single plane. If $c > 0$, taking square roots gives

$$x - y - z = \pm\sqrt{c},$$

so the set is two distinct parallel planes.

Conclusion The classification is $\boxed{\emptyset}$, $\boxed{\text{one plane}}$, and $\boxed{\text{two parallel planes}}$ for the three cases. Substitution into the factored form verifies each result.

Question 10

Consider

$$5x^2 - 6xy + 5y^2 - z^2 = 0, \quad u = \frac{x+y}{\sqrt{2}}, \quad v = \frac{x-y}{\sqrt{2}}.$$

Tasks

1. Rewrite the equation in u, v, z .
2. Classify the surface and identify its axis.
3. Derive the traces in $z = c$, $u = 0$, and $v = 0$.
4. Relate the principal directions to the original xy -axes.

Question 10 – Solution

Strategy Express $x^2 + y^2$ and xy in rotated coordinates so the cross term disappears.

Step 1: Invert and substitute We have

$$x = \frac{u+v}{\sqrt{2}}, \quad y = \frac{u-v}{\sqrt{2}}, \quad x^2 + y^2 = u^2 + v^2, \quad xy = \frac{u^2 - v^2}{2}.$$

Therefore

$$5(u^2 + v^2) - 3(u^2 - v^2) - z^2 = 0, \quad \boxed{2u^2 + 8v^2 - z^2 = 0}.$$

Step 2: Classify This is an elliptic cone with vertex at the origin and axis the z -axis.

Step 3: Traces For $z = c \neq 0$,

$$\boxed{\frac{u^2}{c^2/2} + \frac{v^2}{c^2/8} = 1}.$$

For $v = 0$, $z = \pm\sqrt{2}u$; for $u = 0$, $z = \pm 2\sqrt{2}v$. At $z = 0$, only the vertex remains.

Step 4: Interpret rotation The u -axis is $x = y$ and the v -axis is $x = -y$, so the principal horizontal directions are rotated 45° from the original axes.

Question 1

Let

$$f(x, y) = \frac{\sqrt{9 - x^2 - y^2}}{\ln(x - y)}.$$

Tasks

1. Determine the exact domain.
2. State which boundary curves are included or excluded.
3. Test $(2, 0)$, $(0, 0)$, and $(1, 0)$ for membership.

Question 1 – Solution

Strategy Enforce separately the square-root, logarithm, and nonzero-denominator conditions.

Step 1: Square root Real values require $9 - x^2 - y^2 \geq 0$, hence $x^2 + y^2 \leq 9$.

Step 2: Logarithm The logarithm requires $x - y > 0$. Because it is a denominator, $\ln(x - y) \neq 0$, so $x - y \neq 1$.

Step 3: Combine Therefore

$$D = \{(x, y) : x^2 + y^2 \leq 9, x > y, x - y \neq 1\}.$$

The circular boundary is included only where the other restrictions hold; $x = y$ and $x - y = 1$ are excluded.

Step 4: Test points $(2, 0)$ satisfies all conditions. $(0, 0)$ fails $x > y$. $(1, 0)$ makes the denominator $\ln 1 = 0$. Thus only $\boxed{(2, 0)}$ belongs to D .

Question 2

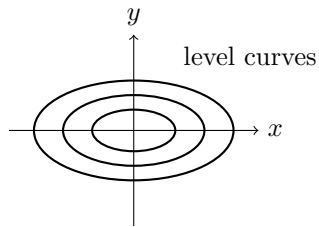
For $f(x, y) = x^2 + 4y^2$, analyze the level curves $f(x, y) = c$.

Tasks

1. Classify them for $c < 0$, $c = 0$, and $c > 0$.
2. Find semiaxes and enclosed area for $c > 0$.
3. Explain how the contours determine the graph's shape.

Question 2 – Solution

Strategy Treat c by sign and normalize the positive-level equation.



Step 1: Cases Because squares are nonnegative, $c < 0$ gives no points. For $c = 0$, both squares vanish, giving $(0, 0)$. For $c > 0$,

$$\frac{x^2}{c} + \frac{y^2}{c/4} = 1.$$

Thus the semiaxes are $\sqrt{c}$ and $\sqrt{c}/2$.

Step 2: Area The enclosed area is

$$A(c) = \pi (\sqrt{c}) \left(\frac{\sqrt{c}}{2} \right) = \boxed{\frac{\pi c}{2}}.$$

Step 3: Graph Higher levels produce nested, expanding ellipses centered at the origin. Since $z = x^2 + 4y^2 \geq 0$ and increases quadratically, the graph is an upward elliptic paraboloid, steeper in the y -direction.

Question 3

For positive inputs define

$$F(x, y) = \frac{xy}{x + y}.$$

Tasks

1. Show how F changes when both inputs are scaled by $s > 0$.
2. Derive the level curve $F = c > 0$.
3. State its domain branches and asymptotes.

Question 3 – Solution

Strategy Substitute scaled inputs, then solve the level equation explicitly for y .

Step 1: Scaling Direct substitution gives

$$F(sx, sy) = \frac{s^2 xy}{s(x+y)} = sF(x, y).$$

Thus the output has degree-one scaling.

Step 2: Level curve Set $xy/(x+y) = c$:

$$xy = cx + cy \implies y(x - c) = cx \implies \boxed{y = \frac{cx}{x - c}}.$$

For positive x, y , the denominator must be positive, so $x > c$; symmetry also implies $y > c$.

Step 3: Geometry Rewrite $y = c + c^2/(x - c)$. The asymptotes are $\boxed{x = c}$ and $\boxed{y = c}$, and the positive-input level is the branch with $x > c, y > c$. Substitution back into F returns c .

Question 4

A temperature field is

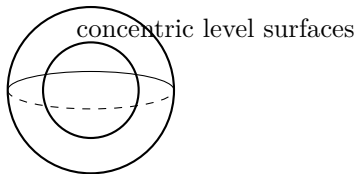
$$T(x, y, z) = 100e^{-(x^2+y^2+z^2)/4}.$$

Tasks

1. Describe the level surface $T = T_0$ for $0 < T_0 \leq 100$.
2. Find the 50-degree level radius.
3. Along $(t, t, 0)$, find when the temperature first reaches 50.

Question 4 – Solution

Strategy Take natural logarithms after isolating the exponential.



Step 1: General level From $T_0/100 = e^{-r^2/4}$,

$$\ln(T_0/100) = -r^2/4 \implies \boxed{r^2 = 4 \ln(100/T_0)}.$$

Thus levels are concentric spheres; $T_0 = 100$ degenerates to the origin.

Step 2: Fifty-degree surface For $T_0 = 50$, $r^2 = 4 \ln 2$, so $\boxed{r = 2\sqrt{\ln 2}}$.

Step 3: Path On $(t, t, 0)$, $r^2 = 2t^2$. Hence

$$2t^2 = 4 \ln 2 \implies t = \pm \sqrt{2 \ln 2}.$$

Starting at $t = 0$ and moving with increasing t , the first time is $\boxed{t = \sqrt{2 \ln 2}}$.

Question 5

Let $f(x, y) = \sqrt{1 - x^2 - y^2}$ and define $g(u, v) = f(u + v, u - v)$.

Tasks

1. Find a simplified formula and exact domain for g .
2. Describe its level curves.
3. Explain the geometric effect of the coordinate change.

Question 5 – Solution

Strategy Substitute first and expand carefully.

Step 1: Simplify Since

$$(u + v)^2 + (u - v)^2 = 2u^2 + 2v^2,$$

we get $\boxed{g(u, v) = \sqrt{1 - 2u^2 - 2v^2}}$.

Step 2: Domain The radicand must be nonnegative:

$$1 - 2u^2 - 2v^2 \geq 0 \implies \boxed{u^2 + v^2 \leq \frac{1}{2}}.$$

Step 3: Levels For $0 \leq c \leq 1$, $g = c$ implies

$$u^2 + v^2 = \frac{1 - c^2}{2},$$

circles centered at the origin. The map $(u, v) \mapsto (x, y) = (u + v, u - v)$ reflects v to $-v$, then rotates by 45° and scales lengths by $\sqrt{2}$. Its inverse image of the unit disk is the disk of radius $1/\sqrt{2}$. At $c = 1$ the level is just the origin; levels outside $[0, 1]$ are empty.

Question 6

Determine the exact range of

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad (x, y) \neq (0, 0).$$

Tasks

1. Prove upper and lower bounds.
2. Show every intermediate value occurs.
3. Describe the level sets $f = c$.

Question 6 – Solution

Strategy Compare $|x^2 - y^2|$ with $x^2 + y^2$, then solve the level equation.

Step 1: Bounds Since $x^2, y^2 \geq 0$,

$$|x^2 - y^2| \leq x^2 + y^2.$$

The denominator is positive off the origin, so $-1 \leq f \leq 1$.

Step 2: Endpoints and intermediate values On the x -axis, $f = 1$; on the y -axis, $f = -1$. For $-1 < c < 1$,

$$(1 - c)x^2 = (1 + c)y^2.$$

Choosing $y = 1$ and $x = \sqrt{(1 + c)/(1 - c)}$ realizes c . Hence

$$\boxed{\text{range}(f) = [-1, 1]}.$$

Step 3: Levels For $-1 < c < 1$, the level set is the pair of lines

$$\boxed{y = \pm \sqrt{\frac{1 - c}{1 + c}} x},$$

with the origin removed. At $c = 1$ and $c = -1$, the levels are the punctured x - and y -axes.

Question 7

A quadratic response has the form

$$q(x, y) = ax^2 + bxy + cy^2 + d.$$

It satisfies $q(1, 0) = 6$, $q(0, 1) = 4$, $q(1, 1) = 15$, and $q(0, 0) = 1$.

Tasks

1. Determine q .
2. Compute $q(2, -1)$.
3. Classify its level curves after removing the cross term by factoring.

Question 7 – Solution

Strategy Substitute the four samples one at a time.

Step 1: Coefficients From $q(0, 0) = 1$, $d = 1$. Then

$$a + d = 6 \implies a = 5, \quad c + d = 4 \implies c = 3.$$

Finally,

$$a + b + c + d = 15 \implies 5 + b + 3 + 1 = 15 \implies b = 6.$$

Thus $\boxed{q(x, y) = 5x^2 + 6xy + 3y^2 + 1}$.

Step 2: Evaluate $q(2, -1) = 20 - 12 + 3 + 1 = \boxed{12}$.

Step 3: Geometry Complete a square:

$$5x^2 + 6xy + 3y^2 = 5 \left(x + \frac{3}{5}y \right)^2 + \frac{6}{5}y^2.$$

Both coefficients are positive, so $q = k > 1$ gives rotated ellipses, $q = 1$ gives only the origin, and $q < 1$ gives no real points.

Question 8

Analyze

$$f(x, y) = \frac{xy}{1 + x^2 + y^2}.$$

Tasks

1. Prove strict bounds for its range.
2. Show every value between the bounds occurs.
3. Explain why the endpoints are approached but never attained.

Question 8 – Solution

Strategy Use $2|xy| \leq x^2 + y^2$ and then restrict to the lines $y = \pm x$.

Step 1: Bound Let $r^2 = x^2 + y^2$. Then

$$|f| \leq \frac{r^2/2}{1 + r^2} < \frac{1}{2}.$$

Thus the range is contained in $(-1/2, 1/2)$.

Step 2: Realize values On $y = x$,

$$f(x, x) = \frac{x^2}{1 + 2x^2},$$

which increases from 0 toward $1/2$. Solving $c = x^2/(1 + 2x^2)$ gives $x^2 = c/(1 - 2c)$ for every $0 < c < 1/2$. The line $y = -x$ similarly realizes every negative value, and $(0, 0)$ realizes 0.

Step 3: Conclusion Therefore

$$\boxed{\text{range}(f) = (-1/2, 1/2)}.$$

The denominator contains the extra positive 1, making equality in the bound impossible for finite inputs; the endpoints occur only as limits.

Question 9

Let

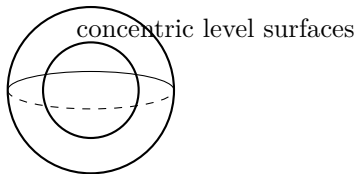
$$F(x, y, z) = \ln(4 - x^2 - y^2 - z^2).$$

Tasks

1. Find its domain and range.
2. Describe every level surface $F = c$.
3. Determine how the level radius changes as c increases.

Question 9 – Solution

Strategy Enforce positivity of the logarithm's argument and solve $F = c$ by exponentiating.



Step 1: Domain We need

$$4 - x^2 - y^2 - z^2 > 0,$$

so the domain is the open ball $\boxed{x^2 + y^2 + z^2 < 4}$.

Step 2: Range The logarithm's argument takes every value in $(0, 4]$. Therefore F takes every value in

$$\boxed{(-\infty, \ln 4]}.$$

The maximum $\ln 4$ occurs at the origin; the function tends to $-\infty$ near the boundary.

Step 3: Levels From $F = c$,

$$4 - r^2 = e^c \implies \boxed{r = \sqrt{4 - e^c}}, \quad c \leq \ln 4.$$

These are concentric spheres, degenerating to the origin at $c = \ln 4$. As c increases, e^c increases and the radius decreases.

Question 10

An open-top box has length x , width y , and height h . A design constraint fixes the sum of these dimensions:

$$x + y + h = 12, \quad x, y, h > 0.$$

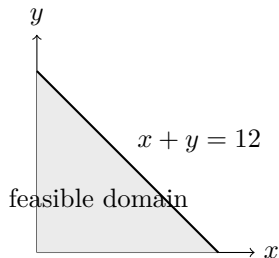
Its volume is $V(x, y) = xy(12 - x - y)$.

Tasks

1. State the natural domain.
2. Find the greatest possible volume without calculus.
3. Prove the maximizing dimensions are unique.

Question 10 – Solution

Strategy Interpret the three positive factors x , y , and $h = 12 - x - y$, then apply AM–GM.



Step 1: Domain Positivity requires

$$\boxed{x > 0, \quad y > 0, \quad x + y < 12}.$$

This is the interior of a triangular region.

Step 2: Bound The three positive numbers x, y, h have fixed sum 12. AM–GM gives

$$\frac{x + y + h}{3} \geq \sqrt[3]{xyh} \implies 4 \geq \sqrt[3]{V}.$$

Cubing yields $V \leq 64$.

Step 3: Equality Equality in AM–GM occurs exactly when

$$x = y = h = 4.$$

Thus the unique maximizing dimensions are $\boxed{x = 4, y = 4, h = 4}$ and

$$\boxed{V_{\max} = 64}.$$

Direct substitution satisfies the material constraint and confirms the volume.

Question 1

Let

$$\mathbf{r}(t) = \left\langle \sqrt{t+2}, \ln(5-t), \frac{1}{t^2-1} \right\rangle.$$

Tasks

1. Determine the exact domain of $\mathbf{r}$.
2. Evaluate $\mathbf{r}(2)$.
3. Explain what fails at every excluded endpoint or isolated value.

Question 1 – Solution

Strategy. A vector function is defined only where all three component functions are defined. We therefore intersect their domains.

Step 1: Square-root condition Real values require

$$t + 2 \geq 0 \implies t \geq -2.$$

Step 2: Logarithm condition The logarithm requires a strictly positive argument:

$$5 - t > 0 \implies t < 5.$$

Step 3: Denominator condition We must have

$$t^2 - 1 \neq 0 \implies t \neq -1, 1.$$

Intersecting the three conditions gives

$$\boxed{\text{Dom}(\mathbf{r}) = [-2, -1) \cup (-1, 1) \cup (1, 5]}.$$

Step 4: Evaluation

$$\boxed{\mathbf{r}(2) = \left\langle 2, \ln 3, \frac{1}{3} \right\rangle}.$$

Verification. At $t = -2$ the square root is zero, so that endpoint is included. At $t = 5$ the logarithm sees zero; below -2 its first component is nonreal; and at $t = \pm 1$ the third component divides by zero.

Question 2

Consider

$$\mathbf{r}(t) = \langle 2t - 1, t^2 + 1, 3t + 4 \rangle, \quad -2 \leq t \leq 2.$$

Tasks

1. Eliminate t and give two Cartesian equations satisfied by the trace.
2. Identify the curve and its containing plane.
3. Find its two endpoints and indicate the direction of travel.

Question 2 – Solution

Strategy. Solve the simplest component for the parameter and substitute it into the other two components.

Step 1: Recover the parameter Since $x = 2t - 1$,

$$t = \frac{x+1}{2}.$$

Step 2: Eliminate t Substitution into $y = t^2 + 1$ gives

$$y = \frac{(x+1)^2}{4} + 1.$$

Also,

$$z = 3\left(\frac{x+1}{2}\right) + 4 = \frac{3}{2}x + \frac{11}{2},$$

or $3x - 2z + 11 = 0$. Hence the trace is the part of the parabola

$$\boxed{y = \frac{(x+1)^2}{4} + 1, \quad 3x - 2z + 11 = 0}$$

for which $-5 \leq x \leq 3$. It lies in the stated plane.

Step 3: Endpoints and orientation

$$\mathbf{r}(-2) = \langle -5, 5, -2 \rangle, \quad \mathbf{r}(2) = \langle 3, 5, 10 \rangle.$$

As t increases, $x = 2t - 1$ increases, so the curve is traversed from the first point to the second.

Verification. Substituting either endpoint into both Cartesian equations confirms both constraints exactly.

Question 3

A particle follows

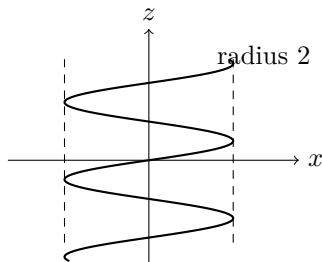
$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t/\pi \rangle, \quad 0 \leq t \leq 4\pi.$$

Without calculus:

1. Describe the trace geometrically and state its axis and radius.
2. Find every point where the particle meets the plane $z = 1$.
3. Determine the vertical rise during one complete revolution and the total number of revolutions.

Question 3 – Solution

Strategy. Use the identity $\cos^2 t + \sin^2 t = 1$ and compare the angular parameter with the height.



Step 1: Cylindrical constraint

$$x^2 + y^2 = 4 \cos^2 t + 4 \sin^2 t = 4.$$

Thus the trace lies on the cylinder of radius 2 about the z -axis. Since $z = t/\pi$ steadily increases while (x, y) rotates counterclockwise, the curve is a circular helix.

Step 2: Plane intersection Setting $z = 1$ gives

$$\frac{t}{\pi} = 1 \implies t = \pi.$$

This value is in the interval, and therefore the only point is

$$\mathbf{r}(\pi) = \langle -2, 0, 1 \rangle.$$

Step 3: Pitch and turns One revolution changes t by 2π , so

$$\Delta z = \frac{t + 2\pi}{\pi} - \frac{t}{\pi} = 2.$$

The interval length 4π contains $4\pi/(2\pi) = 2$ revolutions. Hence the rise per turn is $\boxed{2}$ and the curve makes $\boxed{2}$ complete turns.

Question 4

Two moving objects have position functions

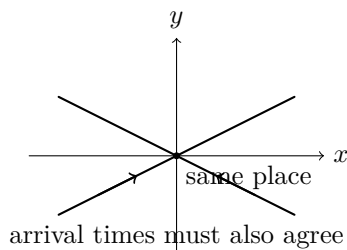
$$\mathbf{r}_1(t) = \langle t, t^2, 1 - t \rangle, \quad \mathbf{r}_2(s) = \langle 2 - s, s, s - 1 \rangle.$$

Tasks

1. Determine whether their geometric paths intersect.
2. If they do, find the parameter value on each path and the intersection point.
3. If both parameters represent the same clock time, decide whether the objects collide.

Question 4 – Solution

Strategy. A path intersection permits different parameters t and s . A collision additionally requires the same time.



Step 1: Match coordinates Equality of the two position vectors requires

$$t = 2 - s, \quad t^2 = s, \quad 1 - t = s - 1.$$

The first and third equations both simplify to

$$t + s = 2.$$

Using $s = t^2$ gives

$$t + t^2 = 2 \implies t^2 + t - 2 = 0 \implies (t - 1)(t + 2) = 0.$$

Step 2: Check both candidates If $t = 1$, then $s = t^2 = 1$, and all equations hold. If $t = -2$, then $s = 4$, but $t + s = 2$ also holds. Thus there are two path intersections:

$$\boxed{\langle 1, 1, 0 \rangle \quad (t, s) = (1, 1)}, \quad \boxed{\langle -2, 4, 3 \rangle \quad (t, s) = (-2, 4)}.$$

Step 3: Collision test With a common clock, a collision requires $t = s$. Only $(t, s) = (1, 1)$ satisfies that requirement. Therefore the objects collide at

$$\boxed{t = 1 \text{ at } \langle 1, 1, 0 \rangle}.$$

Verification. Direct substitution into both vector functions reproduces each path-intersection point.

Question 5

Construct a vector parametrization of the curve where

$$x^2 + y^2 = 9 \quad \text{and} \quad z = x - 2y.$$

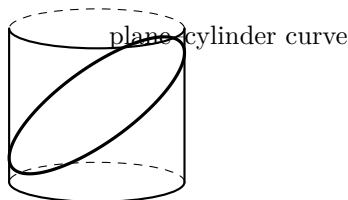
Choose the parametrization so that it begins at $(3, 0, 3)$ and initially moves toward points with $y > 0$. State a parameter interval that traces the curve exactly once.

Tasks

1. Parametrize the cylindrical constraint with the required starting direction.
2. Impose the plane equation to obtain the height component.
3. State and verify an interval that traces the curve exactly once.

Question 5 – Solution

Strategy. Parametrize the cylinder first; then use the plane equation to determine the remaining component.



Step 1: Parametrize the circle The cylinder equation suggests

$$x = 3 \cos t, \quad y = 3 \sin t,$$

because these expressions satisfy $x^2 + y^2 = 9$ identically.

Step 2: Impose the plane Substitute the chosen x and y into $z = x - 2y$:

$$z = 3 \cos t - 6 \sin t.$$

Therefore

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 3 \cos t - 6 \sin t \rangle, \quad 0 \leq t < 2\pi.$$

Step 3: Initial conditions At $t = 0$,

$$\mathbf{r}(0) = \langle 3, 0, 3 \rangle.$$

For small positive t , $\sin t > 0$, so $y = 3 \sin t > 0$; the required initial direction is satisfied.

Verification. Squaring the first two components gives $9(\cos^2 t + \sin^2 t) = 9$, and the third component is exactly the first minus twice the second. The half-open interval avoids repeating the starting point.

Question 6

The vector functions

$$\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t \rangle, \quad 0 \leq t < 2\pi,$$

and

$$\mathbf{q}(u) = \langle \cos u, -\sin u, \cos 2u \rangle, \quad 0 \leq u < 2\pi,$$

are given.

1. Prove that they have the same geometric trace.
2. Compare their orientations starting at $(1, 0, 1)$.
3. Decide whether either parametrization visits a point twice on its stated interval.

Question 6 – Solution

Strategy. Find a parameter substitution connecting the two formulas, then use the first two components to test repeated points.

Step 1: Same trace For $0 < u < 2\pi$, put $t = 2\pi - u$; at $u = 0$, use $t = 0$. For the interior parameters,

$$\cos(2\pi - u) = \cos u, \quad \sin(2\pi - u) = -\sin u,$$

and

$$\cos(2t) = \cos(4\pi - 2u) = \cos 2u.$$

Hence $\mathbf{r}(2\pi - u) = \mathbf{q}(u)$, so every point of one trace occurs on the other. Thus their geometric traces are identical.

Step 2: Orientation Both begin at $(1, 0, 1)$. For small positive t , the second component of $\mathbf{r}(t)$ is positive. For small positive u , the second component of $\mathbf{q}(u)$ is negative. Therefore they traverse the trace in opposite orientations.

Step 3: Repeated-point test Suppose $\mathbf{r}(t_1) = \mathbf{r}(t_2)$. Its first two coordinates give

$$(\cos t_1, \sin t_1) = (\cos t_2, \sin t_2),$$

so $t_1 - t_2$ is an integer multiple of 2π . Since both parameters lie in $[0, 2\pi)$, this forces $t_1 = t_2$. The same argument applies to $\mathbf{q}$.

Thus

each parametrization traces the curve exactly once, in opposite directions.

Question 7

Let

$$\mathbf{a}(t) = \langle t, 1 - t, 2t \rangle, \quad \mathbf{b}(t) = \langle 1, t, t^2 \rangle.$$

Tasks

1. Find the common domain of $\mathbf{a}/(t - 2)$ and $\mathbf{b}/(t + 1)$.
2. Compute $2\mathbf{a}(t) - \mathbf{b}(t)$.
3. Find all t for which $\mathbf{a}(t) \cdot \mathbf{b}(t) = 0$.

Question 7 – Solution

Strategy. Treat scalar division and vector algebra componentwise; for orthogonality, set the dot product equal to zero.

Step 1: Common domain The vector functions themselves are polynomial and defined for all real t . The scalar denominators impose

$$t - 2 \neq 0, \quad t + 1 \neq 0.$$

Therefore the common domain is

$$\boxed{\mathbb{R} \setminus \{-1, 2\}}.$$

Step 2: Vector combination

$$\begin{aligned} 2\mathbf{a}(t) - \mathbf{b}(t) &= \langle 2t, 2 - 2t, 4t \rangle - \langle 1, t, t^2 \rangle \\ &= \boxed{\langle 2t - 1, 2 - 3t, 4t - t^2 \rangle}. \end{aligned}$$

Step 3: Orthogonality

$$\begin{aligned} \mathbf{a}(t) \cdot \mathbf{b}(t) &= t(1) + (1 - t)t + (2t)t^2 \\ &= 2t - t^2 + 2t^3 \\ &= t(2t^2 - t + 2). \end{aligned}$$

The quadratic has discriminant $(-1)^2 - 4(2)(2) = -15 < 0$, so it has no real roots. Consequently

$$\boxed{t = 0}$$

is the only real solution. Directly, $\mathbf{a}(0) = \langle 0, 1, 0 \rangle$ and $\mathbf{b}(0) = \langle 1, 0, 0 \rangle$, whose dot product is zero.

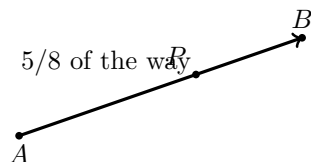
Question 8

Points $A = (-2, 1, 4)$ and $B = (6, -3, 0)$ are endpoints of a directed segment.

1. Construct a vector function $\mathbf{r}(t)$, $0 \leq t \leq 1$, that travels from A to B at a constant parameter rate.
2. Find the point reached when five-eighths of the parameter interval has elapsed.
3. Reparametrize the same directed segment on $-3 \leq u \leq 5$.

Question 8 – Solution

Strategy. A directed segment is an affine blend $(1 - t)A + tB$. A new interval requires a linear map back to $[0, 1]$.



Step 1: Segment formula First compute

$$B - A = \langle 8, -4, -4 \rangle.$$

Thus

$$\boxed{\mathbf{r}(t) = A + t(B - A) = \langle -2 + 8t, 1 - 4t, 4 - 4t \rangle}, \quad 0 \leq t \leq 1.$$

Indeed, $\mathbf{r}(0) = A$ and $\mathbf{r}(1) = B$.

Step 2: Division point At $t = 5/8$,

$$\mathbf{r}\left(\frac{5}{8}\right) = \left\langle -2 + 5, 1 - \frac{5}{2}, 4 - \frac{5}{2} \right\rangle = \boxed{\left\langle 3, -\frac{3}{2}, \frac{3}{2} \right\rangle}.$$

Step 3: New parameter interval The linear map taking $u = -3$ to $t = 0$ and $u = 5$ to $t = 1$ is

$$t = \frac{u + 3}{8}.$$

Substitution gives

$$\boxed{\mathbf{R}(u) = \left\langle u + 1, -\frac{1}{2}u - \frac{1}{2}, \frac{5}{2} - \frac{1}{2}u \right\rangle}, \quad -3 \leq u \leq 5.$$

Checking $u = -3$ and $u = 5$ recovers A and B in the required order.

Question 9

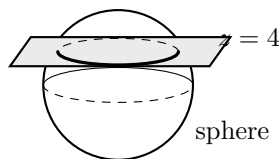
A curve is known to lie on the sphere $x^2 + y^2 + z^2 = 25$ and in the plane $z = 4$. Construct two vector parametrizations that traverse the entire curve exactly once in opposite directions. Choose both to start at $(3, 0, 4)$.

Tasks

1. Identify the intersection curve and its dimensions.
2. Construct the two oppositely oriented parametrizations.
3. Verify the surface constraints, starting point, and one-pass intervals.

Question 9 – Solution

Strategy. Substituting the fixed height into the sphere reveals a circle. The sign of the sine component controls orientation.



Step 1: Identify the circle With $z = 4$, the sphere equation becomes

$$x^2 + y^2 + 16 = 25 \implies x^2 + y^2 = 9.$$

The intersection is a radius-3 circle centered at $(0, 0, 4)$.

Step 2: First orientation A standard counterclockwise parametrization as viewed from the positive z -axis is

$$\mathbf{r}_+(t) = \langle 3 \cos t, 3 \sin t, 4 \rangle, \quad 0 \leq t < 2\pi.$$

It begins at $\mathbf{r}_+(0) = \langle 3, 0, 4 \rangle$ and initially enters the half-space $y > 0$.

Step 3: Opposite orientation Reverse the sign of the sine component:

$$\mathbf{r}_-(t) = \langle 3 \cos t, -3 \sin t, 4 \rangle, \quad 0 \leq t < 2\pi.$$

It has the same starting point but initially enters $y < 0$, so its orientation is opposite.

Verification. For either formula,

$$x^2 + y^2 + z^2 = 9(\cos^2 t + \sin^2 t) + 16 = 25.$$

The half-open interval covers every angular position once without duplicating the start at 2π .

Question 10

Consider the closed space curve

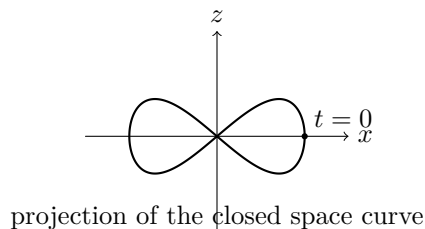
$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, \sin t \cos t \rangle, \quad 0 \leq t < 2\pi.$$

Tasks

1. Find two surfaces whose intersection contains the entire trace.
2. Determine all points where the trace meets the plane $z = 0$.
3. Prove that the parametrization traces each point exactly once, despite the repeated values of the z -component.

Question 10 – Solution

Strategy. Eliminate the parameter using the first two components. For uniqueness, retain the ordered pair $(\cos t, \sin t)$ rather than examining z alone.



Step 1: Eliminate the parameter The first two components give

$$x^2 + y^2 = 4(\cos^2 t + \sin^2 t) = 4.$$

Also,

$$xy = (2 \cos t)(2 \sin t) = 4 \sin t \cos t = 4z,$$

so $z = xy/4$. The trace is therefore contained in

$$\boxed{x^2 + y^2 = 4, \quad z = \frac{xy}{4}}.$$

Conversely, every point of this intersection has $(x/2, y/2)$ on the unit circle and hence equals $(\cos t, \sin t)$ for some t ; thus these equations describe the full trace.

Step 2: Intersections with $z = 0$ Since $z = xy/4$, we need $x = 0$ or $y = 0$. On $x^2 + y^2 = 4$, this yields

$$\boxed{(2, 0, 0), (-2, 0, 0), (0, 2, 0), (0, -2, 0)}.$$

Step 3: One-to-one traversal If $\mathbf{r}(t_1) = \mathbf{r}(t_2)$, then the first two coordinates imply

$$(\cos t_1, \sin t_1) = (\cos t_2, \sin t_2).$$

Thus $t_1 - t_2 = 2\pi k$ for an integer k . Inside $[0, 2\pi)$, only $k = 0$ is possible, so $t_1 = t_2$. Hence no point is repeated.

Question 1

For $t \neq 0$, let

$$\mathbf{r}(t) = \left\langle \frac{\sin 3t}{t}, \frac{e^{2t} - 1}{t}, \frac{1 - \cos t}{t^2} \right\rangle.$$

Tasks

1. Compute $\lim_{t \rightarrow 0} \mathbf{r}(t)$.
2. Define $\mathbf{r}(0)$ so that $\mathbf{r}$ is continuous at 0.
3. Justify why a vector limit exists precisely when all component limits exist.

Question 1 – Solution

Strategy. A vector limit is evaluated component by component. Rewrite each component in terms of a standard scalar limit.

Step 1: First component

$$\frac{\sin 3t}{t} = 3 \frac{\sin 3t}{3t} \longrightarrow 3(1) = 3.$$

Step 2: Second component Put $u = 2t$. Then

$$\frac{e^{2t} - 1}{t} = 2 \frac{e^u - 1}{u} \longrightarrow 2.$$

Step 3: Third component Multiply by the conjugate:

$$\begin{aligned} \frac{1 - \cos t}{t^2} &= \frac{(1 - \cos t)(1 + \cos t)}{t^2(1 + \cos t)} \\ &= \frac{\sin^2 t}{t^2(1 + \cos t)} = \left(\frac{\sin t}{t} \right)^2 \frac{1}{1 + \cos t} \longrightarrow \frac{1}{2}. \end{aligned}$$

Therefore

$$\boxed{\lim_{t \rightarrow 0} \mathbf{r}(t) = \langle 3, 2, \tfrac{1}{2} \rangle}.$$

Step 4: Continuous extension Define $\boxed{\mathbf{r}(0) = \langle 3, 2, \tfrac{1}{2} \rangle}$. This makes the function value equal its limit. The vector limit criterion follows because the distance to a candidate vector tends to zero exactly when each coordinate difference tends to zero.

Question 2

Let

$$\mathbf{r}(t) = \langle t^2 - 1, t^3 - 3t, 2t + 1 \rangle.$$

Tasks

1. Find $\mathbf{r}'(t)$ directly from the vector difference quotient.
2. Find the tangent line at the point corresponding to $t = 1$.
3. Verify that your direction vector is nonzero.

Question 2 – Solution

Strategy. Expand $\mathbf{r}(t+h) - \mathbf{r}(t)$ componentwise, divide by h , and then let $h \rightarrow 0$.

Step 1: Difference quotient

$$\begin{aligned}\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} &= \left\langle \frac{(t+h)^2 - t^2}{h}, \frac{(t+h)^3 - 3(t+h) - (t^3 - 3t)}{h}, \frac{2h}{h} \right\rangle \\ &= \langle 2t+h, 3t^2 + 3th + h^2 - 3, 2 \rangle.\end{aligned}$$

Taking $h \rightarrow 0$ yields

$$\boxed{\mathbf{r}'(t) = \langle 2t, 3t^2 - 3, 2 \rangle}.$$

Step 2: Point and direction At $t = 1$,

$$\mathbf{r}(1) = \langle 0, -2, 3 \rangle, \quad \mathbf{r}'(1) = \langle 2, 0, 2 \rangle.$$

The derivative is nonzero, so it supplies a valid tangent direction.

Step 3: Tangent line Using a new line parameter s ,

$$\boxed{\mathbf{L}(s) = \langle 0, -2, 3 \rangle + s \langle 2, 0, 2 \rangle}.$$

Substitution at $s = 0$ confirms the point, and the line direction is parallel to $\mathbf{r}'(1)$.

Question 3

Let

$$\mathbf{u}(t) = \langle t, t^2, 1 \rangle, \quad \mathbf{v}(t) = \langle \cos t, \sin t, t \rangle.$$

Without expanding first, compute at $t = 0$:

Tasks

1. $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}),$
2. $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}),$
3. $\frac{d}{dt}[(t^2 + 1)\mathbf{v}(t)].$

State the differentiation rule used in each part.

Question 3 – Solution

Strategy. Apply the dot-product, cross-product, and scalar–vector product rules before inserting $t = 0$.

At $t = 0$,

$$\mathbf{u} = \langle 0, 0, 1 \rangle, \quad \mathbf{u}' = \langle 1, 0, 0 \rangle, \quad \mathbf{v} = \langle 1, 0, 0 \rangle, \quad \mathbf{v}' = \langle 0, 1, 1 \rangle.$$

Step 1: Dot product rule

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}', \\ \left. \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) \right|_0 &= \langle 1, 0, 0 \rangle \cdot \langle 1, 0, 0 \rangle + \langle 0, 0, 1 \rangle \cdot \langle 0, 1, 1 \rangle = \boxed{2}. \end{aligned}$$

Step 2: Cross product rule Order matters:

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}', \\ \left. \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) \right|_0 &= \mathbf{0} + \langle 0, 0, 1 \rangle \times \langle 0, 1, 1 \rangle = \boxed{\langle -1, 0, 0 \rangle}. \end{aligned}$$

Step 3: Scalar–vector rule With $f(t) = t^2 + 1$,

$$(f\mathbf{v})' = f'\mathbf{v} + f\mathbf{v}' \implies (f\mathbf{v})'(0) = 0\mathbf{v}(0) + 1\mathbf{v}'(0) = \boxed{\langle 0, 1, 1 \rangle}.$$

Question 4

Find the unique vector function $\mathbf{r}(t)$ satisfying

$$\mathbf{r}'(t) = \left\langle 2te^{t^2}, \frac{3}{1+t^2}, 4t^3 - 2t \right\rangle, \quad \mathbf{r}(0) = \langle 1, -2, 5 \rangle.$$

Tasks

1. Recover $\mathbf{r}(t)$ by componentwise antidifferentiation.
2. Use the initial vector to determine every constant.
3. Differentiate your final expression to verify it.

Question 4 – Solution

Strategy. Antidifferentiate each component and then use the initial vector to determine three constants.

Step 1: First component

$$\int 2te^{t^2} dt = e^{t^2} + C_1,$$

using the substitution $u = t^2$.

Step 2: Remaining components

$$\int \frac{3}{1+t^2} dt = 3 \arctan t + C_2,$$

and

$$\int (4t^3 - 2t) dt = t^4 - t^2 + C_3.$$

Thus

$$\mathbf{r}(t) = \left\langle e^{t^2} + C_1, 3 \arctan t + C_2, t^4 - t^2 + C_3 \right\rangle.$$

Step 3: Initial condition At $t = 0$,

$$\mathbf{r}(0) = \langle 1 + C_1, C_2, C_3 \rangle = \langle 1, -2, 5 \rangle.$$

Hence $C_1 = 0$, $C_2 = -2$, and $C_3 = 5$. Therefore

$$\boxed{\mathbf{r}(t) = \left\langle e^{t^2}, 3 \arctan t - 2, t^4 - t^2 + 5 \right\rangle}.$$

Verification. Componentwise differentiation gives $\left\langle 2te^{t^2}, 3/(1+t^2), 4t^3 - 2t \right\rangle$, and direct substitution gives the required initial vector. These three scalar initial-value problems each have only one integration constant, so the solution is unique.

Question 5

Evaluate exactly

$$\int_0^{\pi/2} \langle t \cos t, t \sin t, \sin t \cos t \rangle dt.$$

Tasks

1. Evaluate all three components exactly, showing each scalar integration step.
2. State the resulting vector.
3. Verify the third component using a double-angle identity.

Question 5 – Solution

Strategy. Integrate componentwise. The first two components require integration by parts; the third can be checked in two different ways.

Step 1: First component Integration by parts with $u = t$, $dv = \cos t \, dt$ gives

$$\begin{aligned}\int_0^{\pi/2} t \cos t \, dt &= [t \sin t]_0^{\pi/2} - \int_0^{\pi/2} \sin t \, dt \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

Step 2: Second component With $u = t$, $dv = \sin t \, dt$,

$$\int_0^{\pi/2} t \sin t \, dt = [-t \cos t]_0^{\pi/2} + \int_0^{\pi/2} \cos t \, dt = 1.$$

Step 3: Third component Since $d(\sin^2 t)/dt = 2 \sin t \cos t$,

$$\int_0^{\pi/2} \sin t \cos t \, dt = \left[\frac{1}{2} \sin^2 t \right]_0^{\pi/2} = \frac{1}{2}.$$

Therefore

$$\boxed{\int_0^{\pi/2} \langle t \cos t, t \sin t, \sin t \cos t \rangle \, dt = \left\langle \frac{\pi}{2} - 1, 1, \frac{1}{2} \right\rangle}.$$

Using $\sin t \cos t = \frac{1}{2} \sin 2t$ gives the same third component: $[-\frac{1}{4} \cos 2t]_0^{\pi/2} = 1/2$.

Question 6

Suppose a differentiable vector function $\mathbf{r}(t)$ satisfies

$$\|\mathbf{r}(t)\| = 7$$

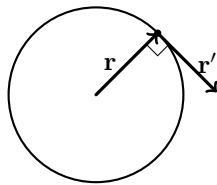
for every t in an interval.

Tasks

1. Prove that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ throughout the interval.
2. Explain the geometric meaning when $\mathbf{r}'(t) \neq \mathbf{0}$.
3. Show that $\mathbf{r} \cdot \mathbf{r}' = 0$ on an interval implies constant length; explain what extra datum fixes that length at 7.

Question 6 – Solution

Strategy. Square the norm to avoid differentiating a square root, then use the dot-product rule.



constant distance $\Rightarrow$ perpendicular change

Step 1: Differentiate the invariant The hypothesis gives

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|^2 = 49.$$

Differentiating both sides,

$$\mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 0.$$

Because the dot product is commutative, the two terms are equal. Hence

$$2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \quad \Longrightarrow \quad \boxed{\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0}.$$

Step 2: Geometry The position vector points radially from the origin. When $\mathbf{r}'(t) \neq \mathbf{0}$, it points along the tangent to the curve. Their zero dot product means the tangent is perpendicular to the radius, as expected for motion constrained to a sphere.

Step 3: Converse Now assume $\mathbf{r} \cdot \mathbf{r}' = 0$. Then

$$\frac{d}{dt} \|\mathbf{r}(t)\|^2 = \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r}) = 2\mathbf{r} \cdot \mathbf{r}' = 0.$$

Therefore $\|\mathbf{r}(t)\|^2$ is constant on the interval, so its nonnegative square root $\|\mathbf{r}(t)\|$ is also constant. The constant is 7 if $\|\mathbf{r}(t_0)\| = 7$ at one point of the interval.

Question 7

Define

$$\mathbf{r}(t) = \begin{cases} \langle t, t^2, t^3 \rangle, & t \leq 0, \\ \langle t, at, t^2 \rangle, & t > 0, \end{cases}$$

where a is a real constant.

Tasks

1. Determine all a for which $\mathbf{r}$ is continuous at 0.
2. Determine all a for which $\mathbf{r}$ is differentiable at 0.
3. For the differentiable case, find $\mathbf{r}'(0)$.

Question 7 – Solution

Strategy. Continuity compares one-sided function values; differentiability compares one-sided vector difference quotients.

Step 1: Continuity The defined value from the first branch is

$$\mathbf{r}(0) = \langle 0, 0, 0 \rangle.$$

From the left, $\langle t, t^2, t^3 \rangle \rightarrow \langle 0, 0, 0 \rangle$. From the right,

$$\langle t, at, t^2 \rangle \rightarrow \langle 0, 0, 0 \rangle$$

for every fixed real a . Therefore $\mathbf{r}$ is continuous at 0 for every real value of a .

Step 2: Left derivative As h approaches zero through negative values,

$$\frac{\mathbf{r}(h) - \mathbf{r}(0)}{h} = \langle 1, h, h^2 \rangle \longrightarrow \langle 1, 0, 0 \rangle.$$

Step 3: Right derivative As h approaches zero through positive values,

$$\frac{\mathbf{r}(h) - \mathbf{r}(0)}{h} = \langle 1, a, h \rangle \longrightarrow \langle 1, a, 0 \rangle.$$

The one-sided derivatives agree exactly when $a = 0$. Hence

$$\boxed{a = 0}, \quad \boxed{\mathbf{r}'(0) = \langle 1, 0, 0 \rangle}.$$

This also shows explicitly why continuity alone does not guarantee differentiability.

Question 8

Let

$$\mathbf{F}(t) = \int_1^{t^2} \left\langle e^u, \frac{1}{u}, \cos(\pi u) \right\rangle du, \quad t > 0.$$

Tasks

1. Find $\mathbf{F}'(t)$ without first evaluating the integral.
2. Evaluate $\mathbf{F}'(1)$ and $\mathbf{F}(1)$.
3. Explain the roles of the Fundamental Theorem of Calculus and the chain rule.

Question 8 – Solution

Strategy. Differentiate the vector integral componentwise. The moving upper limit contributes its derivative by the chain rule.

Step 1: General rule If

$$\mathbf{F}(t) = \int_a^{g(t)} \mathbf{f}(u) du,$$

then the Fundamental Theorem gives the derivative with respect to the upper limit, while the chain rule multiplies by $g'(t)$:

$$\mathbf{F}'(t) = \mathbf{f}(g(t))g'(t).$$

Step 2: Apply the rule Here $g(t) = t^2$, so $g'(t) = 2t$. Therefore

$$\begin{aligned}\mathbf{F}'(t) &= 2t \left\langle e^{t^2}, \frac{1}{t^2}, \cos(\pi t^2) \right\rangle \\ &= \left\langle 2te^{t^2}, \frac{2}{t}, 2t \cos(\pi t^2) \right\rangle.\end{aligned}$$

The restriction $t > 0$ ensures the interval between 1 and t^2 never crosses the singularity $u = 0$.

Step 3: Values at $t = 1$

$$\mathbf{F}'(1) = \langle 2e, 2, -2 \rangle.$$

Also, an integral with identical limits is the zero vector:

$$\mathbf{F}(1) = \int_1^1 \mathbf{f}(u) du = \langle 0, 0, 0 \rangle.$$

These results distinguish accumulated value from its instantaneous rate of change.

Question 9

Let

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \quad t \geq 0,$$

and reparametrize it by $t = s^2 + 1$.

Tasks

1. Write the composite vector function $\mathbf{R}(s) = \mathbf{r}(s^2 + 1)$.
2. Compute $\mathbf{R}'(s)$ directly.
3. Compute it again using the vector chain rule and reconcile the fact that $\mathbf{R}'(0) = \mathbf{0}$ while $\mathbf{r}'(1) \neq \mathbf{0}$.

Question 9 – Solution

Strategy. Compare direct componentwise differentiation with $(\mathbf{r} \circ g)' = \mathbf{r}'(g)g'$.

Step 1: Composition

$$\mathbf{R}(s) = \langle s^2 + 1, (s^2 + 1)^2, (s^2 + 1)^3 \rangle.$$

Step 2: Direct differentiation

$$\begin{aligned}\mathbf{R}'(s) &= \langle 2s, 2(s^2 + 1)(2s), 3(s^2 + 1)^2(2s) \rangle \\ &= \langle 2s, 4s(s^2 + 1), 6s(s^2 + 1)^2 \rangle.\end{aligned}$$

Step 3: Chain-rule check Since

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle, \quad g(s) = s^2 + 1,$$

we obtain

$$\begin{aligned}\mathbf{R}'(s) &= \mathbf{r}'(g(s))g'(s) \\ &= \langle 1, 2(s^2 + 1), 3(s^2 + 1)^2 \rangle (2s),\end{aligned}$$

which is identical to the direct result.

At $s = 0$, $g'(0) = 0$, so $\mathbf{R}'(0) = \mathbf{0}$ even though

$$\mathbf{r}'(1) = \langle 1, 2, 3 \rangle \neq \mathbf{0}.$$

There is no contradiction: the new parameter momentarily stops changing the old parameter. The geometric curve is regular at $t = 1$, but this particular parametrization is not regular at $s = 0$.

Question 10

Find every differentiable vector function $\mathbf{r}(t)$ satisfying

$$\mathbf{r}'(t) = 2t\mathbf{r}(t), \quad \mathbf{r}(0) = \langle a, b, c \rangle,$$

where a, b, c are constants. Then prove directly that any two solutions with the same initial vector must be identical.

Tasks

1. Solve the vector initial-value problem component by component.
2. Verify the resulting formula by differentiation and substitution.
3. Prove uniqueness using an integrating factor applied to the difference of two solutions.

Question 10 – Solution

Strategy. The vector equation is three identical scalar equations. Solve them componentwise, then prove uniqueness with an integrating factor.

Step 1: Split into components Write $\mathbf{r} = \langle x, y, z \rangle$. Then

$$x' = 2tx, \quad y' = 2ty, \quad z' = 2tz.$$

For a nonzero component w , separation gives

$$\frac{w'}{w} = 2t \implies \ln |w| = t^2 + C \implies w = Ke^{t^2}.$$

The formula also includes the zero solution when $K = 0$.

Step 2: Apply the initial vector Since $e^0 = 1$, the constants are a, b, c . Thus

$$\boxed{\mathbf{r}(t) = e^{t^2} \langle a, b, c \rangle}.$$

Indeed,

$$\mathbf{r}'(t) = 2te^{t^2} \langle a, b, c \rangle = 2t\mathbf{r}(t),$$

and $\mathbf{r}(0) = \langle a, b, c \rangle$.

Step 3: Direct uniqueness proof Suppose $\mathbf{r}_1$ and $\mathbf{r}_2$ share the initial value, and let $\mathbf{d} = \mathbf{r}_1 - \mathbf{r}_2$. Then

$$\mathbf{d}' = 2t\mathbf{d}, \quad \mathbf{d}(0) = \mathbf{0}.$$

Multiply by e^{-t^2} and use the product rule:

$$\frac{d}{dt}(e^{-t^2} \mathbf{d}(t)) = -2te^{-t^2} \mathbf{d} + e^{-t^2} \mathbf{d}' = \mathbf{0}.$$

Hence $e^{-t^2} \mathbf{d}$ is constant and equals its value $\mathbf{0}$ at $t = 0$. Therefore $\mathbf{d} = \mathbf{0}$, proving $\boxed{\mathbf{r}_1 = \mathbf{r}_2}$.

Question 1

For

$$\mathbf{r}(t) = \langle t^2, 2t, t^2 - 2t \rangle,$$

find the unit tangent vector at $t = 1$ and write the tangent line there.

Tasks

1. Compute the point and velocity vector.
2. Normalize the velocity to obtain $\mathbf{T}(1)$.
3. Write and verify a tangent-line parametrization.

Question 1 – Solution

Strategy. At a regular point, the unit tangent is $\mathbf{T} = \mathbf{r}' / \|\mathbf{r}'\|$.

Step 1: Point and velocity

$$\mathbf{r}(1) = \langle 1, 2, -1 \rangle, \quad \mathbf{r}'(t) = \langle 2t, 2, 2t - 2 \rangle,$$

so $\mathbf{r}'(1) = \langle 2, 2, 0 \rangle \neq \mathbf{0}$.

Step 2: Normalize

$$\|\mathbf{r}'(1)\| = \sqrt{2^2 + 2^2} = 2\sqrt{2}.$$

Therefore

$$\mathbf{T}(1) = \frac{\langle 2, 2, 0 \rangle}{2\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle.$$

Step 3: Tangent line Either $\mathbf{r}'(1)$ or the parallel unit vector may be used:

$$\mathbf{L}(s) = \langle 1, 2, -1 \rangle + s \langle 1, 1, 0 \rangle.$$

At $s = 0$ the line passes through the curve point, and its direction $\langle 1, 1, 0 \rangle$ is one-half of $\mathbf{r}'(1)$.

Verification. The squared length of $\mathbf{T}(1)$ is $1/2 + 1/2 = 1$, as required.

Question 2

For the circular helix

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle,$$

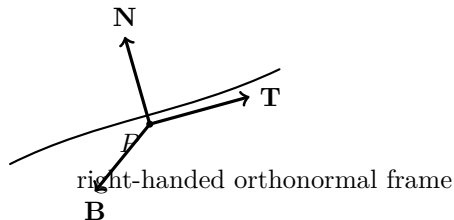
find the complete Frenet frame $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$.

Tasks

1. Normalize $\mathbf{r}'(t)$.
2. Normalize $\mathbf{T}'(t)$.
3. Compute $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ and verify orientation.

Question 2 – Solution

Strategy. The helix has constant speed, which makes each normalization especially transparent.

**Step 1: Unit tangent**

$$\mathbf{r}'(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{9 \sin^2 t + 9 \cos^2 t + 16} = 5.$$

Thus

$$\mathbf{T}(t) = \left\langle -\frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5} \right\rangle.$$

Step 2: Principal unit normal

$$\mathbf{T}'(t) = \left\langle -\frac{3}{5} \cos t, -\frac{3}{5} \sin t, 0 \right\rangle, \quad \|\mathbf{T}'(t)\| = \frac{3}{5}.$$

Therefore

$$\mathbf{N}(t) = \langle -\cos t, -\sin t, 0 \rangle.$$

Step 3: Binormal

$$\begin{aligned} \mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\ &= \left\langle \frac{4}{5} \sin t, -\frac{4}{5} \cos t, \frac{3}{5} \right\rangle. \end{aligned}$$

Its norm is 1, and direct dot products give $\mathbf{T} \cdot \mathbf{N} = \mathbf{T} \cdot \mathbf{B} = \mathbf{N} \cdot \mathbf{B} = 0$. The defining cross product confirms the right-handed orientation.

Question 3

Consider the plane curve

$$\mathbf{r}(t) = \langle t, t^2, 0 \rangle.$$

At $t = 1$, determine $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

Tasks

1. Differentiate and normalize to find $\mathbf{T}(1)$.
2. Differentiate the general $\mathbf{T}(t)$ before evaluating to find $\mathbf{N}(1)$.
3. Compute $\mathbf{B}(1)$ and interpret its direction.

Question 3 – Solution

Strategy. Computing $\mathbf{T}'(1)$ requires differentiating the normalized tangent, not merely normalizing $\mathbf{r}''(1)$.

Step 1: Unit tangent

$$\mathbf{r}'(t) = \langle 1, 2t, 0 \rangle, \quad \mathbf{T}(t) = \frac{\langle 1, 2t, 0 \rangle}{\sqrt{1 + 4t^2}}.$$

Hence

$$\boxed{\mathbf{T}(1) = \frac{1}{\sqrt{5}} \langle 1, 2, 0 \rangle}.$$

Step 2: Differentiate $\mathbf{T}$ Componentwise differentiation gives

$$\mathbf{T}'(t) = \frac{1}{(1 + 4t^2)^{3/2}} \langle -4t, 2, 0 \rangle.$$

At $t = 1$ this vector is a positive multiple of $\langle -2, 1, 0 \rangle$, whose length is $\sqrt{5}$. Therefore

$$\boxed{\mathbf{N}(1) = \frac{1}{\sqrt{5}} \langle -2, 1, 0 \rangle}.$$

Step 3: Binormal

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \frac{1}{5} \langle 1, 2, 0 \rangle \times \langle -2, 1, 0 \rangle = \boxed{\langle 0, 0, 1 \rangle}.$$

The curve lies in the xy -plane, so its binormal points perpendicular to that plane. The three vectors are unit length and mutually perpendicular.

Question 4

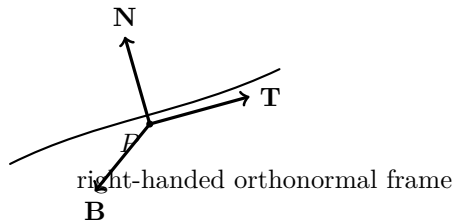
Let $\mathbf{r}$ be a regular curve for which $\mathbf{T}'(t) \neq \mathbf{0}$.

Tasks

1. Prove that $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$.
2. Prove that $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector.
3. Explain why $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is a right-handed orthonormal frame.

Question 4 – Solution

Strategy. Differentiate the invariant $\mathbf{T} \cdot \mathbf{T} = 1$, then use standard cross-product identities.



Step 1: Tangent-normal orthogonality Since $\mathbf{T}$ is a unit vector,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = 1.$$

Differentiate using the dot-product rule:

$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 0 \implies 2\mathbf{T} \cdot \mathbf{T}' = 0.$$

Because $\mathbf{N} = \mathbf{T}' / \|\mathbf{T}'\|$ and the denominator is nonzero,

$$\boxed{\mathbf{T} \cdot \mathbf{N} = 0}.$$

Step 2: Length of $\mathbf{B}$ For vectors separated by angle θ ,

$$\|\mathbf{T} \times \mathbf{N}\| = \|\mathbf{T}\| \|\mathbf{N}\| \sin \theta.$$

Here both lengths are 1 and $\theta = \pi/2$, so

$$\boxed{\|\mathbf{B}\| = 1}.$$

The cross product is perpendicular to both factors; hence $\mathbf{B}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

Step 3: Orientation The ordered relation $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ is exactly the right-hand rule. Thus the three mutually perpendicular unit vectors form a right-handed orthonormal frame.

Question 5

The curve

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$$

is reparametrized by $t = 2u + 1$. Let $\mathbf{R}(u) = \mathbf{r}(2u + 1)$.

Tasks

1. Compare the unit tangents of $\mathbf{r}$ at $t = 2u + 1$ and $\mathbf{R}$ at u .
2. Compare their principal normals.
3. State what changes if the reparametrization is instead $t = 1 - 2u$.

Question 5 – Solution

Strategy. Apply the chain rule and keep track of the sign of the parameter derivative during normalization.

Step 1: Increasing reparametrization The chain rule gives

$$\mathbf{R}'(u) = 2\mathbf{r}'(2u + 1).$$

Since the factor 2 is positive,

$$\frac{\mathbf{R}'(u)}{\|\mathbf{R}'(u)\|} = \frac{2\mathbf{r}'(2u + 1)}{2\|\mathbf{r}'(2u + 1)\|} = \boxed{\mathbf{T}_r(2u + 1)}.$$

Thus an increasing reparametrization preserves the unit tangent.

Step 2: Principal normal Differentiating $\mathbf{T}_R(u) = \mathbf{T}_r(2u + 1)$ gives

$$\mathbf{T}'_R(u) = 2\mathbf{T}'_r(2u + 1).$$

Again the positive factor cancels upon normalization, so

$$\boxed{\mathbf{N}_R(u) = \mathbf{N}_r(2u + 1)}.$$

Step 3: Reversed parameter If $t = 1 - 2u$, then $dt/du = -2$. Consequently

$$\boxed{\mathbf{T}_R = -\mathbf{T}_r},$$

at corresponding points. Differentiating this relation with respect to u introduces a second negative factor, so $\mathbf{T}'_R = 2\mathbf{T}'_r$ and

$$\boxed{\mathbf{N}_R = \mathbf{N}_r}.$$

Therefore $\mathbf{B}_R = \mathbf{T}_R \times \mathbf{N}_R = -\mathbf{B}_r$. Reversing orientation changes $\mathbf{T}$ and $\mathbf{B}$, but not the principal normal.

Question 6

At a regular point of a curve, suppose

$$\mathbf{r}'(t_0) = \langle 1, 2, 2 \rangle, \quad \mathbf{r}''(t_0) = \langle 2, 0, 1 \rangle.$$

Tasks

1. Find the unit tangent $\mathbf{T}(t_0)$.
2. Find the binormal using $\mathbf{r}' \times \mathbf{r}''$.
3. Obtain the principal normal from $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and verify the frame.

Question 6 – Solution

Strategy. The cross product $\mathbf{r}' \times \mathbf{r}''$ gives the binormal direction without differentiating a normalized vector.

Step 1: Tangent

$$\|\mathbf{r}'(t_0)\| = \sqrt{1+4+4} = 3, \quad \boxed{\mathbf{T}(t_0) = \frac{1}{3} \langle 1, 2, 2 \rangle}.$$

Step 2: Binormal direction

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \langle 2, 3, -4 \rangle.$$

Its length is $\sqrt{4+9+16} = \sqrt{29}$, so

$$\boxed{\mathbf{B}(t_0) = \frac{1}{\sqrt{29}} \langle 2, 3, -4 \rangle}.$$

Step 3: Principal normal In a right-handed frame, $\mathbf{B} \times \mathbf{T} = \mathbf{N}$. Thus

$$\begin{aligned} \mathbf{N}(t_0) &= \frac{1}{3\sqrt{29}} \langle 2, 3, -4 \rangle \times \langle 1, 2, 2 \rangle \\ &= \boxed{\frac{1}{3\sqrt{29}} \langle 14, -8, 1 \rangle}. \end{aligned}$$

Its squared numerator length is $196 + 64 + 1 = 261 = 9(29)$, so it has unit length. Direct dot products with $\mathbf{T}$ and $\mathbf{B}$ are zero, verifying the frame.

Question 7

For

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle,$$

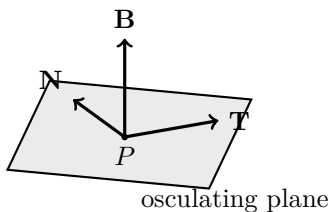
find equations of the osculating plane and normal plane at $t = 1$.

Tasks

1. Compute suitable directions for $\mathbf{T}$ and $\mathbf{B}$.
2. Use $\mathbf{B}$ as the osculating-plane normal.
3. Use $\mathbf{T}$ as the normal-plane normal and verify both planes pass through the curve point.

Question 7 – Solution

Strategy. The osculating plane is spanned by $\mathbf{T}$ and $\mathbf{N}$, so its normal is $\mathbf{B}$. The normal plane is perpendicular to $\mathbf{T}$.

**Step 1: Point and tangent direction**

$$P = \mathbf{r}(1) = (1, 1, 1), \quad \mathbf{r}'(1) = \langle 1, 2, 3 \rangle.$$

We may use $\langle 1, 2, 3 \rangle$ as the normal vector of the normal plane.

Step 2: Binormal direction

$$\mathbf{r}''(t) = \langle 0, 2, 6t \rangle, \quad \mathbf{r}''(1) = \langle 0, 2, 6 \rangle.$$

Then

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 3 \rangle \times \langle 0, 2, 6 \rangle = \langle 6, -6, 2 \rangle = 2 \langle 3, -3, 1 \rangle.$$

Hence an osculating-plane normal is $\langle 3, -3, 1 \rangle$.

Step 3: Plane equations Through $P = (1, 1, 1)$, the osculating plane is

$$3(x - 1) - 3(y - 1) + (z - 1) = 0 \implies \boxed{3x - 3y + z = 1}.$$

The normal plane is

$$(x - 1) + 2(y - 1) + 3(z - 1) = 0 \implies \boxed{x + 2y + 3z = 6}.$$

Substitution of $(1, 1, 1)$ verifies both equations, and their normals have the required Frenet directions.

Question 8

Consider

$$\mathbf{r}(t) = \langle t, t^3, 0 \rangle.$$

Tasks

1. Find $\mathbf{T}(0)$.
2. Determine whether the principal normal and binormal are defined at $t = 0$ by the standard Frenet formulas.
3. Compute the one-sided limiting normal directions and explain the geometric issue.

Question 8 – Solution

Strategy. Regularity of $\mathbf{r}$ guarantees $\mathbf{T}$, but defining $\mathbf{N}$ additionally requires $\mathbf{T}' \neq \mathbf{0}$.

Step 1: Unit tangent

$$\mathbf{r}'(t) = \langle 1, 3t^2, 0 \rangle, \quad \mathbf{T}(t) = \frac{\langle 1, 3t^2, 0 \rangle}{\sqrt{1 + 9t^4}}.$$

Thus $\boxed{\mathbf{T}(0) = \langle 1, 0, 0 \rangle}$.

Step 2: Differentiate $\mathbf{T}$ Differentiation and simplification give

$$\mathbf{T}'(t) = \frac{1}{(1 + 9t^4)^{3/2}} \langle -18t^3, 6t, 0 \rangle = \frac{6t}{(1 + 9t^4)^{3/2}} \langle -3t^2, 1, 0 \rangle.$$

Therefore $\mathbf{T}'(0) = \mathbf{0}$. The expression

$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|}$$

would divide by zero, so $\boxed{\mathbf{N}(0) \text{ and } \mathbf{B}(0) \text{ are not defined by the Frenet formulas}}$.

Step 3: One-sided behavior For $t > 0$, normalization yields a vector tending to $\langle 0, 1, 0 \rangle$. For $t < 0$, the scalar factor $6t$ reverses direction, giving the limit $\langle 0, -1, 0 \rangle$. The unequal limits show that the preferred bending direction flips at the inflection point. Although the curve has a tangent there, it has no single principal-normal direction.

Question 9

Let

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle.$$

Find all $t \in [0, 2\pi)$ for which the unit tangent is perpendicular to the fixed vector $\mathbf{a} = \langle 1, 1, 0 \rangle$, and find the corresponding points.

Tasks

1. Explain why it is enough to test $\mathbf{r}'(t) \cdot \mathbf{a} = 0$.
2. Solve the resulting trigonometric equation exactly.
3. Evaluate the curve at every admissible parameter.

Question 9 – Solution

Strategy. Normalizing a nonzero vector multiplies it by a positive scalar, which does not change whether its dot product with $\mathbf{a}$ is zero.

Step 1: Tangent direction

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 1 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{5}.$$

Thus $\mathbf{T} \cdot \mathbf{a} = 0$ exactly when $\mathbf{r}' \cdot \mathbf{a} = 0$.

Step 2: Solve

$$\begin{aligned} \mathbf{r}'(t) \cdot \mathbf{a} &= -2 \sin t + 2 \cos t = 0 \\ \implies \sin t &= \cos t \\ \implies \tan t &= 1. \end{aligned}$$

On $[0, 2\pi)$,

$$t = \frac{\pi}{4}, \frac{5\pi}{4}.$$

Neither solution is lost by using tangent because $\cos t \neq 0$ at both values.

Step 3: Points

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left\langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4} \right\rangle,$$

and

$$\mathbf{r}\left(\frac{5\pi}{4}\right) = \left\langle -\sqrt{2}, -\sqrt{2}, \frac{5\pi}{4} \right\rangle.$$

At either parameter, substituting into $-2 \sin t + 2 \cos t$ gives zero, verifying perpendicularity.

Question 10

At a point $P = (1, -1, 2)$ on a regular curve, a unit tangent and a candidate principal normal are

$$\mathbf{T} = \frac{1}{3} \langle 2, 1, 2 \rangle, \quad \mathbf{N} = \frac{1}{3\sqrt{5}} \langle -4, 7, \frac{1}{2} \rangle.$$

Tasks

1. Determine whether the supplied vectors can belong to a Frenet frame, and diagnose any defect.
2. Replace $\mathbf{N}$ by the unit vector in the direction of the component of $\langle -4, 7, 1 \rangle$ perpendicular to $\mathbf{T}$.
3. Find the corrected $\mathbf{B}$ and the osculating plane at P .

Question 10 – Solution

Strategy. First audit unit length and orthogonality. Then use vector projection to remove the tangential component from the proposed bending direction.

Step 1: Diagnose the supplied normal The tangent is unit because $(4 + 1 + 4)/9 = 1$. But

$$\mathbf{T} \cdot \mathbf{N} = \frac{2(-4) + 1(7) + 2(1/2)}{9\sqrt{5}} = 0,$$

while

$$\|\mathbf{N}\|^2 = \frac{16 + 49 + 1/4}{45} = \frac{29}{20} \neq 1.$$

Thus it is perpendicular to $\mathbf{T}$ but is not a unit vector, so the data are not a Frenet frame.

Step 2: Correct the bending direction Let $\mathbf{w} = \langle -4, 7, 1 \rangle$. Its tangential scalar component is

$$\mathbf{w} \cdot \mathbf{T} = \frac{-8 + 7 + 2}{3} = \frac{1}{3}.$$

Therefore

$$\mathbf{w}_\perp = \mathbf{w} - (\mathbf{w} \cdot \mathbf{T})\mathbf{T} = \langle -4, 7, 1 \rangle - \frac{1}{9} \langle 2, 1, 2 \rangle = \frac{1}{9} \langle -38, 62, 7 \rangle.$$

Since $\sqrt{(-38)^2 + 62^2 + 7^2} = \sqrt{5337} = 3\sqrt{593}$,

$$\hat{\mathbf{N}} = \frac{1}{3\sqrt{593}} \langle -38, 62, 7 \rangle.$$

Step 3: Binormal and osculating plane

$$\mathbf{B} = \mathbf{T} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{593}} \langle -13, -10, 18 \rangle.$$

The numerator has length $\sqrt{169 + 100 + 324} = \sqrt{593}$. Since $\mathbf{B}$ is normal to the osculating plane through P ,

$$-13(x - 1) - 10(y + 1) + 18(z - 2) = 0,$$

or

$$13x + 10y - 18z + 33 = 0.$$

Substitution of P gives zero, and the construction guarantees a right-handed orthonormal frame.

Question 1

Find the exact length of

$$\mathbf{r}(t) = \left\langle 3t, 4t, \frac{1}{2}t^2 \right\rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Compute the speed and simplify it completely.
2. Set up and evaluate the arc-length integral.
3. Check that the result exceeds the straight-line distance between the endpoints.

Question 1 – Solution

Strategy. Arc length is the integral of speed: $L = \int_a^b \|\mathbf{r}'(t)\| dt$.

Step 1: Velocity and speed

$$\mathbf{r}'(t) = \langle 3, 4, t \rangle.$$

Therefore

$$\|\mathbf{r}'(t)\| = \sqrt{3^2 + 4^2 + t^2} = \sqrt{25 + t^2}.$$

Step 2: Integrate Use

$$\int \sqrt{t^2 + a^2} dt = \frac{1}{2} \left(t\sqrt{t^2 + a^2} + a^2 \ln \left| t + \sqrt{t^2 + a^2} \right| \right).$$

With $a = 5$,

$$\begin{aligned} L &= \int_0^2 \sqrt{25 + t^2} dt \\ &= \frac{1}{2} \left[t\sqrt{25 + t^2} + 25 \ln \left(t + \sqrt{25 + t^2} \right) \right]_0^2 \\ &= \boxed{\sqrt{29} + \frac{25}{2} \ln \left(\frac{2 + \sqrt{29}}{5} \right)}. \end{aligned}$$

Step 3: Endpoint check The displacement is

$$\mathbf{r}(2) - \mathbf{r}(0) = \langle 6, 8, 2 \rangle,$$

whose length is $\sqrt{36 + 64 + 4} = 2\sqrt{26} \approx 10.198$. The arc length is approximately 10.261, which is slightly larger, as a nonstraight path requires.

Question 2

A helix is parametrized by

$$\mathbf{r}(t) = \langle 5 \cos t, 5 \sin t, 12t \rangle, \quad 0 \leq t \leq 3\pi.$$

Tasks

1. Find its exact length.
2. Determine the length traveled during one complete revolution.
3. Explain why equal parameter intervals produce equal arc lengths here.

Question 2 – Solution

Strategy. Differentiate and look for constant speed.

Step 1: Speed

$$\mathbf{r}'(t) = \langle -5 \sin t, 5 \cos t, 12 \rangle.$$

Hence

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{25 \sin^2 t + 25 \cos^2 t + 144} \\ &= \sqrt{25 + 144} = 13.\end{aligned}$$

Step 2: Total length

$$L = \int_0^{3\pi} 13 \, dt = 13[t]_0^{3\pi} = \boxed{39\pi}.$$

Step 3: One revolution A complete revolution changes t by 2π . Therefore

$$L_{\text{rev}} = \int_t^{t+2\pi} 13 \, du = \boxed{26\pi}.$$

Because the speed is the same constant 13 at every parameter value, an interval of width Δt always contributes length $13\Delta t$. Thus equal parameter intervals have equal arc lengths.

Verification. The interval $[0, 3\pi]$ contains $3/2$ revolutions, and $(3/2)(26\pi) = 39\pi$, confirming the total.

Question 3

Determine the exact length of

$$\mathbf{r}(t) = \left\langle t - \frac{t^3}{3}, t^2, t + \frac{t^3}{3} \right\rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Compute the speed without prematurely expanding every square.
2. Identify and justify the perfect-square simplification.
3. Evaluate the length and verify the speed is never negative.

Question 3 – Solution

Strategy. Differentiate first and group symmetric components; the radicals collapse to a polynomial.

Step 1: Differentiate

$$\mathbf{r}'(t) = \langle 1 - t^2, 2t, 1 + t^2 \rangle.$$

Thus

$$\begin{aligned}\|\mathbf{r}'(t)\|^2 &= (1 - t^2)^2 + (2t)^2 + (1 + t^2)^2 \\ &= (1 - 2t^2 + t^4) + 4t^2 + (1 + 2t^2 + t^4) \\ &= 2 + 4t^2 + 2t^4 \\ &= 2(1 + t^2)^2.\end{aligned}$$

Step 2: Take the square root

$$\|\mathbf{r}'(t)\| = \sqrt{2}|1 + t^2|.$$

Since $1 + t^2 > 0$ for every real t , this becomes

$$\|\mathbf{r}'(t)\| = \sqrt{2}(1 + t^2).$$

Step 3: Integrate

$$\begin{aligned}L &= \sqrt{2} \int_0^2 (1 + t^2) dt \\ &= \sqrt{2} \left[t + \frac{t^3}{3} \right]_0^2 \\ &= \boxed{\frac{14\sqrt{2}}{3}}.\end{aligned}$$

The speed is strictly positive because $1 + t^2 > 0$, so the curve is regular throughout the interval.

Question 4

For

$$\mathbf{r}(t) = \langle 3t, 4t, 0 \rangle, \quad t \geq 1,$$

let $s(t)$ denote distance traveled from the starting parameter $t = 1$.

Tasks

1. Find $s(t)$ and its inverse $t(s)$.
2. Reparametrize the curve by arc length measured from $t = 1$.
3. Verify directly that the new parametrization has unit speed.

Question 4 – Solution

Strategy. Integrate speed from the specified base parameter, invert the resulting monotone function, and substitute.

Step 1: Arc-length function

$$\mathbf{r}'(t) = \langle 3, 4, 0 \rangle, \quad \|\mathbf{r}'(t)\| = 5.$$

Therefore

$$s(t) = \int_1^t 5 \, du = 5(t - 1).$$

Since $t \geq 1$, we have $s \geq 0$, and solving for t gives

$$t(s) = 1 + \frac{s}{5}.$$

Step 2: Substitute

$$\begin{aligned} \mathbf{R}(s) &= \mathbf{r}\left(1 + \frac{s}{5}\right) \\ &= \left\langle 3 + \frac{3}{5}s, 4 + \frac{4}{5}s, 0 \right\rangle, \quad s \geq 0. \end{aligned}$$

Step 3: Unit-speed verification

$$\mathbf{R}'(s) = \left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle,$$

so

$$\|\mathbf{R}'(s)\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1.$$

Thus an increase of one unit in s corresponds to exactly one unit of distance along the curve.

Question 5

The upper semicircle $x^2 + y^2 = 4$ is represented by

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 0 \rangle, \quad 0 \leq t \leq \pi,$$

and also by

$$\mathbf{R}(u) = \langle 2 \cos(u^3), 2 \sin(u^3), 0 \rangle, \quad 0 \leq u \leq \sqrt[3]{\pi}.$$

Tasks

1. Compute the length using each parametrization.
2. Explain why the nonconstant speed of $\mathbf{R}$ does not change the result.
3. Identify the substitution connecting the two integrals.

Question 5 – Solution

Strategy. Calculate both speed integrals, taking care that $u \geq 0$ when simplifying an absolute value.

Step 1: Standard parametrization

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 0 \rangle, \quad \|\mathbf{r}'(t)\| = 2.$$

Hence

$$L_r = \int_0^\pi 2 \, dt = \boxed{2\pi}.$$

Step 2: Cubic reparametrization

$$\mathbf{R}'(u) = \langle -6u^2 \sin(u^3), 6u^2 \cos(u^3), 0 \rangle.$$

Because $u \geq 0$,

$$\|\mathbf{R}'(u)\| = 6u^2.$$

Therefore

$$L_R = \int_0^{\sqrt[3]{\pi}} 6u^2 \, du = [2u^3]_0^{\sqrt[3]{\pi}} = \boxed{2\pi}.$$

Step 3: Invariance The substitution $t = u^3$ gives $dt = 3u^2 du$, so

$$2 \, dt = 6u^2 \, du.$$

The second parametrization changes the rate at which points are visited, but it is increasing and covers the same semicircle once. The speed factor exactly compensates for the parameter distortion.

Question 6

A path travels from $A = (-2, 0, 0)$ to $C = (0, 2, 0)$ and then to $B = (2, 0, 0)$ according to

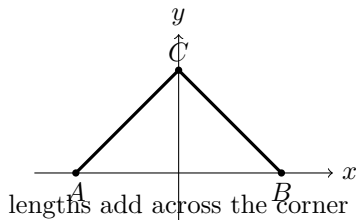
$$\mathbf{r}(t) = \begin{cases} \langle t, t+2, 0 \rangle, & -2 \leq t \leq 0, \\ \langle t, 2-t, 0 \rangle, & 0 < t \leq 2. \end{cases}$$

Tasks

1. Verify continuity at the joining parameter.
2. Compute the total arc length by splitting the integral.
3. Compare it with $\|B - A\|$ and explain the difference.

Question 6 – Solution

Strategy. Arc length is additive across a corner, even though the derivative changes abruptly there.



Step 1: Join check Both formulas give

$$\langle 0, 2, 0 \rangle$$

at $t = 0$ (the second by its right-hand limit), so the path is continuous at C .

Step 2: Length of each segment On the first branch,

$$\mathbf{r}'(t) = \langle 1, 1, 0 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{2}.$$

On the second branch,

$$\mathbf{r}'(t) = \langle 1, -1, 0 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{2}.$$

Therefore

$$\begin{aligned} L &= \int_{-2}^0 \sqrt{2} \, dt + \int_0^2 \sqrt{2} \, dt \\ &= 2\sqrt{2} + 2\sqrt{2} = \boxed{4\sqrt{2}}. \end{aligned}$$

Step 3: Chord comparison The endpoint distance is

$$\|B - A\| = \|\langle 4, 0, 0 \rangle\| = 4.$$

Since $4\sqrt{2} > 4$, the broken path is longer. Equality would require travel directly along the segment from A to B without the detour through C .

Question 7

For

$$\mathbf{r}(t) = \left\langle t^2, \frac{1}{6}(4t+1)^{3/2}, 0 \right\rangle, \quad t \geq 0,$$

find the parameter value $T > 0$ at which the distance traveled from $t = 0$ first equals $35/4$.

Tasks

1. Derive the arc-length function $s(T)$.
2. Solve the resulting equation exactly.
3. Explain why the solution is unique.

Question 7 – Solution

Strategy. Differentiate carefully; the squared speed is a perfect square on the stated domain.

Step 1: Speed

$$\mathbf{r}'(t) = \langle 2t, \sqrt{4t+1}, 0 \rangle.$$

For $t \geq 0$,

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{4t^2 + 4t + 1} \\ &= \sqrt{(2t+1)^2} = |2t+1| = 2t+1.\end{aligned}$$

Step 2: Arc-length function

$$\begin{aligned}s(T) &= \int_0^T (2t+1) dt \\ &= [t^2 + t]_0^T = T^2 + T.\end{aligned}$$

Set this equal to the prescribed distance:

$$T^2 + T = \frac{35}{4}.$$

Multiplying by 4 and factoring,

$$4T^2 + 4T - 35 = (2T-5)(2T+7) = 0.$$

The candidates are $T = 5/2$ and $T = -7/2$; only the first lies in the domain. Hence

$$\boxed{T = \frac{5}{2}}.$$

Step 3: Uniqueness For $T > 0$, $s'(T) = 2T + 1 > 0$, so s is strictly increasing. Therefore the displayed equation has at most one positive solution. Since $s(0) = 0$ and $s(T) \rightarrow \infty$, it has exactly one. Direct substitution gives $s(5/2) = 25/4 + 10/4 = 35/4$, verifying the result.

Question 8

Find the exact length of the semicubical parabola

$$\mathbf{r}(t) = \langle t^2, t^3, 0 \rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Factor the speed using the sign of t .
2. Evaluate the arc-length integral by substitution.
3. Explain why the zero velocity at $t = 0$ does not make the length integral improper.

Question 8 – Solution

Strategy. Factor t^2 from the squared speed and use $t \geq 0$ before integrating.

Step 1: Speed

$$\mathbf{r}'(t) = \langle 2t, 3t^2, 0 \rangle.$$

Hence

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{4t^2 + 9t^4} \\ &= \sqrt{t^2(4 + 9t^2)} \\ &= |t|\sqrt{4 + 9t^2}.\end{aligned}$$

On $[0, 2]$, $|t| = t$, so the speed is $t\sqrt{4 + 9t^2}$.

Step 2: Integrate Let $u = 4 + 9t^2$, so $du = 18t \, dt$:

$$\begin{aligned}L &= \int_0^2 t\sqrt{4 + 9t^2} \, dt \\ &= \frac{1}{18} \int_4^{40} u^{1/2} \, du \\ &= \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_4^{40} \\ &= \frac{1}{27} (40^{3/2} - 4^{3/2}) \\ &= \boxed{\frac{80\sqrt{10} - 8}{27}}.\end{aligned}$$

Step 3: Endpoint regularity Although $\mathbf{r}'(0) = \mathbf{0}$, the speed $t\sqrt{4 + 9t^2}$ is continuous at 0 and equals zero there. A continuous integrand on a closed interval defines an ordinary proper integral; no singularity is present.

Question 9

A wire is wound exactly N times around a cylinder of radius R , rising uniformly through total height H . Model its centerline by

$$\mathbf{r}(t) = \left\langle R \cos t, R \sin t, \frac{H}{2\pi N} t \right\rangle, \quad 0 \leq t \leq 2\pi N.$$

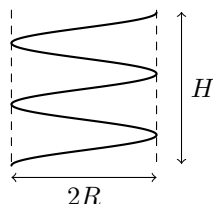
Tasks

1. Derive a formula for the wire length in terms of R, H, N .
2. For $R = 3$, $H = 24$, and length 30π , determine $N > 0$.
3. Verify whether the resulting value represents a whole number of turns.

Question 9 – Solution

Strategy. The model has constant speed; after integration, solve the design equation by squaring positive quantities.

wire wound around a cylinder



Step 1: General formula

$$\mathbf{r}'(t) = \left\langle -R \sin t, R \cos t, \frac{H}{2\pi N} \right\rangle,$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{R^2 + \left(\frac{H}{2\pi N}\right)^2}.$$

Multiplying this constant speed by the parameter length $2\pi N$ gives

$$L = \sqrt{(2\pi NR)^2 + H^2}.$$

Step 2: Apply the data With $R = 3$, $H = 24$, and $L = 30\pi$,

$$(30\pi)^2 = (6\pi N)^2 + 24^2.$$

Therefore

$$\begin{aligned} 36\pi^2 N^2 &= 900\pi^2 - 576, \\ N^2 &= 25 - \frac{16}{\pi^2}. \end{aligned}$$

Since $N > 0$,

$$N = \sqrt{25 - \frac{16}{\pi^2}}.$$

Step 3: Feasibility Numerically, $N \approx 4.835$, which is not an integer. Thus the given continuous model has a positive solution, but the specifications are incompatible with a whole number of complete turns. This is a design-consistency conclusion, not an algebraic failure.

Question 10

Let $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ be continuously differentiable.

Tasks

1. Prove the chord–arc inequality

$$\|\mathbf{r}(b) - \mathbf{r}(a)\| \leq \int_a^b \|\mathbf{r}'(t)\| dt.$$

2. Give a clear geometric interpretation.
3. Show that equality holds if $\mathbf{r}'(t) = \lambda(t)\mathbf{u}$, where $\mathbf{u}$ is fixed unit vector and $\lambda(t) \geq 0$.

Question 10 – Solution

Strategy. Express displacement as the integral of the velocity vector and apply a dot-product estimate in the displacement direction.

Step 1: Zero-displacement case If $\mathbf{r}(b) = \mathbf{r}(a)$, the left side is zero and the inequality follows because the integral of speed is nonnegative.

Now suppose $\mathbf{d} = \mathbf{r}(b) - \mathbf{r}(a) \neq \mathbf{0}$ and define the unit vector $\mathbf{u} = \mathbf{d}/\|\mathbf{d}\|$.

Step 2: Use the Fundamental Theorem

$$\mathbf{d} = \int_a^b \mathbf{r}'(t) dt.$$

Dot both sides with $\mathbf{u}$:

$$\begin{aligned}\|\mathbf{d}\| &= \mathbf{d} \cdot \mathbf{u} \\ &= \int_a^b \mathbf{r}'(t) \cdot \mathbf{u} dt \\ &\leq \int_a^b |\mathbf{r}'(t) \cdot \mathbf{u}| dt \\ &\leq \int_a^b \|\mathbf{r}'(t)\| \|\mathbf{u}\| dt \\ &= \int_a^b \|\mathbf{r}'(t)\| dt.\end{aligned}$$

Therefore

$$\|\mathbf{r}(b) - \mathbf{r}(a)\| \leq \int_a^b \|\mathbf{r}'(t)\| dt.$$

Step 3: Geometry and equality The straight chord is the shortest connection between the endpoints; following a curved or reversing path cannot be shorter. If $\mathbf{r}' = \lambda \mathbf{u}$ with $\lambda \geq 0$, then

$$\mathbf{r}(b) - \mathbf{r}(a) = \mathbf{u} \int_a^b \lambda(t) dt,$$

so its norm is $\int_a^b \lambda(t) dt$. Meanwhile $\|\mathbf{r}'\| = \lambda$, giving equality. The condition describes travel in one fixed direction without backtracking.

Question 1

For the plane curve $\mathbf{r}(t) = \langle t, t^2, 0 \rangle$, find the curvature at $t = 1$.

Tasks

1. Compute $\mathbf{r}'$ and $\mathbf{r}''$.
2. Use $\kappa = \|\mathbf{r}' \times \mathbf{r}''\| / \|\mathbf{r}'\|^3$.
3. State the radius of curvature at the point.

Question 1 – Solution

Strategy. The cross-product formula avoids differentiating the unit tangent.

Step 1: Derivatives

$$\mathbf{r}'(t) = \langle 1, 2t, 0 \rangle, \quad \mathbf{r}''(t) = \langle 0, 2, 0 \rangle.$$

At $t = 1$, $\mathbf{r}'(1) = \langle 1, 2, 0 \rangle$ and $\|\mathbf{r}'(1)\| = \sqrt{5}$.

Step 2: Cross product

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 0 \rangle \times \langle 0, 2, 0 \rangle = \langle 0, 0, 2 \rangle,$$

so its magnitude is 2. Therefore

$$\kappa(1) = \frac{2}{(\sqrt{5})^3} = \frac{2}{5\sqrt{5}}.$$

Step 3: Radius of curvature Since $\rho = 1/\kappa$,

$$\rho(1) = \frac{5\sqrt{5}}{2}.$$

Both values are positive, and their product is 1, verifying the reciprocal relationship.

Question 2

Find the curvature of the circular helix

$$\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle, \quad a > 0,$$

where b is constant.

Tasks

1. Derive $\kappa(t)$ symbolically.
2. Explain why it is constant.
3. Check the special cases $b = 0$ and increasing $|b|$.

Question 2 – Solution

Strategy. Compute the speed and cross product in symbolic form before simplifying.

Step 1: Derivatives and speed

$$\mathbf{r}' = \langle -a \sin t, a \cos t, b \rangle, \quad \mathbf{r}'' = \langle -a \cos t, -a \sin t, 0 \rangle,$$

and

$$\|\mathbf{r}'\| = \sqrt{a^2 + b^2}.$$

Step 2: Cross product

$$\mathbf{r}' \times \mathbf{r}'' = \langle ab \sin t, -ab \cos t, a^2 \rangle.$$

Its magnitude is

$$\sqrt{a^2 b^2 (\sin^2 t + \cos^2 t) + a^4} = a \sqrt{a^2 + b^2}.$$

Thus

$$\kappa(t) = \frac{a \sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}.$$

Step 3: Interpretation No t remains, so curvature is constant. When $b = 0$, the helix becomes a circle of radius a and $\kappa = 1/a$. As $|b|$ grows, $a/(a^2 + b^2)$ decreases: the helix stretches vertically and bends less sharply.

Question 3

For the graph $y = \ln x$, $x > 0$, determine where its curvature is greatest.

Tasks

1. Derive $\kappa(x)$ using the graph formula.
2. Maximize it exactly.
3. Give the point and maximum curvature.

Question 3 – Solution

Strategy. Use $\kappa = |y''|/[1 + (y')^2]^{3/2}$, then simplify before differentiating.

Step 1: Curvature function

$$y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}.$$

Therefore

$$\begin{aligned}\kappa(x) &= \frac{1/x^2}{(1 + 1/x^2)^{3/2}} \\ &= \frac{x}{(x^2 + 1)^{3/2}}, \quad x > 0.\end{aligned}$$

Step 2: Critical point

$$\begin{aligned}\kappa'(x) &= (x^2 + 1)^{-3/2} - 3x^2(x^2 + 1)^{-5/2} \\ &= \frac{1 - 2x^2}{(x^2 + 1)^{5/2}}.\end{aligned}$$

Thus $\kappa'(x) = 0$ when $x = 1/\sqrt{2}$. The numerator is positive before this point and negative after it, so this is the unique maximum.

Step 3: Point and value The point is

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2} \ln 2 \right).$$

At that value,

$$\kappa_{\max} = \frac{1/\sqrt{2}}{(3/2)^{3/2}} = \boxed{\frac{2}{3\sqrt{3}}}.$$

Also $\kappa \rightarrow 0$ as $x \rightarrow 0^+$ or $x \rightarrow \infty$, confirming the interior maximum.

Question 4

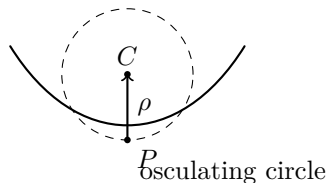
Find the osculating circle of $y = x^2$ at the vertex.

Tasks

1. Compute the curvature and principal-normal direction at $x = 0$.
2. Find the center and radius of the osculating circle.
3. Write its Cartesian equation and verify tangency.

Question 4 – Solution

Strategy. The center lies one radius of curvature from the point in the principal-normal direction.



Step 1: Curvature For $y = x^2$,

$$y' = 2x, \quad y'' = 2,$$

so

$$\kappa(0) = \frac{|2|}{(1+0)^{3/2}} = 2, \quad \rho(0) = \frac{1}{2}.$$

At the vertex the curve bends upward, so $\mathbf{N} = \langle 0, 1 \rangle$.

Step 2: Center With $P = (0, 0)$,

$$C = P + \rho \mathbf{N} = (0, 0) + \frac{1}{2}(0, 1) = \left(0, \frac{1}{2}\right).$$

Step 3: Circle equation Radius $1/2$ and center $(0, 1/2)$ give

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

The circle contains $(0, 0)$. Implicit differentiation gives

$$2x + 2\left(y - \frac{1}{2}\right)y' = 0,$$

so at $(0, 0)$, $y' = 0$, matching the parabola's horizontal tangent. The common curvature at contact gives second-order agreement.

Question 5

The same parabola is parametrized by

$$\mathbf{r}(t) = \langle t, t^2, 0 \rangle \quad \text{and} \quad \mathbf{R}(u) = \langle u^3, u^6, 0 \rangle, \quad u > 0.$$

Tasks

1. Compute curvature from each parametrization.
2. Compare the answers at corresponding points $t = u^3$.
3. Explain why the restriction $u > 0$ is useful.

Question 5 – Solution

Strategy. Apply the cross-product formula twice and simplify all powers before comparing.

Step 1: First parametrization

$$\mathbf{r}' = \langle 1, 2t, 0 \rangle, \quad \mathbf{r}'' = \langle 0, 2, 0 \rangle.$$

Thus

$$\kappa_r(t) = \frac{2}{(1 + 4t^2)^{3/2}}.$$

Step 2: Second parametrization

$$\mathbf{R}' = \langle 3u^2, 6u^5, 0 \rangle, \quad \mathbf{R}'' = \langle 6u, 30u^4, 0 \rangle.$$

The cross product has k -component

$$(3u^2)(30u^4) - (6u^5)(6u) = 54u^6.$$

Also

$$\|\mathbf{R}'\| = 3u^2\sqrt{1 + 4u^6}$$

because $u > 0$. Hence

$$\kappa_R(u) = \frac{54u^6}{27u^6(1 + 4u^6)^{3/2}} = \frac{2}{(1 + 4u^6)^{3/2}}.$$

Step 3: Compare Substituting $t = u^3$ into κ_r gives exactly κ_R . Curvature depends on the geometric curve, not its regular parametrization. The restriction $u > 0$ ensures $\mathbf{R}' \neq \mathbf{0}$ and selects the positive-parameter branch. The speed simplification uses $u^2 \geq 0$, which also holds for negative u .

Question 6

At a point of a regular curve,

$$\mathbf{r}' = \langle 2, -1, 2 \rangle, \quad \mathbf{r}'' = \langle 1, 2, 0 \rangle.$$

Tasks

1. Compute the curvature.
2. Decompose $\mathbf{r}''$ into components parallel and perpendicular to $\mathbf{r}'$.
3. Verify curvature using $\kappa = \|\mathbf{r}''_{\perp}\|/\|\mathbf{r}'\|^2$.

Question 6 – Solution

Strategy. Compare the cross-product formula with the geometric perpendicular-acceleration formula.

Step 1: Cross-product computation

$$\mathbf{r}' \times \mathbf{r}'' = \langle 2, -1, 2 \rangle \times \langle 1, 2, 0 \rangle = \langle -4, 2, 5 \rangle.$$

Its length is $\sqrt{16 + 4 + 25} = 3\sqrt{5}$, while $\|\mathbf{r}'\| = 3$. Therefore

$$\kappa = \frac{3\sqrt{5}}{3^3} = \frac{\sqrt{5}}{9}.$$

Step 2: Parallel component Since

$$\mathbf{r}'' \cdot \mathbf{r}' = 2 - 2 + 0 = 0,$$

the projection of $\mathbf{r}''$ onto $\mathbf{r}'$ is

$$\mathbf{r}''_{\parallel} = \frac{\mathbf{r}'' \cdot \mathbf{r}'}{\|\mathbf{r}'\|^2} \mathbf{r}' = \mathbf{0}.$$

Thus $\mathbf{r}''_{\perp} = \mathbf{r}'' = \langle 1, 2, 0 \rangle$ and $\|\mathbf{r}''_{\perp}\| = \sqrt{5}$.

Step 3: Verification

$$\frac{\|\mathbf{r}''_{\perp}\|}{\|\mathbf{r}'\|^2} = \frac{\sqrt{5}}{9},$$

which agrees exactly with the cross-product result.

Question 7

For the space curve

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle,$$

find all real parameters at which the curvature is zero.

Tasks

1. Compute $\mathbf{r}' \times \mathbf{r}''$.
2. Decide whether its magnitude can vanish.
3. Interpret the conclusion geometrically.

Question 7 – Solution

Strategy. Since the curve is regular, curvature is zero exactly when $\mathbf{r}'$ and $\mathbf{r}''$ are parallel.

Step 1: Derivatives

$$\mathbf{r}' = \langle 1, 2t, 3t^2 \rangle, \quad \mathbf{r}'' = \langle 0, 2, 6t \rangle.$$

The first component of $\mathbf{r}'$ is always 1, so the curve is regular.

Step 2: Cross product

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= \langle 1, 2t, 3t^2 \rangle \times \langle 0, 2, 6t \rangle \\ &= \langle 12t^2 - 6t^2, -6t, 2 \rangle \\ &= \langle 6t^2, -6t, 2 \rangle. \end{aligned}$$

Its squared magnitude is

$$36t^4 + 36t^2 + 4.$$

Every term is nonnegative and the constant term is $4 > 0$. Therefore the cross product never vanishes.

Step 3: Conclusion

$\kappa(t) > 0$ for every real t ; there are no zero-curvature points.

Geometrically, the tangent direction is changing everywhere. In particular, no point of this twisted cubic is locally straight in the curvature sense.

Question 8

Consider $\mathbf{r}(t) = \langle t, t^3, 0 \rangle$.

Tasks

1. Derive the curvature function.
2. Explain why curvature vanishes at the origin even though the curve is regular there.
3. Determine the two points where curvature is greatest.

Question 8 – Solution

Strategy. Use the planar graph formula and maximize the even function by working with $u = t^2$.

Step 1: Curvature

$$y' = 3t^2, \quad y'' = 6t,$$

so

$$\kappa(t) = \frac{6|t|}{(1 + 9t^4)^{3/2}}.$$

At $t = 0$, $\mathbf{r}'(0) = \langle 1, 0, 0 \rangle \neq \mathbf{0}$, so the curve is regular, but $y''(0) = 0$ and therefore $\kappa(0) = 0$. This is an inflection point.

Step 2: Maximize for $t > 0$ There $|t| = t$. Logarithmic differentiation gives

$$\frac{\kappa'}{\kappa} = \frac{1}{t} - \frac{54t^3}{1 + 9t^4}.$$

Setting this equal to zero,

$$1 + 9t^4 = 54t^4 \implies t^4 = \frac{1}{45}.$$

Thus $t = 45^{-1/4}$ on the positive side. By even symmetry, the other maximizer is its negative.

Step 3: Points With $\alpha = 45^{-1/4}$, the two curve points are

$$(\alpha, \alpha^3, 0) \quad \text{and} \quad (-\alpha, -\alpha^3, 0).$$

Curvature tends to zero as $|t| \rightarrow \infty$ and is zero at $t = 0$, so these critical points are global maxima.

Question 9

A helix must have radius 3 and constant curvature $1/5$. It is modeled by

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, bt \rangle, \quad b > 0.$$

Tasks

1. Determine b exactly.
2. Find the vertical rise per complete revolution.
3. Verify the designed curvature directly.

Question 9 – Solution

Strategy. Use the helix curvature formula and then translate parameter change into vertical rise.

Step 1: Solve the design equation For a helix of radius a ,

$$\kappa = \frac{a}{a^2 + b^2}.$$

With $a = 3$ and $\kappa = 1/5$,

$$\begin{aligned}\frac{3}{9 + b^2} &= \frac{1}{5}, \\ 15 &= 9 + b^2, \\ b^2 &= 6.\end{aligned}$$

Since $b > 0$, $\boxed{b = \sqrt{6}}$.

Step 2: Rise per revolution A full revolution changes t by 2π . Since $z = bt$,

$$\Delta z = b(2\pi) = \boxed{2\pi\sqrt{6}}.$$

Step 3: Verification Substituting $b^2 = 6$ into the curvature formula gives

$$\kappa = \frac{3}{3^2 + (\sqrt{6})^2} = \frac{3}{9 + 6} = \boxed{\frac{1}{5}}.$$

The positive root also makes the helix rise as t increases, satisfying the stated orientation.

Question 10

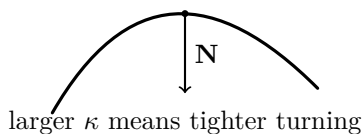
Let $\mathbf{r}(s)$ be a twice-differentiable curve parametrized by arc length, so $\|\mathbf{r}'(s)\| = 1$.

Tasks

1. Prove that $\mathbf{r}''(s)$ is perpendicular to $\mathbf{r}'(s)$.
2. Prove that $\kappa(s) = \|\mathbf{r}''(s)\|$.
3. Explain what $\mathbf{r}''(s) = \mathbf{0}$ means locally.

Question 10 – Solution

Strategy. Differentiate the unit-speed invariant, then compare the definition $\kappa = \|d\mathbf{T}/ds\|$ with $\mathbf{T} = \mathbf{r}'$.



Step 1: Orthogonality Unit speed means

$$\mathbf{r}'(s) \cdot \mathbf{r}'(s) = 1.$$

Differentiate both sides:

$$2\mathbf{r}'(s) \cdot \mathbf{r}''(s) = 0.$$

Therefore

$$\boxed{\mathbf{r}'(s) \cdot \mathbf{r}''(s) = 0}.$$

Thus the change of the unit tangent has no tangential component.

Step 2: Curvature formula Because the parameter is arc length,

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s).$$

Consequently

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''(s)\|,$$

so

$$\boxed{\kappa(s) = \|\mathbf{r}''(s)\|}.$$

Step 3: Zero second derivative If $\mathbf{r}'' = \mathbf{0}$ throughout a neighborhood, then $\mathbf{r}'$ is a constant unit vector there. Integrating gives $\mathbf{r}(s) = \mathbf{p} + s\mathbf{u}$ locally, a straight line traversed at unit speed. At a single point, $\mathbf{r}''(s_0) = \mathbf{0}$ means zero curvature at that point, but does not by itself force an entire neighboring segment to be straight.

Question 1

A particle has position

$$\mathbf{r}(t) = \langle t^2 - 2t, t^3 - 3t, 4t + 1 \rangle, \quad t \geq 0.$$

Tasks

1. Find velocity, acceleration, and speed.
2. Evaluate them at $t = 1$.
3. Find the tangent line to the trajectory at that time.

Question 1 – Solution

Strategy. Differentiate position componentwise; speed is the magnitude of velocity.

Step 1: Velocity and acceleration

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t - 2, 3t^2 - 3, 4 \rangle,$$

and

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 6t, 0 \rangle.$$

Therefore

$$\text{speed} = \|\mathbf{v}(t)\| = \sqrt{(2t - 2)^2 + (3t^2 - 3)^2 + 16}.$$

Step 2: Values at $t = 1$

$$\mathbf{r}(1) = \langle -1, -2, 5 \rangle, \quad \mathbf{v}(1) = \langle 0, 0, 4 \rangle, \quad \mathbf{a}(1) = \langle 2, 6, 0 \rangle.$$

The speed is $\boxed{4}$.

Step 3: Tangent line Since $\mathbf{v}(1) \neq \mathbf{0}$, it gives the tangent direction:

$$\mathbf{L}(s) = \langle -1, -2, 5 \rangle + s \langle 0, 0, 1 \rangle.$$

The simplified direction is parallel to velocity. At $s = 0$ the line passes through the correct position.

Question 2

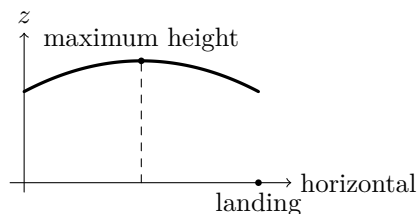
A ball is launched from $(0, 0, 5)$ with initial velocity $\langle 30, 40, 50 \rangle$ ft/s. Gravity is $\langle 0, 0, -32 \rangle$ ft/s² and air resistance is neglected.

Tasks

1. Find position and velocity for $t \geq 0$.
2. Find the maximum height and when it occurs.
3. Find the landing time and horizontal landing point.

Question 2 – Solution

Strategy. Integrate constant acceleration twice and apply the initial data; landing occurs when the vertical coordinate is zero.

**Step 1: Motion functions**

$$\mathbf{v}(t) = \langle 30, 40, 50 - 32t \rangle.$$

Integrating again and using $\mathbf{r}(0) = \langle 0, 0, 5 \rangle$,

$$\mathbf{r}(t) = \langle 30t, 40t, 5 + 50t - 16t^2 \rangle.$$

Step 2: Maximum height The vertical velocity is zero when

$$50 - 32t = 0 \implies t = \frac{25}{16}.$$

Then

$$z = 5 + 50 \left(\frac{25}{16} \right) - 16 \left(\frac{25}{16} \right)^2 = \boxed{\frac{705}{16} \text{ ft}}.$$

Step 3: Landing Set $z = 0$:

$$5 + 50t - 16t^2 = 0 \implies t = \frac{25 \pm \sqrt{705}}{16}.$$

Only the positive root is physical:

$$t_L = \frac{25 + \sqrt{705}}{16}.$$

Thus the horizontal landing point is

$$(x, y) = (30t_L, 40t_L).$$

Substitution into the vertical equation verifies height zero.

Question 3

A particle satisfies

$$\mathbf{a}(t) = \langle 6t, 4e^{2t}, -2 \rangle, \quad \mathbf{v}(0) = \langle 1, -1, 3 \rangle, \quad \mathbf{r}(0) = \langle 2, 0, -4 \rangle.$$

Tasks

1. Recover $\mathbf{v}(t)$.
2. Recover $\mathbf{r}(t)$.
3. Verify both initial conditions and the acceleration.

Question 3 – Solution

Strategy. Integrate componentwise twice, determining a new constant vector at each stage.

Step 1: Velocity

$$\mathbf{v}(t) = \langle 3t^2 + C_1, 2e^{2t} + C_2, -2t + C_3 \rangle.$$

At $t = 0$, comparison with $\langle 1, -1, 3 \rangle$ gives

$$C_1 = 1, \quad 2 + C_2 = -1, \quad C_3 = 3.$$

Therefore

$$\boxed{\mathbf{v}(t) = \langle 3t^2 + 1, 2e^{2t} - 3, 3 - 2t \rangle}.$$

Step 2: Position Integrating velocity,

$$\mathbf{r}(t) = \langle t^3 + t + D_1, e^{2t} - 3t + D_2, 3t - t^2 + D_3 \rangle.$$

Using $\mathbf{r}(0) = \langle 2, 0, -4 \rangle$ gives $D_1 = 2$, $1 + D_2 = 0$, and $D_3 = -4$. Hence

$$\boxed{\mathbf{r}(t) = \langle t^3 + t + 2, e^{2t} - 3t - 1, 3t - t^2 - 4 \rangle}.$$

Step 3: Verification Differentiating $\mathbf{r}$ gives the displayed $\mathbf{v}$; differentiating again gives $\langle 6t, 4e^{2t}, -2 \rangle$. Direct substitution at $t = 0$ returns both required initial vectors.

Question 4

A particle moves on a circle according to

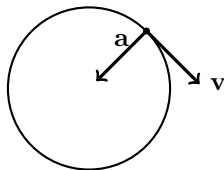
$$\mathbf{r}(t) = \langle R \cos(\omega t), R \sin(\omega t), 0 \rangle, \quad R > 0, \omega > 0.$$

Tasks

1. Find velocity, speed, and acceleration.
2. Prove velocity is perpendicular to acceleration.
3. Express acceleration in terms of position and identify its magnitude.

Question 4 – Solution

Strategy. Differentiate symbolically and use trigonometric identities and dot products.



velocity tangent; acceleration inward

Step 1: Differentiate

$$\mathbf{v} = \langle -R\omega \sin(\omega t), R\omega \cos(\omega t), 0 \rangle,$$

so

$$\|\mathbf{v}\| = R\omega.$$

Differentiating again,

$$\mathbf{a} = \langle -R\omega^2 \cos(\omega t), -R\omega^2 \sin(\omega t), 0 \rangle.$$

Step 2: Orthogonality

$$\mathbf{v} \cdot \mathbf{a} = R^2\omega^3 \sin(\omega t) \cos(\omega t) - R^2\omega^3 \cos(\omega t) \sin(\omega t) = 0.$$

Thus $\mathbf{v} \perp \mathbf{a}$.

Step 3: Inward acceleration Comparing with $\mathbf{r}$ gives

$$\mathbf{a}(t) = -\omega^2 \mathbf{r}(t).$$

It points toward the origin, and

$$\|\mathbf{a}\| = R\omega^2 = \frac{(R\omega)^2}{R} = \frac{v^2}{R}.$$

This is purely normal acceleration; the constant speed confirms there is no tangential component.

Question 5

For the helical motion

$$\mathbf{r}(t) = \langle 3 \cos 2t, 3 \sin 2t, 4t \rangle,$$

determine the velocity, acceleration, speed, and angle between velocity and the positive z -axis.

Tasks

1. Compute $\mathbf{v}$ and $\mathbf{a}$.
2. Show speed and the requested angle are constant.
3. Prove acceleration points horizontally toward the helix axis.

Question 5 – Solution

Strategy. Differentiate, then use the dot-product angle formula with $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Step 1: Motion quantities

$$\mathbf{v}(t) = \langle -6 \sin 2t, 6 \cos 2t, 4 \rangle,$$

and

$$\mathbf{a}(t) = \langle -12 \cos 2t, -12 \sin 2t, 0 \rangle.$$

The speed is

$$\|\mathbf{v}\| = \sqrt{36(\sin^2 2t + \cos^2 2t) + 16} = 2\sqrt{13}.$$

Step 2: Direction angle If θ is the angle with the positive z -axis,

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{k}}{\|\mathbf{v}\| \|\mathbf{k}\|} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}.$$

Therefore

$$\theta = \cos^{-1} \left(\frac{2}{\sqrt{13}} \right).$$

No t occurs, so both speed and angle are constant.

Step 3: Acceleration geometry Its horizontal component is

$$-4 \langle 3 \cos 2t, 3 \sin 2t, 0 \rangle,$$

which is opposite the radial position from the z -axis. Its z -component is zero, so acceleration points horizontally inward. Also $\mathbf{v} \cdot \mathbf{a} = 0$, consistent with constant speed.

Question 6

Two objects move according to

$$\mathbf{r}_1(t) = \langle t, 2t, 3 - t \rangle, \quad \mathbf{r}_2(t) = \langle 4 - t, t + 1, 2t - 2 \rangle, \quad t \geq 0.$$

Tasks

1. Determine whether they collide.
2. Find their velocities and relative velocity.
3. Verify your collision conclusion using relative position.

Question 6 – Solution

Strategy. A collision requires equal position vectors at the same time, not merely intersecting geometric paths.

Step 1: Match coordinates Equality requires

$$t = 4 - t, \quad 2t = t + 1, \quad 3 - t = 2t - 2.$$

The first gives $t = 2$, while the second gives $t = 1$. Since no single time satisfies even these two equations, the objects do not collide:

$$\boxed{\text{no collision for } t \geq 0}.$$

Step 2: Velocities

$$\mathbf{v}_1 = \langle 1, 2, -1 \rangle, \quad \mathbf{v}_2 = \langle -1, 1, 2 \rangle.$$

The velocity of object 1 relative to object 2 is

$$\boxed{\mathbf{v}_{1/2} = \mathbf{v}_1 - \mathbf{v}_2 = \langle 2, 1, -3 \rangle}.$$

Step 3: Relative-position check

$$\mathbf{r}_1 - \mathbf{r}_2 = \langle 2t - 4, t - 1, 5 - 3t \rangle.$$

For this vector to be zero, its first, second, and third components would require respectively $t = 2$, $t = 1$, and $t = 5/3$. The inconsistent times independently confirm that relative position never vanishes.

Question 7

A particle has velocity

$$\mathbf{v}(t) = \langle t - 2, 2t + 1, 2 \rangle, \quad t \geq 0.$$

Tasks

1. Find when its speed is smallest.
2. Find the minimum speed.
3. Show that velocity is perpendicular to acceleration at that instant and explain why.

Question 7 – Solution

Strategy. Minimize squared speed, which has the same minimizers as speed and avoids a square root.

Step 1: Squared speed

$$\begin{aligned}q(t) = \|\mathbf{v}(t)\|^2 &= (t-2)^2 + (2t+1)^2 + 4 \\&= 5t^2 + 9.\end{aligned}$$

On $t \geq 0$, this is minimized at $\boxed{t = 0}$.

Step 2: Minimum speed

$$\mathbf{v}(0) = \langle -2, 1, 2 \rangle, \quad \boxed{\|\mathbf{v}(0)\| = 3}.$$

Step 3: Orthogonality Acceleration is

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 1, 2, 0 \rangle.$$

At $t = 0$,

$$\mathbf{v}(0) \cdot \mathbf{a}(0) = -2 + 2 + 0 = 0.$$

Also $q'(t) = 2\mathbf{v} \cdot \mathbf{a} = 10t$, so the vanishing dot product is exactly the zero derivative of squared speed. Here the minimum occurs at the endpoint, and the right-hand derivative is nonnegative immediately afterward.

Question 8

A particle starts at $P = (1, 0, 2)$ with velocity $\langle 2, -1, 0 \rangle$ and constant acceleration $\langle -1, 2, 3 \rangle$.

Tasks

1. Find position and velocity at time t .
2. Determine when the particle first reaches the plane $z = 8$.
3. Find its position and speed then.

Question 8 – Solution

Strategy. For constant acceleration, use $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}_0 + \frac{1}{2}t^2\mathbf{a}$.

Step 1: Motion functions

$$\mathbf{v}(t) = \langle 2 - t, -1 + 2t, 3t \rangle,$$

and

$$\mathbf{r}(t) = \left\langle 1 + 2t - \frac{1}{2}t^2, -t + t^2, 2 + \frac{3}{2}t^2 \right\rangle.$$

At $t = 0$, these reproduce both initial vectors.

Step 2: Reach $z = 8$

$$2 + \frac{3}{2}t^2 = 8 \quad \implies \quad t^2 = 4.$$

For forward time, $t = 2$.

Step 3: Position and speed

$$\mathbf{r}(2) = \langle 1 + 4 - 2, -2 + 4, 8 \rangle = \langle 3, 2, 8 \rangle.$$

Also

$$\mathbf{v}(2) = \langle 0, 3, 6 \rangle,$$

so

$$\text{speed} = \sqrt{0^2 + 3^2 + 6^2} = 3\sqrt{5}.$$

The vertical coordinate and positive time verify that this is the first forward-time crossing.

Question 9

A particle follows

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle, \quad 0 \leq t \leq 2\pi.$$

Tasks

1. Find its displacement and average velocity over the interval.
2. Find total distance traveled and average speed.
3. Explain why the horizontal motion contributes to distance but not displacement.

Question 9 – Solution

Strategy. Displacement uses endpoints; distance integrates speed. These measure different features of motion.

Step 1: Displacement

$$\mathbf{r}(0) = \langle 3, 0, 0 \rangle, \quad \mathbf{r}(2\pi) = \langle 3, 0, 8\pi \rangle.$$

Thus

$$\Delta \mathbf{r} = \langle 0, 0, 8\pi \rangle.$$

The elapsed time is 2π , so

$$\mathbf{v}_{\text{avg}} = \frac{\Delta \mathbf{r}}{2\pi} = \langle 0, 0, 4 \rangle.$$

Step 2: Distance and average speed

$$\mathbf{v}(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle,$$

so $\|\mathbf{v}(t)\| = \sqrt{9 + 16} = 5$. Hence

$$D = \int_0^{2\pi} 5 \, dt = 10\pi,$$

and

$$\text{average speed} = \frac{10\pi}{2\pi} = 5.$$

Step 3: Interpretation During one revolution, the horizontal coordinates return to their starting values, so their net changes cancel in displacement. Nevertheless the particle continuously moves horizontally around the circle, adding positive length to the traveled path. Therefore distance 10π exceeds displacement magnitude 8π .

Question 10

Let a regular motion have velocity $\mathbf{v}$, speed $v = \|\mathbf{v}\|$, unit tangent $\mathbf{T}$, curvature κ , and principal normal $\mathbf{N}$.

Tasks

1. Derive $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$.
2. Derive formulas for the tangential and normal scalar components using only $\mathbf{v}$ and $\mathbf{a}$.
3. Explain the physical meaning of each component.

Question 10 – Solution

Strategy. Begin with $\mathbf{v} = v\mathbf{T}$, differentiate, and convert $d\mathbf{T}/dt$ through arc length.

Step 1: Differentiate velocity

$$\mathbf{a} = \frac{d}{dt}(v\mathbf{T}) = v'\mathbf{T} + v\frac{d\mathbf{T}}{dt}.$$

By the chain rule,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} = (\kappa\mathbf{N})v.$$

Therefore

$$\boxed{\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}}.$$

Step 2: Tangential component Dot with $\mathbf{T} = \mathbf{v}/v$:

$$a_T = \mathbf{a} \cdot \mathbf{T} = \boxed{\frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}}.$$

It also equals v' , the rate of change of speed.

Step 3: Normal component Orthogonality gives

$$\|\mathbf{a}\|^2 = a_T^2 + a_N^2.$$

Thus

$$\boxed{a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}} = \boxed{\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}} = \kappa v^2.$$

The tangential component changes speed; the normal component changes direction. A motion can therefore accelerate even at constant speed if its path bends.

Question 1

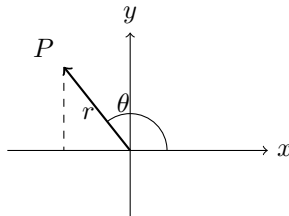
Convert $P = (-3, 3\sqrt{3}, 4)$ from Cartesian to cylindrical coordinates.

Tasks

1. Find r , a principal angle $0 \leq \theta < 2\pi$, and z .
2. Give two other equivalent cylindrical representations.
3. Verify by converting back.

Question 1 – Solution

Strategy. Use $r = \sqrt{x^2 + y^2}$ and determine the quadrant before choosing θ .



Step 1: Radius $r = \sqrt{9 + 27} = 6$.

Step 2: Angle Since $x < 0$ and $y > 0$, the point is in quadrant II. Also $\tan \theta = -\sqrt{3}$, so $\theta = 2\pi/3$. The height is unchanged. Thus

$$(r, \theta, z) = (6, 2\pi/3, 4).$$

Step 3: Equivalent forms Adding 2π gives $(6, 8\pi/3, 4)$. Reversing the radial sign and adding π gives $(-6, 5\pi/3, 4)$.

Verification. $6 \cos(2\pi/3) = -3$, $6 \sin(2\pi/3) = 3\sqrt{3}$, and $z = 4$.

Question 2

Convert $(r, \theta, z) = (4, 7\pi/6, -2)$ to Cartesian coordinates and determine its distance from the z -axis and from the origin.

Tasks

1. Find (x, y, z) .
2. Find both distances.
3. Verify the radial identity.

Question 2 – Solution

Strategy. Use $x = r \cos \theta$, $y = r \sin \theta$; radial distance is r when $r \geq 0$.

Step 1: Convert Since $\cos(7\pi/6) = -\sqrt{3}/2$ and $\sin(7\pi/6) = -1/2$,

$$\boxed{(x, y, z) = (-2\sqrt{3}, -2, -2)}.$$

Step 2: Distances Distance to the z -axis is $\sqrt{x^2 + y^2} = 4$. Distance to the origin is

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{16 + 4} = \boxed{2\sqrt{5}}.$$

Verification. $x^2 + y^2 = 12 + 4 = 16 = r^2$, confirming the conversion and axis distance.

Question 3

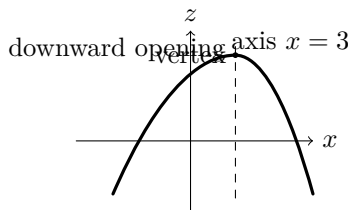
Rewrite $x^2 + y^2 - 6x + 4z = 3$ in cylindrical coordinates and identify the surface.

Tasks

1. Convert and solve for z .
2. Describe horizontal traces.
3. Locate the vertex and axis.

Question 3 – Solution

Strategy. Replace $x^2 + y^2$ by r^2 and x by $r \cos \theta$.



Step 1: Convert

$$r^2 - 6r \cos \theta + 4z = 3, \quad \boxed{z = \frac{3 - r^2 + 6r \cos \theta}{4}}.$$

For geometric identification, complete the Cartesian square:

$$(x - 3)^2 + y^2 = 12 - 4z.$$

Step 2: Identify Equivalently,

$$z = 3 - \frac{(x - 3)^2 + y^2}{4}.$$

This is a downward-opening circular paraboloid with vertical axis $x = 3, y = 0$ and vertex $\boxed{(3, 0, 3)}$.

Step 3: Traces At height $z = c$, the trace is a circle centered at $(3, 0, c)$ with radius $2\sqrt{3 - c}$, present for $c \leq 3$.

Question 4

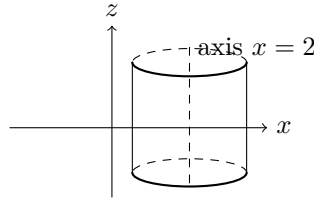
Describe the surface $r = 4 \cos \theta$ in Cartesian coordinates.

Tasks

1. Convert without dividing by r .
2. Identify the surface and its axis.
3. State its intersections with the coordinate axes.

Question 4 – Solution

Strategy. Multiply by r and use $r^2 = x^2 + y^2$ and $r \cos \theta = x$.



Step 1: Convert From $r = 4 \cos \theta$,

$$r^2 = 4r \cos \theta \implies x^2 + y^2 = 4x.$$

Completing the square,

$$\boxed{(x - 2)^2 + y^2 = 4}.$$

Step 2: Identify Since z is unrestricted, this is a circular cylinder of radius 2 whose axis is the vertical line $x = 2, y = 0$.

Step 3: Axis intersections In the xy -plane it meets the x -axis at $(0, 0, 0)$ and $(4, 0, 0)$ and the y -axis only at the origin. More generally, the cylinder contains the vertical lines $(0, 0, z)$ and $(4, 0, z)$.

Question 5

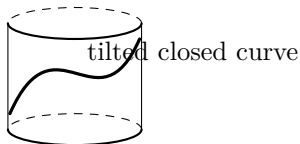
Find a cylindrical parametrization of the curve where $x^2 + y^2 = 9$ and $z = x + y$.

Tasks

1. Express both surfaces cylindrically.
2. Parametrize the entire intersection once.
3. Find its highest and lowest points.

Question 5 – Solution

Strategy. The cylinder fixes r ; let θ be the curve parameter and use the plane for z .



Step 1: Constraints The cylinder gives $r = 3$. The plane becomes

$$z = r \cos \theta + r \sin \theta = 3(\cos \theta + \sin \theta).$$

Thus

$$\mathbf{r}(\theta) = \langle 3 \cos \theta, 3 \sin \theta, 3(\cos \theta + \sin \theta) \rangle, \quad 0 \leq \theta < 2\pi.$$

Step 2: Extrema Use $\cos \theta + \sin \theta = \sqrt{2} \cos(\theta - \pi/4)$. Hence $z_{\max} = 3\sqrt{2}$ at $\theta = \pi/4$, and $z_{\min} = -3\sqrt{2}$ at $\theta = 5\pi/4$.

Step 3: Points

$$P_{\max} = (3/\sqrt{2}, 3/\sqrt{2}, 3\sqrt{2}), \quad P_{\min} = (-3/\sqrt{2}, -3/\sqrt{2}, -3\sqrt{2}).$$

Both satisfy the original cylinder and plane equations.

Question 6

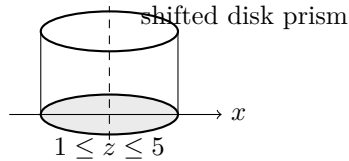
Convert the region $x^2 + y^2 \leq 4x$, $1 \leq z \leq 5$ to cylindrical inequalities.

Tasks

1. Determine the correct angular interval.
2. Give radial and vertical bounds.
3. Explain the geometry and why negative radial bounds are unnecessary.

Question 6 – Solution

Strategy. Convert the disk inequality and enforce the convention $r \geq 0$.



Step 1: Convert The inequality becomes

$$r^2 \leq 4r \cos \theta.$$

With $r \geq 0$, this is $0 \leq r \leq 4 \cos \theta$. The upper bound must be nonnegative, so $\cos \theta \geq 0$.

Step 2: Bounds One nonoverlapping description is

$$\boxed{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 4 \cos \theta, \quad 1 \leq z \leq 5}.$$

Step 3: Geometry Completing the square gives $(x - 2)^2 + y^2 \leq 4$: a radius-2 disk centered at $(2, 0)$, extended from height 1 to 5. Negative r is unnecessary because directions with $\cos \theta < 0$ would duplicate points already represented using nonnegative r and opposite angles.

Question 7

Find the distance between cylindrical points $P = (2, \pi/6, 1)$ and $Q = (5, 5\pi/6, 4)$ without first listing both Cartesian points.

Tasks

1. Derive a cylindrical distance formula.
2. Evaluate it exactly.
3. Check the horizontal contribution geometrically.

Question 7 – Solution

Strategy. Expand $(x_1 - x_2)^2 + (y_1 - y_2)^2$ and use the cosine difference identity.

Step 1: Formula For cylindrical points,

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2) + (z_1 - z_2)^2.$$

This follows from the planar law of cosines plus the vertical difference.

Step 2: Substitute Here $\theta_1 - \theta_2 = -2\pi/3$, so its cosine is $-1/2$. Thus

$$\begin{aligned} d^2 &= 2^2 + 5^2 - 2(2)(5)(-1/2) + (1 - 4)^2 \\ &= 4 + 25 + 10 + 9 = 48. \end{aligned}$$

Therefore $\boxed{d = 4\sqrt{3}}$.

Step 3: Check The horizontal squared distance is $4 + 25 + 10 = 39$; adding vertical squared distance 9 gives 48, consistent with perpendicular horizontal and vertical components.

Question 8

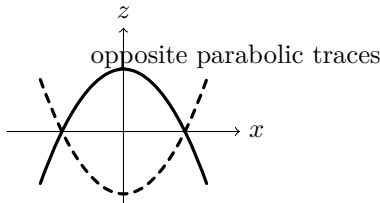
Analyze the surface $z = r^2 \cos 2\theta$.

Tasks

1. Convert it to Cartesian form.
2. Identify its principal vertical traces.
3. Describe its signs by angular sector and its symmetry.

Question 8 – Solution

Strategy. Use $r^2 \cos 2\theta = x^2 - y^2$.



Step 1: Convert Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$,

$$r^2 \cos 2\theta = r^2 \cos^2 \theta - r^2 \sin^2 \theta = x^2 - y^2.$$

Thus $\boxed{z = x^2 - y^2}$, a hyperbolic paraboloid.

Step 2: Traces In $y = 0$, $z = x^2$ opens upward. In $x = 0$, $z = -y^2$ opens downward. On $y = \pm x$, $z = 0$.

Step 3: Angular behavior The height is positive where $\cos 2\theta > 0$, negative where it is negative, and zero at $\theta = \pi/4 + k\pi/2$. The equation is unchanged by $\theta \mapsto \theta + \pi$ and by reflections across either coordinate axis, matching the Cartesian symmetries.

Question 9

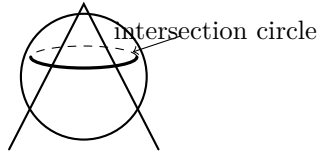
A point has cylindrical coordinates (r, θ, z) with $r = 2z$, $z \geq 0$, and lies on the sphere $x^2 + y^2 + z^2 = 45$.

Tasks

1. Determine r and z .
2. Describe all possible points.
3. Find the horizontal circle containing them.

Question 9 – Solution

Strategy. Convert the sphere to $r^2 + z^2 = 45$ and combine it with the cone relation.



Step 1: Solve Substitute $r = 2z$:

$$(2z)^2 + z^2 = 45 \implies 5z^2 = 45.$$

Since $z \geq 0$, $\boxed{z = 3}$ and $\boxed{r = 6}$.

Step 2: Angular freedom No condition restricts θ , so

$$\boxed{(r, \theta, z) = (6, \theta, 3), \quad 0 \leq \theta < 2\pi}.$$

In Cartesian form the points are $(6 \cos \theta, 6 \sin \theta, 3)$.

Step 3: Locus They form the horizontal circle

$$\boxed{x^2 + y^2 = 36, \quad z = 3}.$$

Verification gives $36 + 9 = 45$ and $r = 6 = 2(3)$.

Question 10

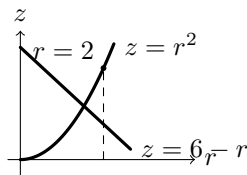
A solid is described cylindrically by $0 \leq r \leq 2 + \cos \theta$, $0 \leq \theta < 2\pi$, and $r^2 \leq z \leq 6 - r$.

Tasks

1. Determine whether the vertical bounds are consistent everywhere in the stated radial range.
2. Correct the radial upper bound if necessary.
3. Give a precise final description and explain which constraint is active.

Question 10 – Solution

Strategy. Vertical consistency requires lower height not exceed upper height.

**Step 1: Compare heights**

$$r^2 \leq 6 - r \iff r^2 + r - 6 \leq 0 \iff (r + 3)(r - 2) \leq 0.$$

With $r \geq 0$, this requires $0 \leq r \leq 2$.

Step 2: Combine radial restrictions The stated planar bound is $r \leq 2 + \cos \theta$, which ranges from 1 to 3. Both conditions must hold, so

$$0 \leq r \leq \min\{2, 2 + \cos \theta\}.$$

Equivalently, the upper bound is 2 when $\cos \theta \geq 0$ and $2 + \cos \theta$ when $\cos \theta < 0$.

Step 3: Final description

$$0 \leq \theta < 2\pi, \quad 0 \leq r \leq \min\{2, 2 + \cos \theta\}, \quad r^2 \leq z \leq 6 - r.$$

The height surfaces meet at $r = 2$. For angles on the right half-plane the height condition truncates the planar radius; on the left half-plane the limaçon-type boundary is already smaller than 2.

Question 1

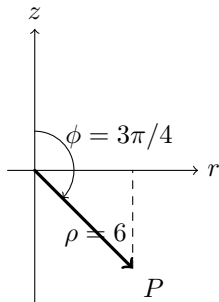
Convert the Cartesian point $P = (-3, 3, -3\sqrt{2})$ to spherical coordinates using $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$.

Tasks

1. Find ρ , θ , and ϕ .
2. Give one equivalent spherical representation outside the principal angular range.
3. Verify by converting back.

Question 1 – Solution

Strategy. Find radial distance first, use the quadrant of (x, y) for θ , and use $\cos \phi = z/\rho$.



Step 1: Radius

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{9 + 9 + 18} = 6.$$

Step 2: Angles The projection $(-3, 3)$ lies in quadrant II, so $\theta = 3\pi/4$. Also

$$\cos \phi = \frac{-3\sqrt{2}}{6} = -\frac{\sqrt{2}}{2}, \quad 0 \leq \phi \leq \pi,$$

giving $\phi = 3\pi/4$. Hence

$$\boxed{(\rho, \theta, \phi) = (6, 3\pi/4, 3\pi/4)}.$$

Step 3: Equivalence and verification Adding 2π to the azimuth gives $(6, 11\pi/4, 3\pi/4)$. Moreover,

$$6 \sin(3\pi/4) \cos(3\pi/4) = -3, \quad 6 \sin(3\pi/4) \sin(3\pi/4) = 3, \quad 6 \cos(3\pi/4) = -3\sqrt{2}.$$

Question 2

Convert $(\rho, \theta, \phi) = (8, 5\pi/6, \pi/3)$ to Cartesian and cylindrical coordinates.

Tasks

1. Find (x, y, z) exactly.
2. Find the corresponding principal cylindrical coordinates.
3. Verify both radial identities.

Question 2 – Solution

Strategy. Use $r = \rho \sin \phi$ and $z = \rho \cos \phi$, then resolve r using θ .

Step 1: Cylindrical data

$$r = 8 \sin(\pi/3) = 4\sqrt{3}, \quad z = 8 \cos(\pi/3) = 4.$$

Thus the principal cylindrical coordinates are

$$(r, \theta, z) = (4\sqrt{3}, 5\pi/6, 4).$$

Step 2: Cartesian data

$$x = r \cos \theta = (4\sqrt{3})(-\sqrt{3}/2) = -6, \quad y = r \sin \theta = (4\sqrt{3})(1/2) = 2\sqrt{3}.$$

Therefore

$$(x, y, z) = (-6, 2\sqrt{3}, 4).$$

Verification. $x^2 + y^2 = 36 + 12 = 48 = r^2$, and $x^2 + y^2 + z^2 = 48 + 16 = 64 = \rho^2$.

Question 3

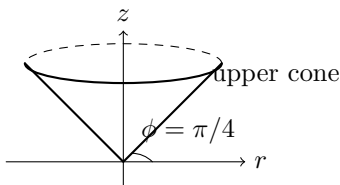
Describe geometrically the coordinate surfaces $\rho = 5$, $\theta = 2\pi/3$, and $\phi = \pi/4$.

Tasks

1. Identify each full coordinate surface.
2. Describe the pairwise intersections.
3. Find their common point.

Question 3 – Solution

Strategy. Interpret ρ as distance, θ as azimuth, and ϕ as inclination from the positive z -axis.



Step 1: Surfaces $\rho = 5$ is the sphere of radius 5. The equation $\theta = 2\pi/3$ is a vertical half-plane from the z -axis in azimuth $2\pi/3$. The equation $\phi = \pi/4$ is the upper nappe of the cone making angle $\pi/4$ with the positive z -axis.

Step 2: Pairwise intersections Sphere–half-plane and sphere–cone intersections are semicircle and horizontal circle, respectively. Half-plane–cone is a ray from the origin.

Step 3: Common point Substitution gives

$$x = 5\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) = -\frac{5\sqrt{2}}{4}, \quad y = 5\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{6}}{4}, \quad z = \frac{5\sqrt{2}}{2}.$$

Thus

$$P = (-5\sqrt{2}/4, 5\sqrt{6}/4, 5\sqrt{2}/2).$$

Question 4

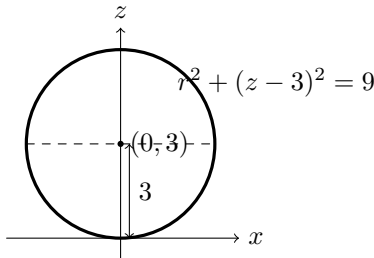
Rewrite the sphere $x^2 + y^2 + z^2 = 6z$ in spherical coordinates and identify its geometry.

Tasks

1. Obtain the nontrivial spherical equation.
2. Find its Cartesian center and radius.
3. Explain the role of $\rho = 0$ when simplifying.

Question 4 – Solution

Strategy. Substitute $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$, while preserving any solution lost by division.

**Step 1: Convert**

$$\rho^2 = 6\rho \cos \phi \iff \rho(\rho - 6 \cos \phi) = 0.$$

The usual surface equation is

$$\boxed{\rho = 6 \cos \phi}, \quad 0 \leq \phi \leq \frac{\pi}{2},$$

and it includes the origin at $\phi = \pi/2$.

Step 2: Identify Completing the square in Cartesian form gives

$$x^2 + y^2 + (z - 3)^2 = 9.$$

It is the sphere centered at $\boxed{(0, 0, 3)}$ with radius $\boxed{3}$.

Step 3: Origin check Division by ρ would temporarily discard $\rho = 0$. The origin satisfies the original equation and is recovered by $\rho = 6 \cos(\pi/2) = 0$; thus no point is absent from the final parametrized surface.

Question 5

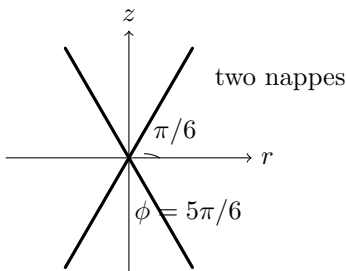
Convert the surface $z^2 = 3(x^2 + y^2)$ to spherical coordinates and determine exactly which cone it represents.

Tasks

1. Derive all permitted constant values of ϕ .
2. Identify both nappes.
3. State the status of the origin.

Question 5 – Solution

Strategy. Substitute $z = \rho \cos \phi$ and $x^2 + y^2 = \rho^2 \sin^2 \phi$, treating $\rho = 0$ separately.



Step 1: Convert For $\rho > 0$,

$$\rho^2 \cos^2 \phi = 3\rho^2 \sin^2 \phi \implies \tan^2 \phi = \frac{1}{3}.$$

On $0 \leq \phi \leq \pi$, the solutions are

$$\boxed{\phi = \frac{\pi}{6} \quad \text{or} \quad \phi = \frac{5\pi}{6}}.$$

Step 2: Geometry The first value is the upper nappe, $z \geq 0$; the second is the lower nappe, $z \leq 0$. Each makes an acute angle $\pi/6$ with its adjacent half of the z -axis.

Step 3: Origin At $\rho = 0$, every angle represents the origin, which satisfies the Cartesian equation. Therefore the two nappes meet at the included vertex $\boxed{(0, 0, 0)}$.

Question 6

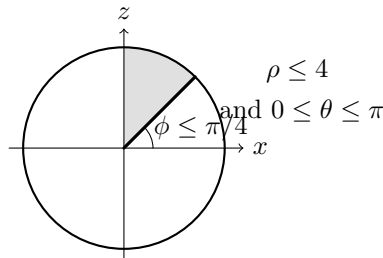
Describe in spherical inequalities the solid inside $x^2 + y^2 + z^2 \leq 16$, above the cone $z = \sqrt{x^2 + y^2}$, and in the half-space $y \geq 0$.

Tasks

1. Determine bounds for ρ , θ , and ϕ .
2. Explain each angular bound geometrically.
3. State which boundary pieces meet at the origin.

Question 6 – Solution

Strategy. Translate the sphere to a radial bound, the half-space to azimuth, and compare z with the cylindrical radius to bound inclination.



Step 1: Radial bound The sphere gives $0 \leq \rho \leq 4$.

Step 2: Azimuth Since $y = \rho \sin \phi \sin \theta$ and $\sin \phi \geq 0$, $y \geq 0$ is represented without overlap by $0 \leq \theta \leq \pi$.

Step 3: Inclination Above the cone means

$$\rho \cos \phi \geq \rho \sin \phi.$$

For $\rho > 0$ this gives $0 \leq \phi \leq \pi/4$. Consequently

$$\boxed{0 \leq \rho \leq 4, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq \pi/4}.$$

The sphere is the outer boundary; the cone and the two half-plane faces $\theta = 0, \pi$ all meet at the origin. Angular multiplicity there does not duplicate physical volume.

Question 7

Find the distance between $P = (3, \pi/6, \pi/3)$ and $Q = (5, 5\pi/6, 2\pi/3)$ in spherical coordinates without first converting both points to Cartesian form.

Tasks

1. Derive the angle between their radial directions.
2. Compute the distance exactly.
3. Check that the result obeys the triangle bounds.

Question 7 – Solution

Strategy. Take the dot product of the two unit radial directions and apply the three-dimensional law of cosines.

Step 1: Direction angle If γ is the angle between the radial directions, then

$$\cos \gamma = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2).$$

Here $\cos \phi_1 \cos \phi_2 = -1/4$, $\sin \phi_1 \sin \phi_2 = 3/4$, and $\cos(-2\pi/3) = -1/2$, so $\cos \gamma = -5/8$.

Step 2: Distance Therefore

$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos \gamma = 9 + 25 - 30(-5/8) = \frac{211}{4}.$$

Thus

$$d = \frac{\sqrt{211}}{2}.$$

Verification. The triangle inequality requires $|5-3| \leq d \leq 5+3$. Numerically $d \approx 7.26$, which lies between 2 and 8.

Question 8

Analyze all spherical-coordinate representations of points on the z -axis under the conventions $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$.

Tasks

1. Characterize a point $(0, 0, a)$ for $a > 0$, $a < 0$, and $a = 0$.
2. Identify which coordinates are nonunique.
3. Explain why the conversion formulas remain consistent.

Question 8 – Solution

Strategy. On the axis, the horizontal projection has length $\rho \sin \phi = 0$, so analyze the poles and origin separately.

Step 1: Positive axis If $a > 0$, then $\rho = a$, $\phi = 0$, and θ is arbitrary:

$$\boxed{(a, \theta, 0), \quad 0 \leq \theta < 2\pi}.$$

Step 2: Negative axis If $a < 0$, then $\rho = |a|$, $\phi = \pi$, and again θ is arbitrary:

$$\boxed{(|a|, \theta, \pi), \quad 0 \leq \theta < 2\pi}.$$

At the origin, $\rho = 0$ and both angles are arbitrary in their stated ranges.

Step 3: Consistency At $\phi = 0$ or π , $\sin \phi = 0$, so $x = y = 0$ regardless of θ , while $z = \rho \cos \phi$ has the correct sign. At $\rho = 0$, all three Cartesian coordinates vanish regardless of either angle. Thus the nonuniqueness is coordinate degeneracy, not geometric ambiguity.

Question 9

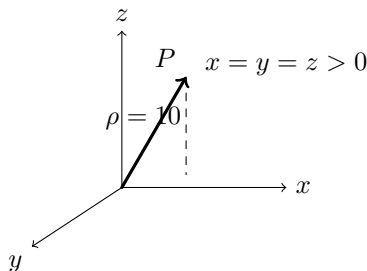
A point P has spherical coordinates (ρ, θ, ϕ) with $\rho = 10$ and is equally distant from the positive x -, y -, and z -axes. Assume P lies in the first octant.

Tasks

1. Determine the Cartesian coordinates of P .
2. Find its principal spherical angles.
3. Verify the three axis distances directly.

Question 9 – Solution

Strategy. Express squared distance to each coordinate axis and use first-octant signs.



Step 1: Equal-distance conditions Squared distances to the x -, y -, and z -axes are $y^2 + z^2$, $x^2 + z^2$, and $x^2 + y^2$. Equality implies $x^2 = y^2 = z^2$. In the first octant, $x = y = z$.

Step 2: Radius condition Since $x^2 + y^2 + z^2 = 100$, we obtain $3x^2 = 100$ and

$$P = (10/\sqrt{3}, 10/\sqrt{3}, 10/\sqrt{3}).$$

Then $\theta = \pi/4$, and $\cos \phi = z/\rho = 1/\sqrt{3}$, so

$$(\rho, \theta, \phi) = (10, \pi/4, \arccos(1/\sqrt{3})).$$

Verification. Every squared axis distance is $100/3 + 100/3 = 200/3$, so every distance is $10\sqrt{2/3}$.

Question 10

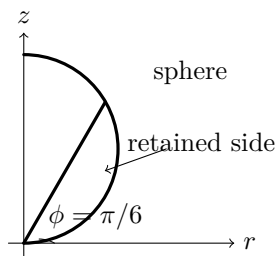
A solid is proposed in spherical coordinates by $0 \leq \rho \leq 6 \cos \phi$, $\pi/6 \leq \phi \leq 2\pi/3$, and $0 \leq \theta < 2\pi$.

Tasks

1. Determine where the radial bounds are consistent.
2. Correct the angular interval while retaining all actual points described.
3. Identify the solid and describe its boundary circle.

Question 10 – Solution

Strategy. A nonnegative radial variable can lie below $6 \cos \phi$ only when that upper bound is nonnegative.



Step 1: Consistency Since $\rho \geq 0$, we require $6 \cos \phi \geq 0$. On $0 \leq \phi \leq \pi$, this means $0 \leq \phi \leq \pi/2$. Intersecting with the proposed range gives

$$\boxed{\pi/6 \leq \phi \leq \pi/2}.$$

Angles $\pi/2 < \phi \leq 2\pi/3$ contribute no points because their proposed upper radial bound is negative.

Step 2: Correct description

$$\boxed{0 \leq \theta < 2\pi, \quad \pi/6 \leq \phi \leq \pi/2, \quad 0 \leq \rho \leq 6 \cos \phi}.$$

The radial surface is the sphere $x^2 + y^2 + (z - 3)^2 = 9$. The lower angular cutoff removes the portion closer than $\pi/6$ to the positive z -axis, leaving the part of the ball outside that narrow cone.

Step 3: Boundary circle On $\phi = \pi/6$ and the sphere, $\rho = 3\sqrt{3}$. Hence

$$z = \rho \cos \phi = \frac{9}{2}, \quad r = \rho \sin \phi = \frac{3\sqrt{3}}{2}.$$

The surfaces meet in

$$\boxed{x^2 + y^2 = 27/4, \quad z = 9/2}.$$