

Calculus III

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Question 1

A point $P = (x, y, z)$ is known only through its octant and its distances to the coordinate axes.

Given

$$x < 0, \quad y > 0, \quad z < 0,$$

$$d(P, x\text{-axis}) = \sqrt{41},$$

$$d(P, y\text{-axis}) = \sqrt{34},$$

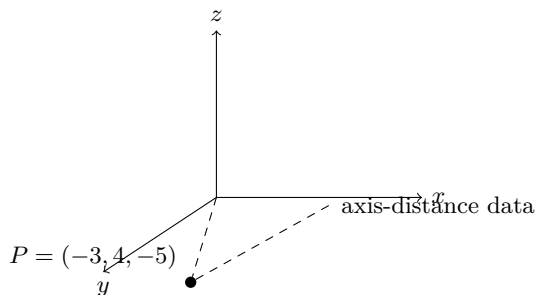
$$d(P, z\text{-axis}) = 5.$$

Tasks

1. Determine the coordinates of P .
2. Verify your result using all three axis distances.
3. Explain why only two of the three distance measurements would not determine P uniquely.

Question 1 – Solution

Strategy Squared distance to an axis is the sum of the squares of the other two coordinates. Solve first for x^2, y^2, z^2 , then use the octant to determine signs.



Equations The data give

$$y^2 + z^2 = 41, \quad x^2 + z^2 = 34, \quad x^2 + y^2 = 25.$$

Adding the last two equations and subtracting the first yields $2x^2 = 18$, so $x^2 = 9$. Similarly,

$$2y^2 = 41 + 25 - 34 = 32, \quad 2z^2 = 41 + 34 - 25 = 50.$$

Thus $|x| = 3$, $|y| = 4$, and $|z| = 5$. The stated octant fixes the signs:

$$\boxed{P = (-3, 4, -5)}.$$

Verification The resulting axis distances are

$$\sqrt{4^2 + (-5)^2} = \sqrt{41}, \quad \sqrt{(-3)^2 + (-5)^2} = \sqrt{34}, \quad \sqrt{(-3)^2 + 4^2} = 5.$$

With only two distances, one squared coordinate would remain free, so the third is essential.

Question 2

A sphere passes through four specified points in three-dimensional space.

Given

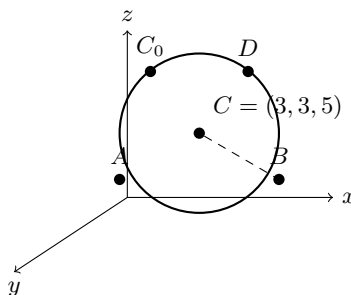
$$\begin{aligned}A &= (1, 0, 2), & B &= (5, 0, 2), \\C &= (1, 6, 2), & D &= (1, 0, 8).\end{aligned}$$

Tasks

1. Find the center and radius of the sphere.
2. Write its Cartesian equation.
3. Explain why the four given points determine a unique sphere.
4. Verify that all four points satisfy your equation.

Question 2 – Solution

Strategy A sphere's center is equidistant from all four points. Perpendicular-bisector conditions can be obtained by equating squared distances.



Center Let the center be $K = (h, k, \ell)$. From $d(A, K)^2 = d(B, K)^2$,

$$(h - 1)^2 + k^2 + (\ell - 2)^2 = (h - 5)^2 + k^2 + (\ell - 2)^2,$$

so $h = 3$. Comparing A with C gives $k = 3$, and comparing A with D gives $\ell = 5$. Hence the center is $(3, 3, 5)$ and

$$r^2 = (3 - 1)^2 + (3 - 0)^2 + (5 - 2)^2 = 22.$$

Therefore

$$\boxed{(x - 3)^2 + (y - 3)^2 + (z - 5)^2 = 22}.$$

Uniqueness and verification The three difference vectors from A point independently in the x -, y -, and z -directions, so their perpendicular-bisector planes meet in exactly one point. Substitution shows that each given point produces squared distance 22.

Question 3

Consider the fixed points

$$A = (-2, 0, 0), \quad B = (4, 0, 0),$$

and let $P = (x, y, z)$ satisfy the distance condition

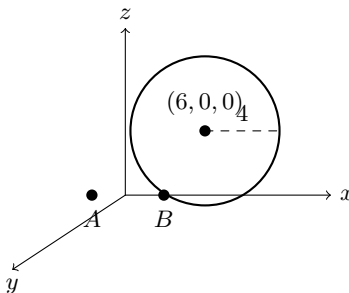
$$d(P, A) = 2d(P, B).$$

Tasks

1. Derive a Cartesian equation for the locus of all possible points P .
2. Identify the surface by stating its center and radius.
3. Determine which of A and B lies inside the surface.
4. Explain geometrically what the factor 2 requires of points on the locus.

Question 3 – Solution

Strategy Square the distance relation, expand, and complete the square. The resulting surface is an Apollonius sphere.



Derivation The condition is

$$(x + 2)^2 + y^2 + z^2 = 4((x - 4)^2 + y^2 + z^2).$$

After collecting terms and dividing by 3,

$$x^2 - 12x + y^2 + z^2 + 20 = 0.$$

Completing the square gives

$$\boxed{(x - 6)^2 + y^2 + z^2 = 16}.$$

Thus the locus is the sphere centered at $(6, 0, 0)$ with radius 4.

Interpretation Point B is 2 units from the center and lies inside; A is 8 units from the center and lies outside. A point must remain closer to B because its distance to A is required to be twice as large.

Verification At the axial endpoints $(2, 0, 0)$ and $(10, 0, 0)$, the distance pairs are $(4, 2)$ and $(12, 6)$, respectively, confirming the ratio.

Question 4

Let S be the set of points whose distance to the xy -plane is no greater than their distance to the z -axis:

$$S = \{P : d(P, xy\text{-plane}) \leq d(P, z\text{-axis})\}.$$

Tasks

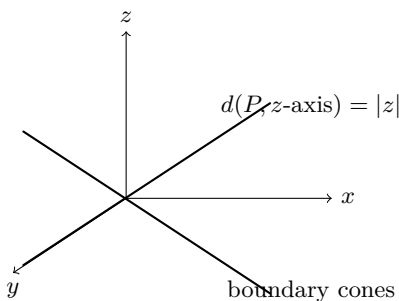
1. Derive a Cartesian inequality describing S .
2. Describe the boundary surface of S .
3. Determine whether each test point belongs to S :

$$P_1 = (1, 2, 3), \quad P_2 = (2, -2, 1).$$

4. Describe the reflectional and rotational symmetries of S .

Question 4 – Solution

Strategy Translate each geometric distance into coordinates: distance to the xy -plane is $|z|$, while distance to the z -axis is the radial distance $\sqrt{x^2 + y^2}$.



Inequality and boundary We require

$$|z| \leq \sqrt{x^2 + y^2} \iff \boxed{z^2 \leq x^2 + y^2}.$$

The boundary $z^2 = x^2 + y^2$ is a double cone with vertex at the origin. The set S is the region on or outside that cone in the radial direction.

Membership For $(1, 2, 3)$, $z^2 = 9 > 1^2 + 2^2 = 5$, so it is not in S . For $(2, -2, 1)$, $z^2 = 1 \leq 8$, so it is in S .

Symmetry and verification The inequality uses only squares, so changing any coordinate's sign preserves membership; rotations about the z -axis also preserve $x^2 + y^2$. On the xy -plane, every point satisfies it, while nonzero points on the z -axis do not, matching the distance interpretation.

Question 5

A rectangular laboratory is aligned with the coordinate axes and has opposite corners

$$O = (0, 0, 0), \quad V = (8, 6, 4).$$

A wireless transmitter will be installed at a point P inside the laboratory. Define

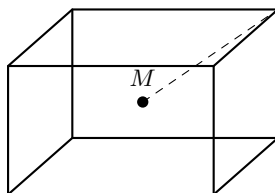
$$R(P) = \max\{d(P, C) : C \text{ is one of the eight corners}\}.$$

Design objective

1. Find the location of P that minimizes $R(P)$.
2. Find the minimum possible value of $R(P)$.
3. Prove that no other transmitter location can produce a smaller greatest distance.

Question 5 – Solution

Strategy Opposite corners force a lower bound: for any transmitter location, at least one of its distances to an opposite pair must be at least half the pair's separation. The box center attains that bound for every pair.



Lower bound The space diagonal has length

$$d(O, V) = \sqrt{8^2 + 6^2 + 4^2} = \sqrt{116} = 2\sqrt{29}.$$

For any location P , the triangle inequality gives

$$d(O, V) \leq d(O, P) + d(P, V).$$

Therefore at least one of $d(O, P)$ and $d(P, V)$ is at least $\sqrt{29}$.

Attaining the bound The midpoint of every space diagonal is

$$M = (4, 3, 2).$$

Each corner differs from M by coordinates $\pm 4, \pm 3, \pm 2$, so every corner is exactly

$$\sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$$

units away. Hence the optimal location is $\boxed{(4, 3, 2)}$, and the minimized greatest distance is $\boxed{\sqrt{29}}$.

Verification The lower bound and attained value agree, which proves global optimality rather than merely checking a plausible point.

Question 6

Two points determine a directed line segment:

$$A = (-4, 1, 7), \quad B = (6, -3, -1).$$

Tasks

1. Find the point P that divides $\overline{AB}$ internally in the ratio

$$AP : PB = 2 : 3.$$

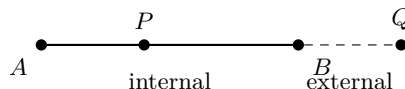
2. Find the point Q that divides the same line externally in the ratio

$$AQ : QB = 2 : 3.$$

3. Verify both ratios using three-dimensional distances, and distinguish clearly between internal and external division.

Question 6 – Solution

Strategy Parameterize the line by $X(t) = A + t(B - A)$. Internal division has $0 < t < 1$; external division requires t outside that interval.



Internal point Since $B - A = (10, -4, -8)$ and $AP/AB = 2/(2 + 3) = 2/5$,

$$P = A + \frac{2}{5}(B - A) = (-4, 1, 7) + (4, -8/5, -16/5) = \boxed{(0, -3/5, 19/5)}.$$

External point For $Q = A + t(B - A)$ beyond A , take $t < 0$. The ratio is

$$\frac{AQ}{QB} = \frac{-t}{1-t} = \frac{2}{3},$$

which gives $t = -2$. Thus

$$Q = A - 2(B - A) = \boxed{(-24, 9, 23)}.$$

Verification Because $P - A = (2/5)(B - A)$ and $B - P = (3/5)(B - A)$, their lengths have ratio 2 : 3. Also $Q - A = -2(B - A)$ and $B - Q = 3(B - A)$, again giving $AQ : QB = 2 : 3$.

Question 7

Find every sphere satisfying all of the following conditions:

1. The sphere lies entirely in the octant

$$x \geq 0, \quad y \leq 0, \quad z \geq 0.$$

2. It is tangent to each of the three coordinate planes.

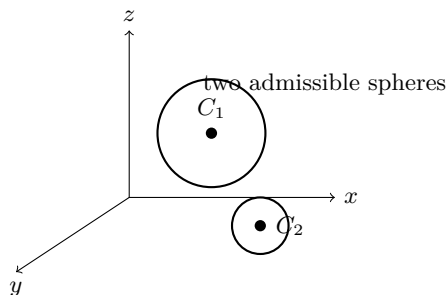
3. It passes through the point

$$P = (1, -2, 3).$$

For each sphere, state its center, radius, and Cartesian equation. Then give a geometric explanation for why two different spheres satisfy the same conditions.

Question 7 – Solution

Strategy Tangency to all three coordinate planes makes the center's coordinate magnitudes equal to the radius. The specified octant fixes their signs.



Center condition If the radius is $r > 0$, then the center must be

$$C = (r, -r, r).$$

Since P lies on the sphere,

$$(1 - r)^2 + (-2 + r)^2 + (3 - r)^2 = r^2.$$

Expanding gives $2r^2 - 12r + 14 = 0$, or

$$r^2 - 6r + 7 = 0.$$

Hence $r = 3 \pm \sqrt{2}$, both positive.

Results The two spheres are

$$(x - r)^2 + (y + r)^2 + (z - r)^2 = r^2, \quad r = 3 \pm \sqrt{2}.$$

Why two? Along the line of possible centers $(r, -r, r)$, the distance to P first decreases and then increases, so the equation “distance to center equals r ” can be met at two scales. Direct substitution verifies both roots, and each sphere has extreme coordinates touching—but not crossing—the three coordinate planes.

Question 8

A transformation T is performed in this order:

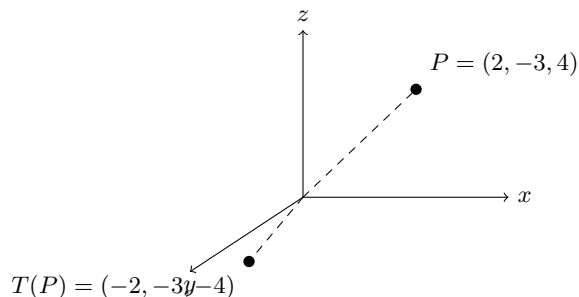
1. reflect across the yz -plane;
2. then reflect the result across the xy -plane.

Tasks

1. Find a coordinate formula for $T(x, y, z)$.
2. Compute $T(2, -3, 4)$.
3. Identify the single rigid motion equivalent to this composition.
4. Prove directly from the three-dimensional distance formula that T preserves the distance between every pair of points.

Question 8 – Solution

Strategy Track which coordinate changes sign under each reflection. Then compare squared distances before and after the transformation.



Coordinate rule Reflection across the yz -plane changes x to $-x$; reflection across the xy -plane changes z to $-z$. Therefore

$$\boxed{T(x, y, z) = (-x, y, -z)}, \quad T(2, -3, 4) = \boxed{(-2, -3, -4)}.$$

Geometric interpretation The y -coordinate is fixed while both perpendicular coordinates reverse. This is a rotation through 180° about the y -axis.

Distance preservation For $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$,

$$d(TP, TQ)^2 = (-x_1 + x_2)^2 + (y_1 - y_2)^2 + (-z_1 + z_2)^2 = d(P, Q)^2.$$

Both distances are nonnegative, so they are equal.

Verification Applying T twice restores every coordinate, as expected for a half-turn.

Question 9

A rectangular box has edges parallel to the coordinate axes and opposite vertices

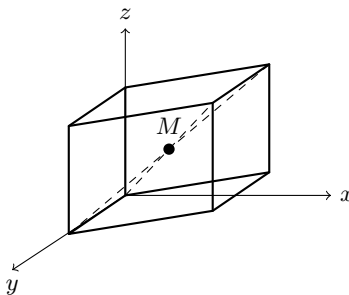
$$A = (1, -2, 3), \quad G = (7, 4, 13).$$

Tasks

1. List all eight vertices systematically.
2. Show that the four space diagonals have a common midpoint, and find it.
3. Find the center and radius of the sphere passing through all eight vertices.
4. Verify that every vertex lies on this sphere without substituting the vertices one at a time.

Question 9 – Solution

Strategy Each vertex independently chooses one of the two available values for each coordinate. Opposite vertices choose complementary values.



Vertices The coordinate choices are $x \in \{1, 7\}$, $y \in \{-2, 4\}$, $z \in \{3, 13\}$. Thus the vertices are

$$(1, -2, 3), (1, -2, 13), (1, 4, 3), (1, 4, 13),$$

$$(7, -2, 3), (7, -2, 13), (7, 4, 3), (7, 4, 13).$$

Common midpoint Every opposite pair averages the low and high value in each coordinate, giving

$$M = \left(\frac{1+7}{2}, \frac{-2+4}{2}, \frac{3+13}{2} \right) = \boxed{(4, 1, 8)}.$$

Hence all four space diagonals bisect one another at M .

Circumsphere From M to any vertex, the coordinate changes have magnitudes 3, 3, 5. Therefore

$$\boxed{r = \sqrt{3^2 + 3^2 + 5^2} = \sqrt{43}}.$$

Verification Every vertex has one of the sign patterns $(\pm 3, \pm 3, \pm 5)$ relative to M , so all eight lie on the sphere.

Question 10

A proposed sphere is required to satisfy all three conditions below:

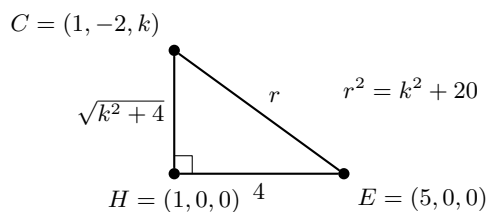
1. It intersects the x -axis at $A = (-3, 0, 0)$ and $B = (5, 0, 0)$.
2. Its center lies in the plane $y = -2$.
3. It is tangent to the xy -plane.

Consistency test

Determine whether such a sphere exists. If it does, find every possible center, radius, and equation. If it does not, prove precisely which requirements are incompatible.

Question 10 – Solution

Strategy The center lies on the perpendicular-bisector plane of the two axial intersection points. Tangency to the xy -plane relates the radius to the center's height.



Center coordinates The midpoint of the two intersections has x -coordinate 1, so write the center as

$$C = (1, -2, k).$$

The squared radius, using $(5, 0, 0)$, is

$$r^2 = (5 - 1)^2 + (0 + 2)^2 + k^2 = 20 + k^2.$$

Tangency to the xy -plane requires $r = |k|$, hence $r^2 = k^2$. Combining these equations would give $k^2 = 20 + k^2$, an impossibility.

Conclusion Therefore no sphere satisfies all the stated conditions.

Consistency review The two axial intersection points already force every possible center onto $x = 1$. Requiring $y = -2$ means the center has a nonzero horizontal offset from the x -axis. Consequently the radius needed to reach either axial point is strictly greater than $|k|$, while tangency to the xy -plane requires it to equal $|k|$. The contradiction is geometric, not an algebraic accident.

Question 1

A line passes through the points

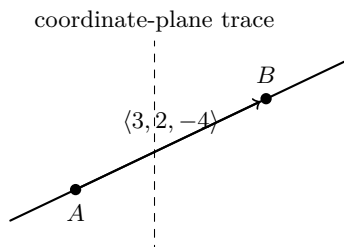
$$A = (-2, 1, 3), \quad B = (4, 5, -5).$$

Tasks

1. Write the line in vector, parametric, and symmetric forms using a primitive integer direction vector.
2. Find the line's intersection with each coordinate plane.
3. Verify that the three plane-intersection points lie on your line.

Question 1 – Solution

Strategy Subtract the given points to obtain a direction, reduce it, and set one coordinate at a time equal to zero to find the coordinate-plane traces.



Line equations Since $B - A = \langle 6, 4, -8 \rangle = 2 \langle 3, 2, -4 \rangle$,

$$\vec{r} = \langle -2, 1, 3 \rangle + t \langle 3, 2, -4 \rangle,$$

$$x = -2 + 3t, \quad y = 1 + 2t, \quad z = 3 - 4t,$$

$$\frac{x + 2}{3} = \frac{y - 1}{2} = \frac{z - 3}{-4}.$$

Coordinate-plane traces Setting $z = 0$, $x = 0$, and $y = 0$ gives, respectively,

$$L \cap xy = \left(\frac{1}{4}, \frac{5}{2}, 0 \right), \quad L \cap yz = \left(0, \frac{7}{3}, \frac{1}{3} \right), \quad L \cap xz = \left(-\frac{7}{2}, 0, 5 \right).$$

Verification These correspond to $t = 3/4$, $2/3$, and $-1/2$ in the same parametric equations, so all three points lie on the line.

Question 2

Consider the line

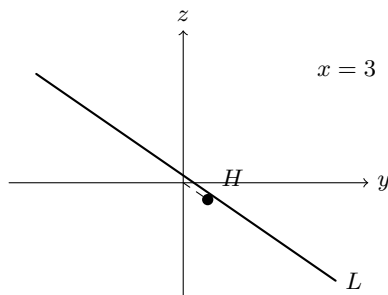
$$L : \quad x = 3, \quad \frac{y+2}{4} = \frac{z-1}{-3}.$$

Tasks

1. Write vector and parametric equations for L .
2. Explain geometrically what the equation $x = 3$ says about the line.
3. Find the point on L closest to the origin and determine that minimum distance.

Question 2 – Solution

Strategy Use the common symmetric ratio as a parameter. For the closest point, minimize the squared distance to avoid unnecessary square roots.



Representations Let the common ratio equal t . Then

$$\vec{r} = \langle 3, -2, 1 \rangle + t \langle 0, 4, -3 \rangle, \quad x = 3, \quad y = -2 + 4t, \quad z = 1 - 3t.$$

The zero x -component of the direction means the entire line lies in the plane $x = 3$ and is parallel to the yz -plane.

Closest point The squared distance from the origin is

$$D^2(t) = 9 + (-2 + 4t)^2 + (1 - 3t)^2 = 25t^2 - 22t + 14.$$

Its minimum occurs at $t = 22/50 = 11/25$. Hence

$$H = \left(3, -\frac{6}{25}, -\frac{8}{25} \right), \quad d(O, L) = \frac{\sqrt{229}}{5}.$$

Verification The vector H from the origin has dot product 0 with $\langle 0, 4, -3 \rangle$, confirming that OH is perpendicular to L .

Question 3

Points

$$A = (1, -2, 3), \quad B = (5, 4, -1), \quad P = (4, u, v)$$

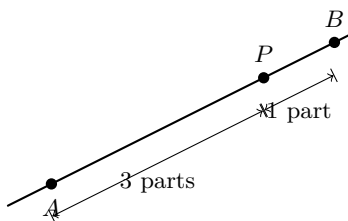
are collinear, and P lies between A and B .

Tasks

1. Determine u and v .
2. Write a parametric equation of the common line.
3. Find the ratio $AP : PB$ and verify it using distances.

Question 3 – Solution

Strategy Parameterize the segment from A to B . The known x -coordinate of P determines the parameter, after which the other coordinates and division ratio follow.



Parameter and coordinates Since $B - A = \langle 4, 6, -4 \rangle$,

$$\vec{r}(t) = \langle 1, -2, 3 \rangle + t \langle 4, 6, -4 \rangle.$$

At P , $1 + 4t = 4$, so $t = 3/4$. Therefore

$$u = -2 + 6\left(\frac{3}{4}\right) = \frac{5}{2}, \quad v = 3 - 4\left(\frac{3}{4}\right) = 0,$$

and

$$\boxed{P = \left(4, \frac{5}{2}, 0\right)}.$$

Division ratio The parameter places P three-fourths of the way from A to B , so

$$\boxed{AP : PB = 3 : 1}.$$

Verification We have $P - A = \frac{3}{4}(B - A)$ and $B - P = \frac{1}{4}(B - A)$; taking lengths preserves the ratio $3 : 1$.

Question 4

For a real parameter k , consider

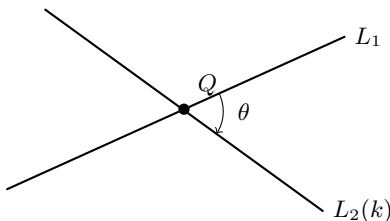
$$\begin{aligned}L_1 : \vec{r} &= \langle 1, 2, 0 \rangle + s \langle 2, -1, 3 \rangle, \\L_2(k) : \vec{r} &= \langle 4, -2, 7 \rangle + t \langle 1, k, -1 \rangle.\end{aligned}$$

Tasks

1. Find the value of k for which the lines intersect.
2. Find the intersection point.
3. For that value of k , find the acute angle between the lines.

Question 4 – Solution

Strategy The x - and z -coordinate equations determine s and t without k ; the y -coordinate then determines the required parameter.



Intersection Equating x - and z -coordinates gives

$$1 + 2s = 4 + t, \quad 3s = 7 - t.$$

The unique solution is $s = 2$, $t = 1$. The y -coordinates then require

$$2 - s = -2 + kt \implies 0 = -2 + k,$$

so $k = 2$. Both lines meet at $Q = (5, 0, 6)$.

Angle The directions are $\vec{d}_1 = \langle 2, -1, 3 \rangle$ and $\vec{d}_2 = \langle 1, 2, -1 \rangle$. Thus

$$\cos \theta = \frac{|\vec{d}_1 \cdot \vec{d}_2|}{\|\vec{d}_1\| \|\vec{d}_2\|} = \frac{3}{\sqrt{14}\sqrt{6}} = \frac{3}{2\sqrt{21}}.$$

Hence $\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right) \approx 70.9^\circ$.

Verification Substitution of $s = 2$, $t = 1$, and $k = 2$ produces $(5, 0, 6)$ in both equations.

Question 5

Let

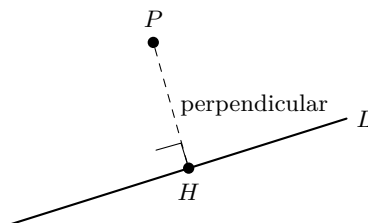
$$P = (4, 3, -1), \quad L : \vec{r} = \langle 1, -1, 2 \rangle + t \langle 2, 1, 2 \rangle.$$

Tasks

1. Find the point H on L closest to P .
2. Write an equation of the line through P that intersects L perpendicularly at H .
3. Find the distance from P to L .

Question 5 – Solution

Strategy At the closest point, the displacement from the line to P is perpendicular to the line's direction vector.



Perpendicular foot Write $A = (1, -1, 2)$ and $\vec{d} = \langle 2, 1, 2 \rangle$. For $H = A + t\vec{d}$,

$$(P - H) \cdot \vec{d} = 0 \implies (P - A - t\vec{d}) \cdot \vec{d} = 0.$$

Since $(P - A) \cdot \vec{d} = 4$ and $\vec{d} \cdot \vec{d} = 9$, $t = 4/9$. Therefore

$$H = \left(\frac{17}{9}, -\frac{5}{9}, \frac{26}{9} \right).$$

Perpendicular line and distance A direction from P to H is $\langle -19, -32, 35 \rangle$ after multiplying by 9. Thus

$$\vec{r} = \langle 4, 3, -1 \rangle + u \langle -19, -32, 35 \rangle.$$

Also,

$$d(P, L) = \|P - H\| = \frac{\sqrt{2610}}{9} = \frac{\sqrt{290}}{3}.$$

Verification $\langle -19, -32, 35 \rangle \cdot \langle 2, 1, 2 \rangle = -38 - 32 + 70 = 0$, and H satisfies the original line equation.

Question 6

Consider the two lines

$$L_1 : \vec{r} = s \langle 1, 2, 0 \rangle ,$$

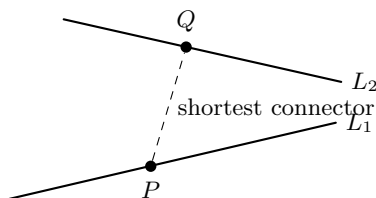
$$L_2 : \vec{r} = \langle 1, 0, 3 \rangle + t \langle 2, -1, 1 \rangle .$$

Tasks

1. Show that the lines are skew.
2. Find the point on each line at which their shortest connecting segment begins and ends.
3. Determine the distance between the lines.

Question 6 – Solution

Strategy For closest points $P \in L_1$ and $Q \in L_2$, the connector $Q - P$ must be perpendicular to both line directions.



Classification The directions $\langle 1, 2, 0 \rangle$ and $\langle 2, -1, 1 \rangle$ are not parallel. Solving the first two intersection equations gives $s = 1/5$, $t = -2/5$, but the third coordinates are then 0 and $13/5$, so the lines do not intersect. They are skew.

Closest points Let

$$P = (s, 2s, 0), \quad Q = (1 + 2t, -t, 3 + t).$$

The conditions $(Q - P) \cdot \langle 1, 2, 0 \rangle = 0$ and $(Q - P) \cdot \langle 2, -1, 1 \rangle = 0$ yield

$$s = \frac{1}{5}, \quad t = -\frac{5}{6}.$$

Thus

$$\boxed{P = \left(\frac{1}{5}, \frac{2}{5}, 0\right)}, \quad \boxed{Q = \left(-\frac{2}{3}, \frac{5}{6}, \frac{13}{6}\right)}.$$

Distance and verification Since $Q - P = \frac{13}{30} \langle -2, 1, 5 \rangle$,

$$\boxed{d(L_1, L_2) = \frac{13\sqrt{30}}{30} = \frac{13}{\sqrt{30}}}.$$

The vector $\langle -2, 1, 5 \rangle$ has zero dot product with both line directions.

Question 7

A line through

$$P = (-1, 2, 0)$$

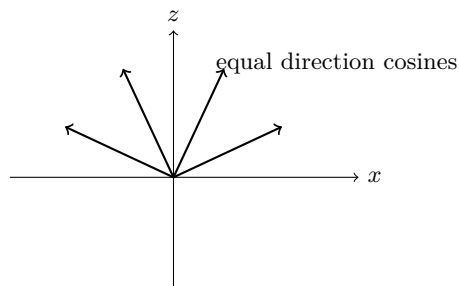
makes equal acute angles with the x -, y -, and z -axes (treated as unoriented lines). Here the first octant includes its boundary planes.

Tasks

1. Find all geometrically distinct lines through P having this property.
2. Determine which of these lines contain at least one point in the first octant.
3. State the parameter intervals corresponding to the portions lying in the first octant.

Question 7 – Solution

Strategy Equal inclination means the three direction cosines have equal absolute value. Direction vectors that differ only by an overall nonzero factor define the same line.



All lines Up to reversal, there are four sign patterns. The distinct lines are

$$\vec{r} = \langle -1, 2, 0 \rangle + t\vec{d}$$

where

$$\vec{d} \in \{ \langle 1, 1, 1 \rangle, \langle 1, 1, -1 \rangle, \langle 1, -1, 1 \rangle, \langle 1, -1, -1 \rangle \}.$$

Each direction has direction cosines with absolute value $1/\sqrt{3}$.

First-octant test For $\langle 1, 1, 1 \rangle$, the coordinates are $(-1 + t, 2 + t, t)$, all nonnegative when $t \geq 1$. For $\langle 1, -1, 1 \rangle$, they are $(-1 + t, 2 - t, t)$, all nonnegative when $1 \leq t \leq 2$.

For $\langle 1, 1, -1 \rangle$ and $\langle 1, -1, -1 \rangle$, the requirements $x \geq 0$ and $z \geq 0$ force incompatible signs of t . Thus exactly the first and third listed lines enter the first octant.

Verification The stated intervals satisfy all three coordinate inequalities, including the boundary planes.

Question 8

Two lines intersect at $Q = (1, -2, 3)$ and have direction vectors

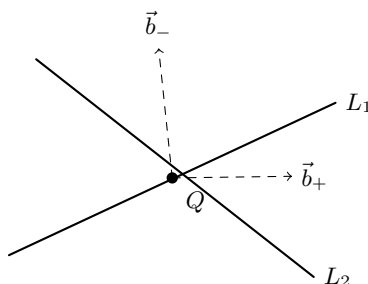
$$\vec{d}_1 = \langle 3, 0, 4 \rangle, \quad \vec{d}_2 = \langle 0, 4, 3 \rangle.$$

Tasks

1. Find equations of both angle-bisector lines through Q .
2. Determine which bisector corresponds to the acute angle between the given lines.
3. Prove that the two bisector lines are perpendicular.

Question 8 – Solution

Strategy Angle-bisector directions are the sum and difference of unit direction vectors. Here the given vectors already have equal length.



Bisector directions Since $\|\vec{d}_1\| = \|\vec{d}_2\| = 5$, the sum and difference give

$$\vec{b}_+ = \langle 3, 4, 7 \rangle, \quad \vec{b}_- = \langle 3, -4, 1 \rangle.$$

Therefore the bisectors are

$$\boxed{\vec{r} = \langle 1, -2, 3 \rangle + t \langle 3, 4, 7 \rangle}, \quad \boxed{\vec{r} = \langle 1, -2, 3 \rangle + u \langle 3, -4, 1 \rangle}.$$

Acute-angle bisector Because $\vec{d}_1 \cdot \vec{d}_2 = 12 > 0$, the directions form an acute angle. Their sum $\vec{b}_+$ lies between them and bisects that acute angle; $\vec{b}_-$ bisects the supplementary angle.

Perpendicularity Directly,

$$\vec{b}_+ \cdot \vec{b}_- = 3(3) + 4(-4) + 7(1) = 9 - 16 + 7 = 0.$$

Hence the two angle-bisector lines are perpendicular.

Verification Dotting $\vec{b}_+$ with each original direction gives the same normalized cosine, confirming equal angles.

Question 9

A line intersects the xy -plane at

$$A = (2, -1, 0)$$

and the yz -plane at

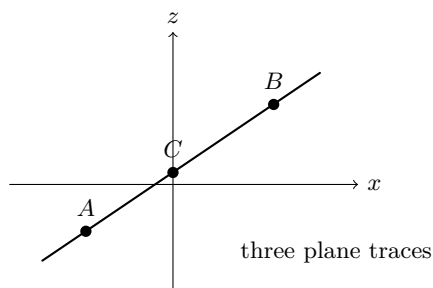
$$B = (0, 3, 4).$$

Tasks

1. Write vector, parametric, and symmetric equations of the line.
2. Find its intersection with the xz -plane.
3. Determine the ratio in which the xz -plane intersection divides $\overline{AB}$.

Question 9 – Solution

Strategy The two traces determine the line. Parameterize from A toward B , then impose $y = 0$ for the xz -plane.



Line equations A direction is $B - A = \langle -2, 4, 4 \rangle = 2 \langle -1, 2, 2 \rangle$. Thus

$$\vec{r} = \langle 2, -1, 0 \rangle + t \langle -1, 2, 2 \rangle,$$

$$x = 2 - t, \quad y = -1 + 2t, \quad z = 2t,$$

and

$$\frac{x - 2}{-1} = \frac{y + 1}{2} = \frac{z}{2}.$$

xz -plane trace Set $y = 0$. Then $-1 + 2t = 0$, so $t = 1/2$ and

$$C = \left(\frac{3}{2}, 0, 1 \right).$$

Division ratio In this reduced parameterization, B occurs at $t = 2$. Thus C lies one-half unit from A in parameter and three-halves units from B , giving

$$AC : CB = 1 : 3.$$

Verification C has $y = 0$, and substituting its coordinates into the symmetric equation gives the common value $1/2$.

Question 10

Two particles move for $t \geq 0$ according to

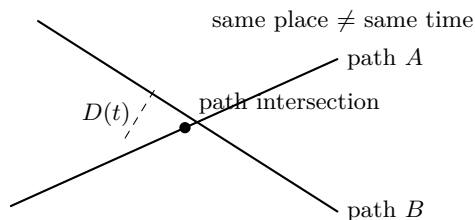
$$\begin{aligned}\vec{r}_A(t) &= \langle 0, 1, 2 \rangle + t \langle 2, -1, 1 \rangle, \\ \vec{r}_B(t) &= \langle 6, -2, 5 \rangle + t \langle -1, 1, 0 \rangle.\end{aligned}$$

Tasks

1. Determine whether their geometric paths intersect.
2. Determine whether the particles collide.
3. Find the time and value of their minimum separation for $t \geq 0$.

Question 10 – Solution

Strategy Path intersection allows different parameters, whereas collision uses the same time. Minimize the squared norm of the relative-position vector.



Paths versus collision Solving $\vec{r}_A(s) = \vec{r}_B(u)$ gives $s = 3$, $u = 0$, so the paths intersect at $\boxed{(6, -2, 5)}$. The particles reach that point at different times and therefore do not collide. Equating positions at the same t would require all components of the relative vector below to vanish, which never occurs.

Separation At a common time t ,

$$\vec{r}_A(t) - \vec{r}_B(t) = \langle -6 + 3t, 3 - 2t, -3 + t \rangle.$$

Hence

$$D^2(t) = (-6 + 3t)^2 + (3 - 2t)^2 + (-3 + t)^2 = 14t^2 - 54t + 54.$$

The vertex occurs at

$$t = \frac{54}{28} = \frac{27}{14} > 0.$$

Substitution gives $D_{\min}^2 = 27/14$, so

$$\boxed{D_{\min} = \sqrt{\frac{27}{14}} = \frac{3\sqrt{42}}{14}}.$$

Verification The quadratic opens upward, and the minimizing time lies in the allowed domain. The nonzero minimum confirms independently that no collision occurs.

Question 1

A plane passes through

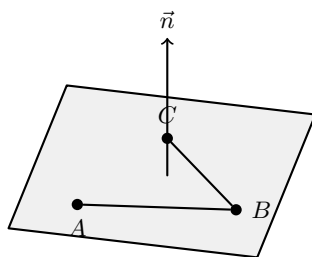
$$A = (1, 0, 2), \quad B = (3, 1, -1), \quad C = (-1, 4, 0).$$

Tasks

1. Derive a Cartesian equation of the plane from two in-plane vectors.
2. Write a unit normal vector.
3. Find all three coordinate-axis intercepts and verify the three given points.

Question 1 – Solution

Strategy Form $\overrightarrow{AB}$ and $\overrightarrow{AC}$; their cross product is normal to the plane.



Step 1: In-plane vectors Subtract coordinates component by component:

$$\begin{aligned}\overrightarrow{AB} &= B - A = \langle 3 - 1, 1 - 0, -1 - 2 \rangle = \langle 2, 1, -3 \rangle, \\ \overrightarrow{AC} &= C - A = \langle -1 - 1, 4 - 0, 0 - 2 \rangle = \langle -2, 4, -2 \rangle.\end{aligned}$$

Step 2: Normal vector Their cross product is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1(-2) - (-3)(4), (-3)(-2) - 2(-2), 2(4) - 1(-2) \rangle = \langle 10, 10, 10 \rangle.$$

Divide by 10 to use the primitive normal $\vec{n} = \langle 1, 1, 1 \rangle$. The point-normal equation through A is

$$(x - 1) + (y - 0) + (z - 2) = 0.$$

Therefore

$$\boxed{x + y + z = 3}, \quad \boxed{\hat{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle}.$$

Step 3: Intercepts On the x -axis, $y = z = 0$, so $x = 3$. The other axes work similarly:

$$\boxed{(3, 0, 0), (0, 3, 0), (0, 0, 3)}.$$

Verification Substitution of A , B , and C gives 3 each time. The two in-plane vectors dot to zero with $\langle 1, 1, 1 \rangle$, independently confirming the orientation.

Question 2

A plane contains the line

$$L : \vec{r} = \langle 1, -2, 0 \rangle + t \langle 2, 1, -1 \rangle$$

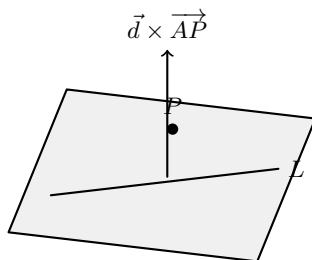
and the point $P = (0, 1, 4)$.

Tasks

1. Find a Cartesian equation of the plane.
2. Explain why the data determine a unique plane.
3. Verify that the entire line and the point lie in the plane.

Question 2 – Solution

Strategy Combine the line direction with a vector from a point on the line to P . Their cross product is a plane normal.



Step 1: Two directions in the plane Take $A = (1, -2, 0)$ from the line. Its direction is $\vec{d} = \langle 2, 1, -1 \rangle$, while

$$\overrightarrow{AP} = P - A = \langle 0 - 1, 1 - (-2), 4 - 0 \rangle = \langle -1, 3, 4 \rangle.$$

Step 2: Find a normal Compute

$$\vec{d} \times \overrightarrow{AP} = \langle 1(4) - (-1)(3), (-1)(-1) - 2(4), 2(3) - 1(-1) \rangle = \langle 7, -7, 7 \rangle.$$

Thus $\vec{n} = \langle 1, -1, 1 \rangle$. Using point A ,

$$(x - 1) - (y + 2) + (z - 0) = 0,$$

so the plane is

$$\boxed{x - y + z = 3}.$$

Uniqueness The vectors $\vec{d}$ and $\overrightarrow{AP}$ are not parallel, so they span exactly one plane.

Verification Point P gives $0 - 1 + 4 = 3$. A general point on L is $(1 + 2t, -2 + t, -t)$, and

$$(1 + 2t) - (-2 + t) - t = 3$$

for every t , proving that the whole line is contained in the plane.

Question 3

For a real parameter k , let

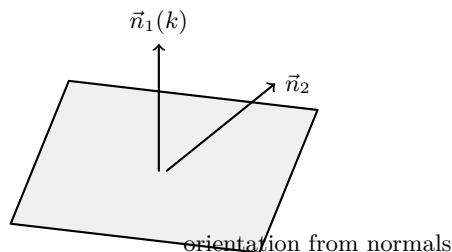
$$\Pi_k : kx + 2y - z = 5, \quad \Sigma : 2x - y + 2z = 7.$$

Tasks

1. Find k if the planes are perpendicular.
2. Determine whether any k makes the planes parallel.
3. Find every k for which the acute angle between the planes is 60° .

Question 3 – Solution

Strategy Plane orientation is controlled by the normals $\vec{n}_k = \langle k, 2, -1 \rangle$ and $\vec{m} = \langle 2, -1, 2 \rangle$.



Step 1: Perpendicularity Calculate

$$\vec{n}_k \cdot \vec{m} = k(2) + 2(-1) + (-1)(2) = 2k - 4.$$

Perpendicular normals require $2k - 4 = 0$, hence $\boxed{k = 2}$.

Step 2: Parallelism Parallelism would require $\langle k, 2, -1 \rangle = \lambda \langle 2, -1, 2 \rangle$. From $2 = -\lambda$, $\lambda = -2$; but the third coordinate would require $-1 = 2(-2) = -4$, a contradiction. Hence $\boxed{\text{no real } k \text{ makes them parallel}}$.

Step 3: The 60° condition The angle formula gives

$$\frac{|2k - 4|}{3\sqrt{k^2 + 5}} = \frac{1}{2}.$$

Cross-multiply and square:

$$2|2k - 4| = 3\sqrt{k^2 + 5} \implies 16(k - 2)^2 = 9(k^2 + 5).$$

Expanding gives $16k^2 - 64k + 64 = 9k^2 + 45$, so

$$7k^2 - 64k + 19 = 0.$$

The quadratic formula yields

$$k = \frac{64 \pm \sqrt{64^2 - 4(7)(19)}}{14} = \frac{64 \pm 18\sqrt{11}}{14},$$

and therefore

$$\boxed{k = \frac{32 \pm 9\sqrt{11}}{7}}.$$

Verification Neither root makes the original denominator zero, and substitution into the unsquared equation gives the positive value $1/2$, so no extraneous root remains.

Question 4

Consider the planes

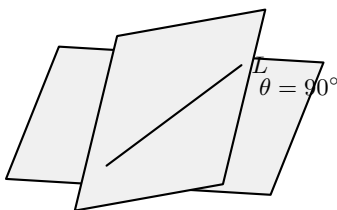
$$\Pi_1 : x + y - z = 2, \quad \Pi_2 : 2x - y + z = 1.$$

Tasks

1. Find a vector and parametric equation for their line of intersection.
2. Find the dihedral angle between the planes.
3. Verify the line equation in both plane equations.

Question 4 – Solution

Strategy The intersection direction is perpendicular to both normals. Solve the two plane equations for one point and one free parameter.



Step 1: Solve simultaneously Add the equations:

$$(x + y - z) + (2x - y + z) = 2 + 1 \implies 3x = 3 \implies x = 1.$$

Substitute $x = 1$ into $x + y - z = 2$:

$$1 + y - z = 2 \implies y = 1 + z.$$

Let the free coordinate be $z = t$. Then $y = 1 + t$, so

$$\boxed{\vec{r} = \langle 1, 1, 0 \rangle + t \langle 0, 1, 1 \rangle}.$$

Step 2: Confirm the direction The normals are $\vec{n}_1 = \langle 1, 1, -1 \rangle$ and $\vec{n}_2 = \langle 2, -1, 1 \rangle$. Their cross product is $\langle 0, -3, -3 \rangle$, parallel to $\langle 0, 1, 1 \rangle$.

Step 3: Angle Compute $\vec{n}_1 \cdot \vec{n}_2 = 2 - 1 - 1 = 0$. Therefore

$$\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|} = 0, \quad \boxed{\theta = 90^\circ}.$$

Verification Substitution of $(1, 1 + t, t)$ gives 2 in Π_1 and 1 in Π_2 for every t .

Question 5

Let $P = (4, -1, 5)$ and

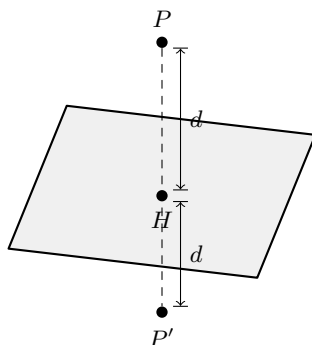
$$\Pi : 2x - y + 2z = 3.$$

Tasks

1. Find the perpendicular projection H of P onto Π .
2. Find the distance from P to Π .
3. Reflect P across Π and find the image point P' .
4. Verify the midpoint and perpendicularity conditions.

Question 5 – Solution

Strategy Move from P along the normal $\vec{n} = \langle 2, -1, 2 \rangle$. The signed equation value determines the required multiple.



Step 1: Signed displacement Write the plane as $F(x, y, z) = 2x - y + 2z - 3 = 0$. At P ,

$$2(4) - (-1) + 2(5) - 3 = 16, \quad \|\vec{n}\|^2 = 9.$$

Step 2: Projection A point on the perpendicular through P is $P - s\vec{n}$. To reach the plane,

$$F(P - s\vec{n}) = F(P) - s\|\vec{n}\|^2 = 16 - 9s = 0,$$

so $s = 16/9$. Hence

$$H = P - \frac{16}{9}\vec{n} = \left(4 - \frac{32}{9}, -1 + \frac{16}{9}, 5 - \frac{32}{9}\right) = \boxed{\left(\frac{4}{9}, \frac{7}{9}, \frac{13}{9}\right)}.$$

Step 3: Distance Since $P - H = (16/9)\vec{n}$,

$$d(P, \Pi) = \frac{16}{9}\|\vec{n}\| = \frac{16}{9}(3), \quad \boxed{d(P, \Pi) = \frac{16}{3}}.$$

Step 4: Reflection Since H is the midpoint of PP' ,

$$P' = 2H - P = P - \frac{32}{9}\vec{n} = \boxed{\left(-\frac{28}{9}, \frac{23}{9}, -\frac{19}{9}\right)}.$$

Verification Substituting H into the plane gives 3. Also $P - H = (16/9)\vec{n}$, so PH is perpendicular to the plane, and direct averaging of P and P' gives H .

Question 6

Among all points on the plane

$$\Pi : x + 2y + 2z = 9,$$

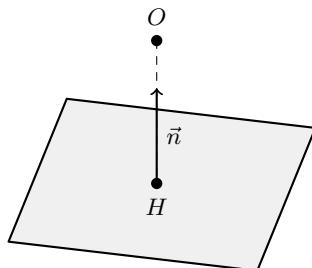
find the point closest to the origin.

Tasks

1. Determine the closest point and minimum distance.
2. Prove global minimality without using calculus.
3. Find the sphere centered at the origin tangent to Π .

Question 6 – Solution

Strategy The shortest segment from a point to a plane follows the plane's normal. Use Cauchy–Schwarz for a global proof.



Step 1: Follow the normal The normal is $\vec{n} = \langle 1, 2, 2 \rangle$ and

$$\|\vec{n}\| = \sqrt{1^2 + 2^2 + 2^2} = 3.$$

A point on the normal line through the origin has form $(t, 2t, 2t)$. Imposing the plane equation gives

$$t + 2(2t) + 2(2t) = 9 \implies 9t = 9 \implies t = 1.$$

Hence

$$\boxed{H = (1, 2, 2)}, \quad \boxed{d(O, \Pi) = \|H\| = 3}.$$

Step 2: Global proof For any $X = (x, y, z)$ on Π , $\vec{n} \cdot X = 9$. Cauchy–Schwarz gives

$$9 = |\vec{n} \cdot X| \leq \|\vec{n}\| \|X\| = 3\|X\|,$$

so $\|X\| \geq 3$. Equality occurs precisely when X is parallel to $\vec{n}$, as H is.

Step 3: Tangent sphere A sphere centered at the origin and reaching H has radius 3, so its equation is

$$\boxed{x^2 + y^2 + z^2 = 9}.$$

Its radius to H is normal to Π , so the plane is tangent there.

Question 7

Let

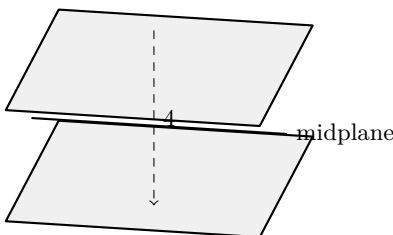
$$\Pi_1 : 2x - 2y + z = 7, \quad \Pi_2 : 2x - 2y + z = -5.$$

Tasks

1. Find the distance between the parallel planes.
2. Derive the plane exactly halfway between them.
3. Find endpoints of one shortest segment joining the planes and verify its midpoint lies on the halfway plane.

Question 7 – Solution

Strategy Put the equations over the same normal. Their constant difference divided by the normal length is the separation.



Step 1: Distance Both planes have normal $\vec{n} = \langle 2, -2, 1 \rangle$, whose length is

$$\|\vec{n}\| = \sqrt{2^2 + (-2)^2 + 1^2} = 3.$$

Because their left sides are identical, the distance is the absolute difference of constants divided by $\|\vec{n}\|$:

$$d(\Pi_1, \Pi_2) = \frac{|7 - (-5)|}{3} = 4.$$

Step 2: Halfway plane A parallel plane has form $2x - 2y + z = c$. Equal distance from c to 7 and -5 requires

$$7 - c = c - (-5) \implies 2c = 2 \implies c = 1.$$

Thus

$$2x - 2y + z = 1.$$

Step 3: Shortest connector Choose $A = (3, 0, 1)$; indeed $2(3) - 2(0) + 1 = 7$. The unit normal is $\vec{n}/3$, so moving four units toward Π_2 gives

$$B = A - \frac{4}{3} \langle 2, -2, 1 \rangle = \left(\frac{1}{3}, \frac{8}{3}, -\frac{1}{3} \right).$$

Substitution gives $2(1/3) - 2(8/3) - 1/3 = -5$, so $B \in \Pi_2$. Its midpoint with A is

$$M = \left(\frac{5}{3}, \frac{4}{3}, \frac{1}{3} \right).$$

Verification $2(5/3) - 2(4/3) + 1/3 = 1$. Also $B - A$ is parallel to the common normal, so AB is perpendicular to both planes and is therefore a shortest connector.

Question 8

All planes containing the intersection of

$$\Pi_1 : x + y + z = 1 \quad \text{and} \quad \Pi_2 : x - y + 2z = 3$$

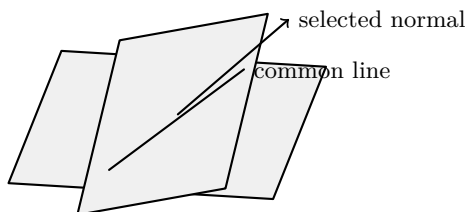
form a pencil of planes.

Tasks

1. Write a one-parameter equation for this pencil.
2. Find the member perpendicular to $\Sigma : x + 2y - z = 4$.
3. Verify that the selected plane contains the entire common line of Π_1 and Π_2 .

Question 8 – Solution

Strategy Every combination $F_1 + \lambda F_2 = 0$ vanishes wherever both original plane expressions vanish. Choose λ using normal-vector perpendicularity.



Step 1: Build the pencil Move all terms to the left:

$$F_1 = x + y + z - 1, \quad F_2 = x - y + 2z - 3.$$

At every point on the common line, $F_1 = F_2 = 0$. Therefore every equation

$$F_1 + \lambda F_2 = 0 \quad (\lambda \in \mathbb{R}), \quad \text{together with } F_2 = 0$$

contains that line. The member $F_2 = 0$ is also needed for the full pencil; its normal has dot product -3 with $\langle 1, 2, -1 \rangle$, so it is not the requested perpendicular member. Collecting coefficients shows its normal is

$$\vec{n}_\lambda = \langle 1 + \lambda, 1 - \lambda, 1 + 2\lambda \rangle.$$

Step 2: Impose perpendicularity The normal of Σ is $\vec{m} = \langle 1, 2, -1 \rangle$. Perpendicular planes have perpendicular normals, so

$$\vec{n}_\lambda \cdot \langle 1, 2, -1 \rangle = 2 - 3\lambda = 0,$$

because $(1 + \lambda) + 2(1 - \lambda) - (1 + 2\lambda) = 2 - 3\lambda$. Thus $\lambda = 2/3$. Multiplying by 3 gives

$$3F_1 + 2F_2 = 0,$$

and expanding yields

$$5x + y + 7z = 9.$$

Verification Any point common to Π_1 and Π_2 has $F_1 = F_2 = 0$, so it satisfies $3F_1 + 2F_2 = 0$. Its normal $\langle 5, 1, 7 \rangle$ dots with $\langle 1, 2, -1 \rangle$ to 0.

Question 9

Find the locus of points equidistant from the intersecting planes

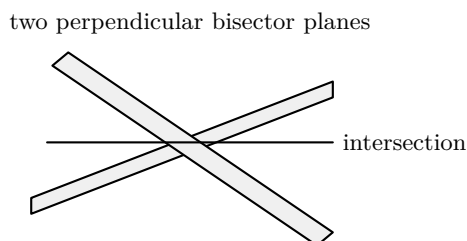
$$\Pi_1 : x - 2y + 2z - 3 = 0, \quad \Pi_2 : 2x + 2y - z + 6 = 0.$$

Tasks

1. Derive equations for both components of the locus.
2. Explain why they are angle-bisector planes.
3. Prove that the two resulting planes are perpendicular.

Question 9 – Solution

Strategy Use the point-to-plane distance formula with absolute values. Equal normal lengths simplify the two sign cases.



Step 1: Distance equation The normal lengths are

$$\sqrt{1^2 + (-2)^2 + 2^2} = 3, \quad \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

Thus equality of point-to-plane distances becomes

$$|x - 2y + 2z - 3| = |2x + 2y - z + 6|.$$

Step 2: Resolve the absolute values Let the two expressions be F_1 and F_2 . The equation $|F_1| = |F_2|$ gives two cases.

If $F_1 = F_2$, then

$$x - 2y + 2z - 3 = 2x + 2y - z + 6 \implies x + 4y - 3z + 9 = 0.$$

If $F_1 = -F_2$, then

$$x - 2y + 2z - 3 = -2x - 2y + z - 6 \implies 3x + z + 3 = 0.$$

Therefore the locus consists of

$$\boxed{x + 4y - 3z + 9 = 0}, \quad \boxed{3x + z + 3 = 0}.$$

Interpretation On either plane, the signed perpendicular displacements from Π_1 and Π_2 have equal magnitude. These are precisely the internal and external angle-bisector planes, and both contain the original planes' intersection line.

Perpendicularity Normals may be chosen as $\langle 1, 4, -3 \rangle$ and $\langle 3, 0, 1 \rangle$. Their dot product is $3 - 3 = 0$, so the bisector planes are perpendicular.

Verification Substituting any point from either boxed plane into the absolute-value equation recovers equal distances.

Question 10

A plane cuts the positive coordinate axes at A , B , and C . The centroid of the intercept triangle ABC is

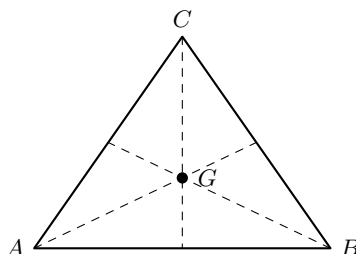
$$G = (2, 3, 4).$$

Tasks

1. Determine A , B , and C .
2. Find the plane in intercept and Cartesian forms.
3. Find the area of triangle ABC .
4. Find the volume of the tetrahedron bounded by the plane and the coordinate planes.

Question 10 – Solution

Strategy Write $A = (a, 0, 0)$, $B = (0, b, 0)$, and $C = (0, 0, c)$. Their centroid directly determines the three intercepts.



Step 1: Recover the intercepts Write $A = (a, 0, 0)$, $B = (0, b, 0)$, and $C = (0, 0, c)$. Then

$$G = \frac{A + B + C}{3} = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right),$$

so $a/3 = 2$, $b/3 = 3$, and $c/3 = 4$. Hence

$$A = (6, 0, 0), \quad B = (0, 9, 0), \quad C = (0, 0, 12).$$

Step 2: Plane equation A plane with intercepts a, b, c has equation $x/a + y/b + z/c = 1$. Therefore

$$\frac{x}{6} + \frac{y}{9} + \frac{z}{12} = 1, \quad 6x + 4y + 3z = 36.$$

Step 3: Triangle area Compute

$$\begin{aligned} \overrightarrow{AB} &= \langle -6, 9, 0 \rangle, & \overrightarrow{AC} &= \langle -6, 0, 12 \rangle, \\ \overrightarrow{AB} \times \overrightarrow{AC} &= \langle 9(12), 0(-6) - (-6)(12), (-6)(0) - 9(-6) \rangle = \langle 108, 72, 54 \rangle. \end{aligned}$$

Its length is

$$\sqrt{108^2 + 72^2 + 54^2} = \sqrt{19764} = 18\sqrt{61}.$$

Hence $\text{Area}(ABC) = 9\sqrt{61}.$

Step 4: Tetrahedron volume The coordinate-axis edges from the origin are mutually perpendicular with lengths 6, 9, 12. A tri-rectangular tetrahedron has one-sixth the corresponding box volume, so

$$V = \frac{1}{6}(6)(9)(12) = 108.$$

Verification Substitution of each intercept into $6x + 4y + 3z = 36$ succeeds, and their coordinate average is $(2, 3, 4)$.

Question 1

Classify and sketch the surface

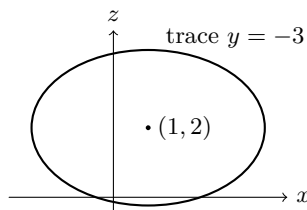
$$4x^2 + y^2 + 9z^2 - 8x + 6y - 36z + 4 = 0.$$

Tasks

1. Convert to standard form and classify it.
2. Find its center and semiaxes.
3. Derive its principal traces.
4. Find all coordinate-axis intercepts.

Question 1 – Solution

Strategy Group like variables and complete each square.



Step 1: Complete squares Write

$$4(x^2 - 2x) + (y^2 + 6y) + 9(z^2 - 4z) + 4 = 0.$$

Using $x^2 - 2x = (x - 1)^2 - 1$, $y^2 + 6y = (y + 3)^2 - 9$, and $z^2 - 4z = (z - 2)^2 - 4$ gives

$$4(x - 1)^2 + (y + 3)^2 + 9(z - 2)^2 = 45.$$

Divide by 45:

$$\frac{(x - 1)^2}{45/4} + \frac{(y + 3)^2}{45} + \frac{(z - 2)^2}{5} = 1.$$

Step 2: Geometry This is an ellipsoid centered at $(1, -3, 2)$ with semiaxes $3\sqrt{5}/2$, $3\sqrt{5}$, and $\sqrt{5}$. Its six axis endpoints relative to the center are obtained by adding and subtracting those lengths in the corresponding coordinate.

Step 3: Principal traces Through the center, the traces are

$$\begin{aligned} x = 1 : \frac{(y + 3)^2}{45} + \frac{(z - 2)^2}{5} &= 1, & y = -3 : \frac{(x - 1)^2}{45/4} + \frac{(z - 2)^2}{5} &= 1, \\ z = 2 : \frac{(x - 1)^2}{45/4} + \frac{(y + 3)^2}{45} &= 1. \end{aligned}$$

Step 4: Axis intercepts Set the other two coordinates to zero in the original equation. This gives

$$(1, 0, 0), \quad (0, -3 \pm \sqrt{5}, 0), \quad (0, 0, 2 \pm 4\sqrt{2}/3).$$

Substitution verifies each intercept and trace.

Question 2

Analyze

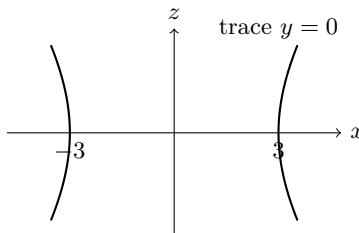
$$\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{16} = 1.$$

Tasks

1. Identify the surface and its axis.
2. Derive all three coordinate-plane traces.
3. Find the waist dimensions.
4. Explain why the surface is connected.

Question 2 – Solution

Strategy Read the signs and inspect traces one coordinate at a time.



Step 1: Classification Two positive squared terms and one negative term equal 1, so this is a hyperboloid of one sheet. The negative term is the z -term, hence the axis is the z -axis.

Step 2: Horizontal traces Setting $z = c$ gives

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 + \frac{c^2}{16}.$$

Every right side is positive, so every horizontal plane cuts an ellipse. At $c = 0$ the waist ellipse has semiaxes 3 and 2.

Step 3: Vertical traces At $y = 0$,

$$\frac{x^2}{9} - \frac{z^2}{16} = 1,$$

and at $x = 0$,

$$\frac{y^2}{4} - \frac{z^2}{16} = 1.$$

Both are hyperbolas opening perpendicular to the z -axis.

Conclusion Because a nonempty ellipse exists for every real $z = c$ and varies continuously with c , the surface is connected.

Question 3

For real k , classify

$$x^2 + y^2 - kz^2 = 1.$$

Tasks

1. Classify the surface for $k < 0$, $k = 0$, and $k > 0$.
2. Describe boundedness in each case.
3. Compare horizontal and vertical traces across the transition.

Question 3 – Solution

Strategy The sign of the z^2 coefficient controls the canonical type.

Case 1: $k < 0$ Write $k = -a$ with $a > 0$. Then $x^2 + y^2 + az^2 = 1$, an ellipsoid of revolution, bounded in every direction.

Case 2: $k = 0$ The equation becomes $x^2 + y^2 = 1$ with unrestricted z , so it is a circular cylinder of radius 1 parallel to the z -axis, unbounded in z .

Case 3: $k > 0$ The equation has two positive terms and one negative term, so it is a one-sheet hyperboloid about the z -axis. At $z = c$, the circle has radius $\sqrt{1 + kc^2}$; the surface is unbounded in all three coordinates.

Verification In every case the xy -trace is the unit circle. Vertical traces change from ellipses, to parallel lines, to hyperbolas, confirming the transition.

Question 4

Analyze the hyperbolic paraboloid

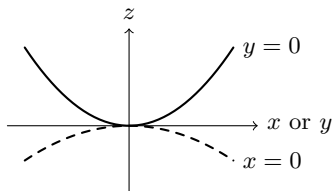
$$z = \frac{x^2}{4} - \frac{y^2}{9}.$$

Tasks

1. Derive the principal vertical traces.
2. Classify horizontal traces for $z = c$.
3. Find every line through the origin contained in the surface.
4. Justify the term “saddle.”

Question 4 – Solution

Strategy Fix one coordinate at a time, then factor the level-zero equation.



Step 1: Vertical traces If $y = 0$, then $z = x^2/4$, an upward parabola. If $x = 0$, then $z = -y^2/9$, a downward parabola. Their opposite concavities create the saddle.

Step 2: Horizontal traces At $z = c$,

$$\frac{x^2}{4} - \frac{y^2}{9} = c.$$

For $c > 0$ the hyperbola opens in the x -direction; for $c < 0$ it opens in the y -direction.

Step 3: Lines through the origin A contained line $(x, y, z) = t(a, b, c)$ must satisfy $tc = t^2(a^2/4 - b^2/9)$ for all t , so $c = 0$ and $a^2/4 = b^2/9$. Factoring gives

$$\left(\frac{x}{2} - \frac{y}{3}\right)\left(\frac{x}{2} + \frac{y}{3}\right) = 0.$$

Thus the two lines are $\boxed{z = 0, y = \pm 3x/2}$. Direct substitution confirms every point on each line lies on the surface.

Question 5

A centered quadric has horizontal traces only for $|z| \geq 3$, where

$$x^2 + y^2 = 4 \left(\frac{z^2}{9} - 1 \right).$$

Tasks

1. Reconstruct and classify the quadric.
2. Find its vertices and axis.
3. Derive both principal vertical traces.
4. Explain the empty central region.

Question 5 – Solution

Strategy Rearrange the trace family and interpret both its signs and its existence condition.

Step 1: Reconstruct Move terms and divide by 4:

$$x^2 + y^2 = \frac{4}{9}z^2 - 4 \implies \boxed{\frac{z^2}{9} - \frac{x^2}{4} - \frac{y^2}{4} = 1}.$$

Step 2: Geometry One positive square and two negative squares equal 1, so this is a two-sheet hyperboloid with z -axis. Setting $x = y = 0$ gives vertices $\boxed{(0, 0, \pm 3)}$.

Step 3: Vertical traces Setting $y = 0$ and $x = 0$ gives

$$\boxed{\frac{z^2}{9} - \frac{x^2}{4} = 1}, \quad \boxed{\frac{z^2}{9} - \frac{y^2}{4} = 1}.$$

Step 4: Gap Since $x^2 + y^2 \geq 0$, the given trace requires $z^2/9 - 1 \geq 0$, hence $|z| \geq 3$. No surface points lie in the central slab $-3 < z < 3$.

Question 6

Find and describe the intersection of

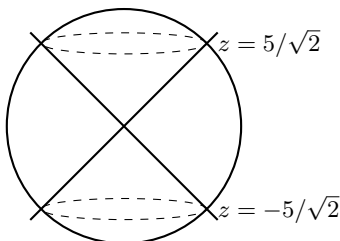
$$x^2 + y^2 + z^2 = 25 \quad \text{and} \quad z^2 = x^2 + y^2.$$

Tasks

1. Determine every component of the intersection.
2. Find each component's exact height and radius.
3. Verify the result in both original equations.

Question 6 – Solution

Strategy Substitute the cone relation into the sphere, then retain both signs of z .



Step 1: Solve for height Since $x^2 + y^2 = z^2$, the sphere gives

$$z^2 + z^2 = 25 \implies 2z^2 = 25 \implies z = \pm \frac{5}{\sqrt{2}}.$$

Step 2: Solve for radius On either level,

$$x^2 + y^2 = z^2 = \frac{25}{2}.$$

Thus the intersection consists of two circles:

$$\boxed{z = \pm \frac{5}{\sqrt{2}}, \quad x^2 + y^2 = \frac{25}{2}}.$$

Each has radius $\boxed{5/\sqrt{2}}$. Both signs are essential because the cone is double. Substitution verifies both circles in both original equations.

Question 7

The ellipsoid

$$\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

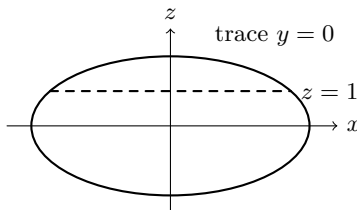
is cut by the plane $z = 1$.

Tasks

1. Find the section in standard form.
2. Determine its semiaxes and area.
3. Compare it quantitatively with the central section $z = 0$.

Question 7 – Solution

Strategy Substitute the fixed height and rewrite the remaining equation in standard ellipse form.



Step 1: Substitute With $z = 1$,

$$\frac{x^2}{16} + \frac{y^2}{9} + \frac{1}{4} = 1 \implies \frac{x^2}{16} + \frac{y^2}{9} = \frac{3}{4}.$$

Divide by $3/4$:

$$\boxed{\frac{x^2}{12} + \frac{y^2}{27/4} = 1}.$$

The semiaxes are $\boxed{2\sqrt{3}}$ and $\boxed{3\sqrt{3}/2}$.

Step 2: Area Therefore

$$A = \pi ab = \pi(2\sqrt{3})(3\sqrt{3}/2) = \boxed{9\pi}.$$

Comparison At $z = 0$, the semiaxes are 4 and 3, so the central area is 12π . The section at $z = 1$ has area $3/4$ as large, matching the right side $3/4$ before normalization.

Question 8

An elliptic paraboloid has vertex $(1, -2, 3)$, axis parallel to the z -axis, opens downward, and its trace in $z = -1$ is

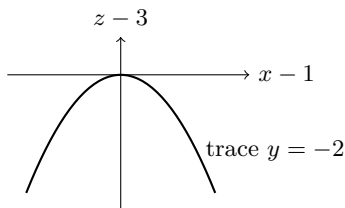
$$\frac{(x-1)^2}{8} + \frac{(y+2)^2}{4} = 1.$$

Tasks

1. Reconstruct the surface equation.
2. Find the trace at $z = 2$ in standard form.
3. State that trace's semiaxes and verify the supplied trace.

Question 8 – Solution

Strategy Use translated form $z - 3 = -A(x - 1)^2 - B(y + 2)^2$ and fit the known level.



Step 1: Use the known trace At $z = -1$, the vertical drop is 4. The given ellipse is equivalent to

$$\frac{1}{2}(x - 1)^2 + (y + 2)^2 = 4.$$

Hence the surface equation is

$$z - 3 = -\frac{1}{2}(x - 1)^2 - (y + 2)^2.$$

Step 2: New trace Set $z = 2$:

$$1 = \frac{1}{2}(x - 1)^2 + (y + 2)^2,$$

or

$$\frac{(x - 1)^2}{2} + (y + 2)^2 = 1.$$

Its semiaxes are $\sqrt{2}$ and 1. Substituting $z = -1$ back into the derived surface recovers the supplied ellipse.

Question 9

Classify the real point sets

$$x^2 + y^2 + z^2 - 2xy - 2xz + 2yz = c$$

Tasks

1. Factor the quadratic expression.
2. Classify the real point set for $c < 0$, $c = 0$, and $c > 0$.
3. Explain the degeneration geometrically.

Question 9 – Solution

Strategy Test whether the quadratic expression is a perfect square.

Step 1: Factor Expanding

$$(x - y - z)^2 = x^2 + y^2 + z^2 - 2xy - 2xz + 2yz.$$

Thus the equation is simply

$$(x - y - z)^2 = c.$$

Case analysis If $c < 0$, no real square can equal c , so the set is empty. If $c = 0$, then $x - y - z = 0$, a single plane. If $c > 0$, taking square roots gives

$$x - y - z = \pm\sqrt{c},$$

so the set is two distinct parallel planes.

Conclusion The classification is $\boxed{\emptyset}$, $\boxed{\text{one plane}}$, and $\boxed{\text{two parallel planes}}$ for the three cases. Substitution into the factored form verifies each result.

Question 10

Consider

$$5x^2 - 6xy + 5y^2 - z^2 = 0, \quad u = \frac{x+y}{\sqrt{2}}, \quad v = \frac{x-y}{\sqrt{2}}.$$

Tasks

1. Rewrite the equation in u, v, z .
2. Classify the surface and identify its axis.
3. Derive the traces in $z = c$, $u = 0$, and $v = 0$.
4. Relate the principal directions to the original xy -axes.

Question 10 – Solution

Strategy Express $x^2 + y^2$ and xy in rotated coordinates so the cross term disappears.

Step 1: Invert and substitute We have

$$x = \frac{u+v}{\sqrt{2}}, \quad y = \frac{u-v}{\sqrt{2}}, \quad x^2 + y^2 = u^2 + v^2, \quad xy = \frac{u^2 - v^2}{2}.$$

Therefore

$$5(u^2 + v^2) - 3(u^2 - v^2) - z^2 = 0, \quad \boxed{2u^2 + 8v^2 - z^2 = 0}.$$

Step 2: Classify This is an elliptic cone with vertex at the origin and axis the z -axis.

Step 3: Traces For $z = c \neq 0$,

$$\boxed{\frac{u^2}{c^2/2} + \frac{v^2}{c^2/8} = 1}.$$

For $v = 0$, $z = \pm\sqrt{2}u$; for $u = 0$, $z = \pm 2\sqrt{2}v$. At $z = 0$, only the vertex remains.

Step 4: Interpret rotation The u -axis is $x = y$ and the v -axis is $x = -y$, so the principal horizontal directions are rotated 45° from the original axes.

Question 1

Let

$$f(x, y) = \frac{\sqrt{9 - x^2 - y^2}}{\ln(x - y)}.$$

Tasks

1. Determine the exact domain.
2. State which boundary curves are included or excluded.
3. Test $(2, 0)$, $(0, 0)$, and $(1, 0)$ for membership.

Question 1 – Solution

Strategy Enforce separately the square-root, logarithm, and nonzero-denominator conditions.

Step 1: Square root Real values require $9 - x^2 - y^2 \geq 0$, hence $x^2 + y^2 \leq 9$.

Step 2: Logarithm The logarithm requires $x - y > 0$. Because it is a denominator, $\ln(x - y) \neq 0$, so $x - y \neq 1$.

Step 3: Combine Therefore

$$D = \{(x, y) : x^2 + y^2 \leq 9, x > y, x - y \neq 1\}.$$

The circular boundary is included only where the other restrictions hold; $x = y$ and $x - y = 1$ are excluded.

Step 4: Test points $(2, 0)$ satisfies all conditions. $(0, 0)$ fails $x > y$. $(1, 0)$ makes the denominator $\ln 1 = 0$. Thus only $\boxed{(2, 0)}$ belongs to D .

Question 2

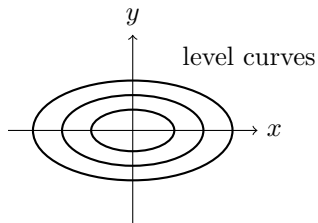
For $f(x, y) = x^2 + 4y^2$, analyze the level curves $f(x, y) = c$.

Tasks

1. Classify them for $c < 0$, $c = 0$, and $c > 0$.
2. Find semiaxes and enclosed area for $c > 0$.
3. Explain how the contours determine the graph's shape.

Question 2 – Solution

Strategy Treat c by sign and normalize the positive-level equation.



Step 1: Cases Because squares are nonnegative, $c < 0$ gives no points. For $c = 0$, both squares vanish, giving $(0, 0)$. For $c > 0$,

$$\frac{x^2}{c} + \frac{y^2}{c/4} = 1.$$

Thus the semiaxes are $\sqrt{c}$ and $\sqrt{c}/2$.

Step 2: Area The enclosed area is

$$A(c) = \pi (\sqrt{c}) \left(\frac{\sqrt{c}}{2} \right) = \boxed{\frac{\pi c}{2}}.$$

Step 3: Graph Higher levels produce nested, expanding ellipses centered at the origin. Since $z = x^2 + 4y^2 \geq 0$ and increases quadratically, the graph is an upward elliptic paraboloid, steeper in the y -direction.

Question 3

For positive inputs define

$$F(x, y) = \frac{xy}{x + y}.$$

Tasks

1. Show how F changes when both inputs are scaled by $s > 0$.
2. Derive the level curve $F = c > 0$.
3. State its domain branches and asymptotes.

Question 3 – Solution

Strategy Substitute scaled inputs, then solve the level equation explicitly for y .

Step 1: Scaling Direct substitution gives

$$F(sx, sy) = \frac{s^2 xy}{s(x+y)} = sF(x, y).$$

Thus the output has degree-one scaling.

Step 2: Level curve Set $xy/(x+y) = c$:

$$xy = cx + cy \implies y(x - c) = cx \implies \boxed{y = \frac{cx}{x - c}}.$$

For positive x, y , the denominator must be positive, so $x > c$; symmetry also implies $y > c$.

Step 3: Geometry Rewrite $y = c + c^2/(x - c)$. The asymptotes are $\boxed{x = c}$ and $\boxed{y = c}$, and the positive-input level is the branch with $x > c, y > c$. Substitution back into F returns c .

Question 4

A temperature field is

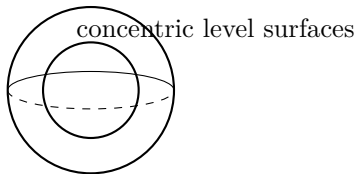
$$T(x, y, z) = 100e^{-(x^2+y^2+z^2)/4}.$$

Tasks

1. Describe the level surface $T = T_0$ for $0 < T_0 \leq 100$.
2. Find the 50-degree level radius.
3. Along $(t, t, 0)$, find when the temperature first reaches 50.

Question 4 – Solution

Strategy Take natural logarithms after isolating the exponential.



Step 1: General level From $T_0/100 = e^{-r^2/4}$,

$$\ln(T_0/100) = -r^2/4 \implies \boxed{r^2 = 4 \ln(100/T_0)}.$$

Thus levels are concentric spheres; $T_0 = 100$ degenerates to the origin.

Step 2: Fifty-degree surface For $T_0 = 50$, $r^2 = 4 \ln 2$, so $\boxed{r = 2\sqrt{\ln 2}}$.

Step 3: Path On $(t, t, 0)$, $r^2 = 2t^2$. Hence

$$2t^2 = 4 \ln 2 \implies t = \pm \sqrt{2 \ln 2}.$$

Starting at $t = 0$ and moving with increasing t , the first time is $\boxed{t = \sqrt{2 \ln 2}}$.

Question 5

Let $f(x, y) = \sqrt{1 - x^2 - y^2}$ and define $g(u, v) = f(u + v, u - v)$.

Tasks

1. Find a simplified formula and exact domain for g .
2. Describe its level curves.
3. Explain the geometric effect of the coordinate change.

Question 5 – Solution

Strategy Substitute first and expand carefully.

Step 1: Simplify Since

$$(u + v)^2 + (u - v)^2 = 2u^2 + 2v^2,$$

we get $\boxed{g(u, v) = \sqrt{1 - 2u^2 - 2v^2}}$.

Step 2: Domain The radicand must be nonnegative:

$$1 - 2u^2 - 2v^2 \geq 0 \implies \boxed{u^2 + v^2 \leq \frac{1}{2}}.$$

Step 3: Levels For $0 \leq c \leq 1$, $g = c$ implies

$$u^2 + v^2 = \frac{1 - c^2}{2},$$

circles centered at the origin. The map $(u, v) \mapsto (x, y) = (u + v, u - v)$ reflects v to $-v$, then rotates by 45° and scales lengths by $\sqrt{2}$. Its inverse image of the unit disk is the disk of radius $1/\sqrt{2}$. At $c = 1$ the level is just the origin; levels outside $[0, 1]$ are empty.

Question 6

Determine the exact range of

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad (x, y) \neq (0, 0).$$

Tasks

1. Prove upper and lower bounds.
2. Show every intermediate value occurs.
3. Describe the level sets $f = c$.

Question 6 – Solution

Strategy Compare $|x^2 - y^2|$ with $x^2 + y^2$, then solve the level equation.

Step 1: Bounds Since $x^2, y^2 \geq 0$,

$$|x^2 - y^2| \leq x^2 + y^2.$$

The denominator is positive off the origin, so $-1 \leq f \leq 1$.

Step 2: Endpoints and intermediate values On the x -axis, $f = 1$; on the y -axis, $f = -1$. For $-1 < c < 1$,

$$(1 - c)x^2 = (1 + c)y^2.$$

Choosing $y = 1$ and $x = \sqrt{(1 + c)/(1 - c)}$ realizes c . Hence

$$\boxed{\text{range}(f) = [-1, 1]}.$$

Step 3: Levels For $-1 < c < 1$, the level set is the pair of lines

$$\boxed{y = \pm \sqrt{\frac{1 - c}{1 + c}} x},$$

with the origin removed. At $c = 1$ and $c = -1$, the levels are the punctured x - and y -axes.

Question 7

A quadratic response has the form

$$q(x, y) = ax^2 + bxy + cy^2 + d.$$

It satisfies $q(1, 0) = 6$, $q(0, 1) = 4$, $q(1, 1) = 15$, and $q(0, 0) = 1$.

Tasks

1. Determine q .
2. Compute $q(2, -1)$.
3. Classify its level curves after removing the cross term by factoring.

Question 7 – Solution

Strategy Substitute the four samples one at a time.

Step 1: Coefficients From $q(0, 0) = 1$, $d = 1$. Then

$$a + d = 6 \implies a = 5, \quad c + d = 4 \implies c = 3.$$

Finally,

$$a + b + c + d = 15 \implies 5 + b + 3 + 1 = 15 \implies b = 6.$$

Thus $\boxed{q(x, y) = 5x^2 + 6xy + 3y^2 + 1}$.

Step 2: Evaluate $q(2, -1) = 20 - 12 + 3 + 1 = \boxed{12}$.

Step 3: Geometry Complete a square:

$$5x^2 + 6xy + 3y^2 = 5 \left(x + \frac{3}{5}y \right)^2 + \frac{6}{5}y^2.$$

Both coefficients are positive, so $q = k > 1$ gives rotated ellipses, $q = 1$ gives only the origin, and $q < 1$ gives no real points.

Question 8

Analyze

$$f(x, y) = \frac{xy}{1 + x^2 + y^2}.$$

Tasks

1. Prove strict bounds for its range.
2. Show every value between the bounds occurs.
3. Explain why the endpoints are approached but never attained.

Question 8 – Solution

Strategy Use $2|xy| \leq x^2 + y^2$ and then restrict to the lines $y = \pm x$.

Step 1: Bound Let $r^2 = x^2 + y^2$. Then

$$|f| \leq \frac{r^2/2}{1 + r^2} < \frac{1}{2}.$$

Thus the range is contained in $(-1/2, 1/2)$.

Step 2: Realize values On $y = x$,

$$f(x, x) = \frac{x^2}{1 + 2x^2},$$

which increases from 0 toward $1/2$. Solving $c = x^2/(1 + 2x^2)$ gives $x^2 = c/(1 - 2c)$ for every $0 < c < 1/2$. The line $y = -x$ similarly realizes every negative value, and $(0, 0)$ realizes 0.

Step 3: Conclusion Therefore

$$\boxed{\text{range}(f) = (-1/2, 1/2)}.$$

The denominator contains the extra positive 1, making equality in the bound impossible for finite inputs; the endpoints occur only as limits.

Question 9

Let

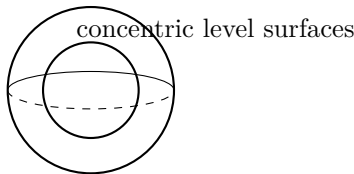
$$F(x, y, z) = \ln(4 - x^2 - y^2 - z^2).$$

Tasks

1. Find its domain and range.
2. Describe every level surface $F = c$.
3. Determine how the level radius changes as c increases.

Question 9 – Solution

Strategy Enforce positivity of the logarithm's argument and solve $F = c$ by exponentiating.



Step 1: Domain We need

$$4 - x^2 - y^2 - z^2 > 0,$$

so the domain is the open ball $\boxed{x^2 + y^2 + z^2 < 4}$.

Step 2: Range The logarithm's argument takes every value in $(0, 4]$. Therefore F takes every value in

$$\boxed{(-\infty, \ln 4]}.$$

The maximum $\ln 4$ occurs at the origin; the function tends to $-\infty$ near the boundary.

Step 3: Levels From $F = c$,

$$4 - r^2 = e^c \implies \boxed{r = \sqrt{4 - e^c}}, \quad c \leq \ln 4.$$

These are concentric spheres, degenerating to the origin at $c = \ln 4$. As c increases, e^c increases and the radius decreases.

Question 10

An open-top box has length x , width y , and height h . A design constraint fixes the sum of these dimensions:

$$x + y + h = 12, \quad x, y, h > 0.$$

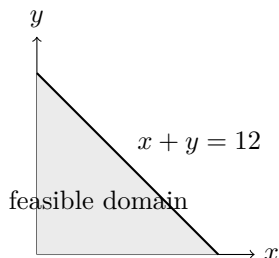
Its volume is $V(x, y) = xy(12 - x - y)$.

Tasks

1. State the natural domain.
2. Find the greatest possible volume without calculus.
3. Prove the maximizing dimensions are unique.

Question 10 – Solution

Strategy Interpret the three positive factors x , y , and $h = 12 - x - y$, then apply AM–GM.



Step 1: Domain Positivity requires

$$\boxed{x > 0, \quad y > 0, \quad x + y < 12}.$$

This is the interior of a triangular region.

Step 2: Bound The three positive numbers x, y, h have fixed sum 12. AM–GM gives

$$\frac{x + y + h}{3} \geq \sqrt[3]{xyh} \implies 4 \geq \sqrt[3]{V}.$$

Cubing yields $V \leq 64$.

Step 3: Equality Equality in AM–GM occurs exactly when

$$x = y = h = 4.$$

Thus the unique maximizing dimensions are $\boxed{x = 4, y = 4, h = 4}$ and

$$\boxed{V_{\max} = 64}.$$

Direct substitution satisfies the material constraint and confirms the volume.

Question 1

Let

$$\mathbf{r}(t) = \left\langle \sqrt{t+2}, \ln(5-t), \frac{1}{t^2-1} \right\rangle.$$

Tasks

1. Determine the exact domain of $\mathbf{r}$.
2. Evaluate $\mathbf{r}(2)$.
3. Explain what fails at every excluded endpoint or isolated value.

Question 1 – Solution

Strategy. A vector function is defined only where all three component functions are defined. We therefore intersect their domains.

Step 1: Square-root condition Real values require

$$t + 2 \geq 0 \implies t \geq -2.$$

Step 2: Logarithm condition The logarithm requires a strictly positive argument:

$$5 - t > 0 \implies t < 5.$$

Step 3: Denominator condition We must have

$$t^2 - 1 \neq 0 \implies t \neq -1, 1.$$

Intersecting the three conditions gives

$$\boxed{\text{Dom}(\mathbf{r}) = [-2, -1) \cup (-1, 1) \cup (1, 5]}.$$

Step 4: Evaluation

$$\boxed{\mathbf{r}(2) = \left\langle 2, \ln 3, \frac{1}{3} \right\rangle}.$$

Verification. At $t = -2$ the square root is zero, so that endpoint is included. At $t = 5$ the logarithm sees zero; below -2 its first component is nonreal; and at $t = \pm 1$ the third component divides by zero.

Question 2

Consider

$$\mathbf{r}(t) = \langle 2t - 1, t^2 + 1, 3t + 4 \rangle, \quad -2 \leq t \leq 2.$$

Tasks

1. Eliminate t and give two Cartesian equations satisfied by the trace.
2. Identify the curve and its containing plane.
3. Find its two endpoints and indicate the direction of travel.

Question 2 – Solution

Strategy. Solve the simplest component for the parameter and substitute it into the other two components.

Step 1: Recover the parameter Since $x = 2t - 1$,

$$t = \frac{x+1}{2}.$$

Step 2: Eliminate t Substitution into $y = t^2 + 1$ gives

$$y = \frac{(x+1)^2}{4} + 1.$$

Also,

$$z = 3\left(\frac{x+1}{2}\right) + 4 = \frac{3}{2}x + \frac{11}{2},$$

or $3x - 2z + 11 = 0$. Hence the trace is the part of the parabola

$$\boxed{y = \frac{(x+1)^2}{4} + 1, \quad 3x - 2z + 11 = 0}$$

for which $-5 \leq x \leq 3$. It lies in the stated plane.

Step 3: Endpoints and orientation

$$\mathbf{r}(-2) = \langle -5, 5, -2 \rangle, \quad \mathbf{r}(2) = \langle 3, 5, 10 \rangle.$$

As t increases, $x = 2t - 1$ increases, so the curve is traversed from the first point to the second.

Verification. Substituting either endpoint into both Cartesian equations confirms both constraints exactly.

Question 3

A particle follows

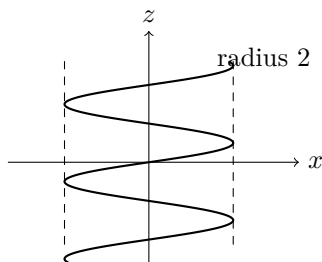
$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t/\pi \rangle, \quad 0 \leq t \leq 4\pi.$$

Without calculus:

1. Describe the trace geometrically and state its axis and radius.
2. Find every point where the particle meets the plane $z = 1$.
3. Determine the vertical rise during one complete revolution and the total number of revolutions.

Question 3 – Solution

Strategy. Use the identity $\cos^2 t + \sin^2 t = 1$ and compare the angular parameter with the height.



Step 1: Cylindrical constraint

$$x^2 + y^2 = 4 \cos^2 t + 4 \sin^2 t = 4.$$

Thus the trace lies on the cylinder of radius 2 about the z -axis. Since $z = t/\pi$ steadily increases while (x, y) rotates counterclockwise, the curve is a circular helix.

Step 2: Plane intersection Setting $z = 1$ gives

$$\frac{t}{\pi} = 1 \implies t = \pi.$$

This value is in the interval, and therefore the only point is

$$\mathbf{r}(\pi) = \langle -2, 0, 1 \rangle.$$

Step 3: Pitch and turns One revolution changes t by 2π , so

$$\Delta z = \frac{t + 2\pi}{\pi} - \frac{t}{\pi} = 2.$$

The interval length 4π contains $4\pi/(2\pi) = 2$ revolutions. Hence the rise per turn is $\boxed{2}$ and the curve makes $\boxed{2}$ complete turns.

Question 4

Two moving objects have position functions

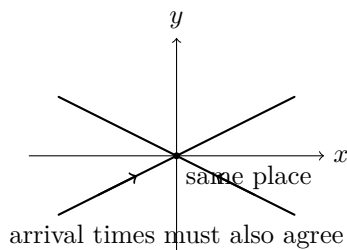
$$\mathbf{r}_1(t) = \langle t, t^2, 1 - t \rangle, \quad \mathbf{r}_2(s) = \langle 2 - s, s, s - 1 \rangle.$$

Tasks

1. Determine whether their geometric paths intersect.
2. If they do, find the parameter value on each path and the intersection point.
3. If both parameters represent the same clock time, decide whether the objects collide.

Question 4 – Solution

Strategy. A path intersection permits different parameters t and s . A collision additionally requires the same time.



Step 1: Match coordinates Equality of the two position vectors requires

$$t = 2 - s, \quad t^2 = s, \quad 1 - t = s - 1.$$

The first and third equations both simplify to

$$t + s = 2.$$

Using $s = t^2$ gives

$$t + t^2 = 2 \implies t^2 + t - 2 = 0 \implies (t - 1)(t + 2) = 0.$$

Step 2: Check both candidates If $t = 1$, then $s = t^2 = 1$, and all equations hold. If $t = -2$, then $s = 4$, but $t + s = 2$ also holds. Thus there are two path intersections:

$$\boxed{\langle 1, 1, 0 \rangle \ (t, s) = (1, 1)}, \quad \boxed{\langle -2, 4, 3 \rangle \ (t, s) = (-2, 4)}.$$

Step 3: Collision test With a common clock, a collision requires $t = s$. Only $(t, s) = (1, 1)$ satisfies that requirement. Therefore the objects collide at

$$\boxed{t = 1 \text{ at } \langle 1, 1, 0 \rangle}.$$

Verification. Direct substitution into both vector functions reproduces each path-intersection point.

Question 5

Construct a vector parametrization of the curve where

$$x^2 + y^2 = 9 \quad \text{and} \quad z = x - 2y.$$

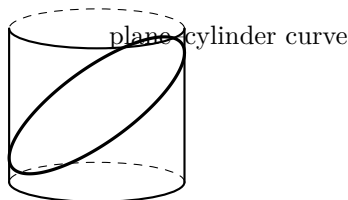
Choose the parametrization so that it begins at $(3, 0, 3)$ and initially moves toward points with $y > 0$. State a parameter interval that traces the curve exactly once.

Tasks

1. Parametrize the cylindrical constraint with the required starting direction.
2. Impose the plane equation to obtain the height component.
3. State and verify an interval that traces the curve exactly once.

Question 5 – Solution

Strategy. Parametrize the cylinder first; then use the plane equation to determine the remaining component.



Step 1: Parametrize the circle The cylinder equation suggests

$$x = 3 \cos t, \quad y = 3 \sin t,$$

because these expressions satisfy $x^2 + y^2 = 9$ identically.

Step 2: Impose the plane Substitute the chosen x and y into $z = x - 2y$:

$$z = 3 \cos t - 6 \sin t.$$

Therefore

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 3 \cos t - 6 \sin t \rangle, \quad 0 \leq t < 2\pi.$$

Step 3: Initial conditions At $t = 0$,

$$\mathbf{r}(0) = \langle 3, 0, 3 \rangle.$$

For small positive t , $\sin t > 0$, so $y = 3 \sin t > 0$; the required initial direction is satisfied.

Verification. Squaring the first two components gives $9(\cos^2 t + \sin^2 t) = 9$, and the third component is exactly the first minus twice the second. The half-open interval avoids repeating the starting point.

Question 6

The vector functions

$$\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t \rangle, \quad 0 \leq t < 2\pi,$$

and

$$\mathbf{q}(u) = \langle \cos u, -\sin u, \cos 2u \rangle, \quad 0 \leq u < 2\pi,$$

are given.

1. Prove that they have the same geometric trace.
2. Compare their orientations starting at $(1, 0, 1)$.
3. Decide whether either parametrization visits a point twice on its stated interval.

Question 6 – Solution

Strategy. Find a parameter substitution connecting the two formulas, then use the first two components to test repeated points.

Step 1: Same trace For $0 < u < 2\pi$, put $t = 2\pi - u$; at $u = 0$, use $t = 0$. For the interior parameters,

$$\cos(2\pi - u) = \cos u, \quad \sin(2\pi - u) = -\sin u,$$

and

$$\cos(2t) = \cos(4\pi - 2u) = \cos 2u.$$

Hence $\mathbf{r}(2\pi - u) = \mathbf{q}(u)$, so every point of one trace occurs on the other. Thus their geometric traces are identical.

Step 2: Orientation Both begin at $(1, 0, 1)$. For small positive t , the second component of $\mathbf{r}(t)$ is positive. For small positive u , the second component of $\mathbf{q}(u)$ is negative. Therefore they traverse the trace in opposite orientations.

Step 3: Repeated-point test Suppose $\mathbf{r}(t_1) = \mathbf{r}(t_2)$. Its first two coordinates give

$$(\cos t_1, \sin t_1) = (\cos t_2, \sin t_2),$$

so $t_1 - t_2$ is an integer multiple of 2π . Since both parameters lie in $[0, 2\pi)$, this forces $t_1 = t_2$. The same argument applies to $\mathbf{q}$.

Thus each parametrization traces the curve exactly once, in opposite directions.

Question 7

Let

$$\mathbf{a}(t) = \langle t, 1 - t, 2t \rangle, \quad \mathbf{b}(t) = \langle 1, t, t^2 \rangle.$$

Tasks

1. Find the common domain of $\mathbf{a}/(t - 2)$ and $\mathbf{b}/(t + 1)$.
2. Compute $2\mathbf{a}(t) - \mathbf{b}(t)$.
3. Find all t for which $\mathbf{a}(t) \cdot \mathbf{b}(t) = 0$.

Question 7 – Solution

Strategy. Treat scalar division and vector algebra componentwise; for orthogonality, set the dot product equal to zero.

Step 1: Common domain The vector functions themselves are polynomial and defined for all real t . The scalar denominators impose

$$t - 2 \neq 0, \quad t + 1 \neq 0.$$

Therefore the common domain is

$$\boxed{\mathbb{R} \setminus \{-1, 2\}}.$$

Step 2: Vector combination

$$\begin{aligned} 2\mathbf{a}(t) - \mathbf{b}(t) &= \langle 2t, 2 - 2t, 4t \rangle - \langle 1, t, t^2 \rangle \\ &= \boxed{\langle 2t - 1, 2 - 3t, 4t - t^2 \rangle}. \end{aligned}$$

Step 3: Orthogonality

$$\begin{aligned} \mathbf{a}(t) \cdot \mathbf{b}(t) &= t(1) + (1 - t)t + (2t)t^2 \\ &= 2t - t^2 + 2t^3 \\ &= t(2t^2 - t + 2). \end{aligned}$$

The quadratic has discriminant $(-1)^2 - 4(2)(2) = -15 < 0$, so it has no real roots. Consequently

$$\boxed{t = 0}$$

is the only real solution. Directly, $\mathbf{a}(0) = \langle 0, 1, 0 \rangle$ and $\mathbf{b}(0) = \langle 1, 0, 0 \rangle$, whose dot product is zero.

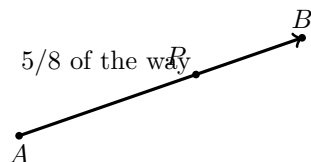
Question 8

Points $A = (-2, 1, 4)$ and $B = (6, -3, 0)$ are endpoints of a directed segment.

1. Construct a vector function $\mathbf{r}(t)$, $0 \leq t \leq 1$, that travels from A to B at a constant parameter rate.
2. Find the point reached when five-eighths of the parameter interval has elapsed.
3. Reparametrize the same directed segment on $-3 \leq u \leq 5$.

Question 8 – Solution

Strategy. A directed segment is an affine blend $(1 - t)A + tB$. A new interval requires a linear map back to $[0, 1]$.



Step 1: Segment formula First compute

$$B - A = \langle 8, -4, -4 \rangle.$$

Thus

$$\boxed{\mathbf{r}(t) = A + t(B - A) = \langle -2 + 8t, 1 - 4t, 4 - 4t \rangle}, \quad 0 \leq t \leq 1.$$

Indeed, $\mathbf{r}(0) = A$ and $\mathbf{r}(1) = B$.

Step 2: Division point At $t = 5/8$,

$$\mathbf{r}\left(\frac{5}{8}\right) = \left\langle -2 + 5, 1 - \frac{5}{2}, 4 - \frac{5}{2} \right\rangle = \boxed{\left\langle 3, -\frac{3}{2}, \frac{3}{2} \right\rangle}.$$

Step 3: New parameter interval The linear map taking $u = -3$ to $t = 0$ and $u = 5$ to $t = 1$ is

$$t = \frac{u + 3}{8}.$$

Substitution gives

$$\boxed{\mathbf{R}(u) = \left\langle u + 1, -\frac{1}{2}u - \frac{1}{2}, \frac{5}{2} - \frac{1}{2}u \right\rangle}, \quad -3 \leq u \leq 5.$$

Checking $u = -3$ and $u = 5$ recovers A and B in the required order.

Question 9

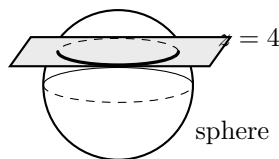
A curve is known to lie on the sphere $x^2 + y^2 + z^2 = 25$ and in the plane $z = 4$. Construct two vector parametrizations that traverse the entire curve exactly once in opposite directions. Choose both to start at $(3, 0, 4)$.

Tasks

1. Identify the intersection curve and its dimensions.
2. Construct the two oppositely oriented parametrizations.
3. Verify the surface constraints, starting point, and one-pass intervals.

Question 9 – Solution

Strategy. Substituting the fixed height into the sphere reveals a circle. The sign of the sine component controls orientation.



Step 1: Identify the circle With $z = 4$, the sphere equation becomes

$$x^2 + y^2 + 16 = 25 \implies x^2 + y^2 = 9.$$

The intersection is a radius-3 circle centered at $(0, 0, 4)$.

Step 2: First orientation A standard counterclockwise parametrization as viewed from the positive z -axis is

$$\mathbf{r}_+(t) = \langle 3 \cos t, 3 \sin t, 4 \rangle, \quad 0 \leq t < 2\pi.$$

It begins at $\mathbf{r}_+(0) = \langle 3, 0, 4 \rangle$ and initially enters the half-space $y > 0$.

Step 3: Opposite orientation Reverse the sign of the sine component:

$$\mathbf{r}_-(t) = \langle 3 \cos t, -3 \sin t, 4 \rangle, \quad 0 \leq t < 2\pi.$$

It has the same starting point but initially enters $y < 0$, so its orientation is opposite.

Verification. For either formula,

$$x^2 + y^2 + z^2 = 9(\cos^2 t + \sin^2 t) + 16 = 25.$$

The half-open interval covers every angular position once without duplicating the start at 2π .

Question 10

Consider the closed space curve

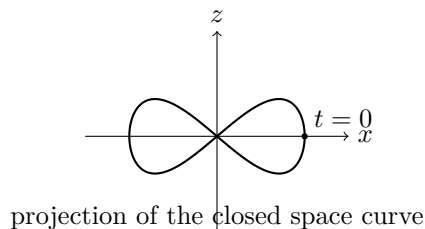
$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, \sin t \cos t \rangle, \quad 0 \leq t < 2\pi.$$

Tasks

1. Find two surfaces whose intersection contains the entire trace.
2. Determine all points where the trace meets the plane $z = 0$.
3. Prove that the parametrization traces each point exactly once, despite the repeated values of the z -component.

Question 10 – Solution

Strategy. Eliminate the parameter using the first two components. For uniqueness, retain the ordered pair $(\cos t, \sin t)$ rather than examining z alone.



Step 1: Eliminate the parameter The first two components give

$$x^2 + y^2 = 4(\cos^2 t + \sin^2 t) = 4.$$

Also,

$$xy = (2 \cos t)(2 \sin t) = 4 \sin t \cos t = 4z,$$

so $z = xy/4$. The trace is therefore contained in

$$\boxed{x^2 + y^2 = 4, \quad z = \frac{xy}{4}}.$$

Conversely, every point of this intersection has $(x/2, y/2)$ on the unit circle and hence equals $(\cos t, \sin t)$ for some t ; thus these equations describe the full trace.

Step 2: Intersections with $z = 0$ Since $z = xy/4$, we need $x = 0$ or $y = 0$. On $x^2 + y^2 = 4$, this yields

$$\boxed{(2, 0, 0), (-2, 0, 0), (0, 2, 0), (0, -2, 0)}.$$

Step 3: One-to-one traversal If $\mathbf{r}(t_1) = \mathbf{r}(t_2)$, then the first two coordinates imply

$$(\cos t_1, \sin t_1) = (\cos t_2, \sin t_2).$$

Thus $t_1 - t_2 = 2\pi k$ for an integer k . Inside $[0, 2\pi)$, only $k = 0$ is possible, so $t_1 = t_2$. Hence no point is repeated.

Question 1

For $t \neq 0$, let

$$\mathbf{r}(t) = \left\langle \frac{\sin 3t}{t}, \frac{e^{2t} - 1}{t}, \frac{1 - \cos t}{t^2} \right\rangle.$$

Tasks

1. Compute $\lim_{t \rightarrow 0} \mathbf{r}(t)$.
2. Define $\mathbf{r}(0)$ so that $\mathbf{r}$ is continuous at 0.
3. Justify why a vector limit exists precisely when all component limits exist.

Question 1 – Solution

Strategy. A vector limit is evaluated component by component. Rewrite each component in terms of a standard scalar limit.

Step 1: First component

$$\frac{\sin 3t}{t} = 3 \frac{\sin 3t}{3t} \longrightarrow 3(1) = 3.$$

Step 2: Second component Put $u = 2t$. Then

$$\frac{e^{2t} - 1}{t} = 2 \frac{e^u - 1}{u} \longrightarrow 2.$$

Step 3: Third component Multiply by the conjugate:

$$\begin{aligned} \frac{1 - \cos t}{t^2} &= \frac{(1 - \cos t)(1 + \cos t)}{t^2(1 + \cos t)} \\ &= \frac{\sin^2 t}{t^2(1 + \cos t)} = \left(\frac{\sin t}{t} \right)^2 \frac{1}{1 + \cos t} \longrightarrow \frac{1}{2}. \end{aligned}$$

Therefore

$$\boxed{\lim_{t \rightarrow 0} \mathbf{r}(t) = \langle 3, 2, \frac{1}{2} \rangle}.$$

Step 4: Continuous extension Define $\boxed{\mathbf{r}(0) = \langle 3, 2, \frac{1}{2} \rangle}$. This makes the function value equal its limit. The vector limit criterion follows because the distance to a candidate vector tends to zero exactly when each coordinate difference tends to zero.

Question 2

Let

$$\mathbf{r}(t) = \langle t^2 - 1, t^3 - 3t, 2t + 1 \rangle.$$

Tasks

1. Find $\mathbf{r}'(t)$ directly from the vector difference quotient.
2. Find the tangent line at the point corresponding to $t = 1$.
3. Verify that your direction vector is nonzero.

Question 2 – Solution

Strategy. Expand $\mathbf{r}(t+h) - \mathbf{r}(t)$ componentwise, divide by h , and then let $h \rightarrow 0$.

Step 1: Difference quotient

$$\begin{aligned}\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} &= \left\langle \frac{(t+h)^2 - t^2}{h}, \frac{(t+h)^3 - 3(t+h) - (t^3 - 3t)}{h}, \frac{2h}{h} \right\rangle \\ &= \langle 2t+h, 3t^2 + 3th + h^2 - 3, 2 \rangle.\end{aligned}$$

Taking $h \rightarrow 0$ yields

$$\boxed{\mathbf{r}'(t) = \langle 2t, 3t^2 - 3, 2 \rangle}.$$

Step 2: Point and direction At $t = 1$,

$$\mathbf{r}(1) = \langle 0, -2, 3 \rangle, \quad \mathbf{r}'(1) = \langle 2, 0, 2 \rangle.$$

The derivative is nonzero, so it supplies a valid tangent direction.

Step 3: Tangent line Using a new line parameter s ,

$$\boxed{\mathbf{L}(s) = \langle 0, -2, 3 \rangle + s \langle 2, 0, 2 \rangle}.$$

Substitution at $s = 0$ confirms the point, and the line direction is parallel to $\mathbf{r}'(1)$.

Question 3

Let

$$\mathbf{u}(t) = \langle t, t^2, 1 \rangle, \quad \mathbf{v}(t) = \langle \cos t, \sin t, t \rangle.$$

Without expanding first, compute at $t = 0$:

Tasks

1. $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}),$
2. $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}),$
3. $\frac{d}{dt}[(t^2 + 1)\mathbf{v}(t)].$

State the differentiation rule used in each part.

Question 3 – Solution

Strategy. Apply the dot-product, cross-product, and scalar–vector product rules before inserting $t = 0$.

At $t = 0$,

$$\mathbf{u} = \langle 0, 0, 1 \rangle, \quad \mathbf{u}' = \langle 1, 0, 0 \rangle, \quad \mathbf{v} = \langle 1, 0, 0 \rangle, \quad \mathbf{v}' = \langle 0, 1, 1 \rangle.$$

Step 1: Dot product rule

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}', \\ \left. \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) \right|_0 &= \langle 1, 0, 0 \rangle \cdot \langle 1, 0, 0 \rangle + \langle 0, 0, 1 \rangle \cdot \langle 0, 1, 1 \rangle = \boxed{2}. \end{aligned}$$

Step 2: Cross product rule Order matters:

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}', \\ \left. \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) \right|_0 &= \mathbf{0} + \langle 0, 0, 1 \rangle \times \langle 0, 1, 1 \rangle = \boxed{\langle -1, 0, 0 \rangle}. \end{aligned}$$

Step 3: Scalar–vector rule With $f(t) = t^2 + 1$,

$$(f\mathbf{v})' = f'\mathbf{v} + f\mathbf{v}' \implies (f\mathbf{v})'(0) = 0\mathbf{v}(0) + 1\mathbf{v}'(0) = \boxed{\langle 0, 1, 1 \rangle}.$$

Question 4

Find the unique vector function $\mathbf{r}(t)$ satisfying

$$\mathbf{r}'(t) = \left\langle 2te^{t^2}, \frac{3}{1+t^2}, 4t^3 - 2t \right\rangle, \quad \mathbf{r}(0) = \langle 1, -2, 5 \rangle.$$

Tasks

1. Recover $\mathbf{r}(t)$ by componentwise antidifferentiation.
2. Use the initial vector to determine every constant.
3. Differentiate your final expression to verify it.

Question 4 – Solution

Strategy. Antidifferentiate each component and then use the initial vector to determine three constants.

Step 1: First component

$$\int 2te^{t^2} dt = e^{t^2} + C_1,$$

using the substitution $u = t^2$.

Step 2: Remaining components

$$\int \frac{3}{1+t^2} dt = 3 \arctan t + C_2,$$

and

$$\int (4t^3 - 2t) dt = t^4 - t^2 + C_3.$$

Thus

$$\mathbf{r}(t) = \left\langle e^{t^2} + C_1, 3 \arctan t + C_2, t^4 - t^2 + C_3 \right\rangle.$$

Step 3: Initial condition At $t = 0$,

$$\mathbf{r}(0) = \langle 1 + C_1, C_2, C_3 \rangle = \langle 1, -2, 5 \rangle.$$

Hence $C_1 = 0$, $C_2 = -2$, and $C_3 = 5$. Therefore

$$\boxed{\mathbf{r}(t) = \left\langle e^{t^2}, 3 \arctan t - 2, t^4 - t^2 + 5 \right\rangle}.$$

Verification. Componentwise differentiation gives $\langle 2te^{t^2}, 3/(1+t^2), 4t^3 - 2t \rangle$, and direct substitution gives the required initial vector. These three scalar initial-value problems each have only one integration constant, so the solution is unique.

Question 5

Evaluate exactly

$$\int_0^{\pi/2} \langle t \cos t, t \sin t, \sin t \cos t \rangle dt.$$

Tasks

1. Evaluate all three components exactly, showing each scalar integration step.
2. State the resulting vector.
3. Verify the third component using a double-angle identity.

Question 5 – Solution

Strategy. Integrate componentwise. The first two components require integration by parts; the third can be checked in two different ways.

Step 1: First component Integration by parts with $u = t$, $dv = \cos t \, dt$ gives

$$\begin{aligned}\int_0^{\pi/2} t \cos t \, dt &= [t \sin t]_0^{\pi/2} - \int_0^{\pi/2} \sin t \, dt \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

Step 2: Second component With $u = t$, $dv = \sin t \, dt$,

$$\int_0^{\pi/2} t \sin t \, dt = [-t \cos t]_0^{\pi/2} + \int_0^{\pi/2} \cos t \, dt = 1.$$

Step 3: Third component Since $d(\sin^2 t)/dt = 2 \sin t \cos t$,

$$\int_0^{\pi/2} \sin t \cos t \, dt = \left[\frac{1}{2} \sin^2 t \right]_0^{\pi/2} = \frac{1}{2}.$$

Therefore

$$\boxed{\int_0^{\pi/2} \langle t \cos t, t \sin t, \sin t \cos t \rangle \, dt = \left\langle \frac{\pi}{2} - 1, 1, \frac{1}{2} \right\rangle}.$$

Using $\sin t \cos t = \frac{1}{2} \sin 2t$ gives the same third component: $[-\frac{1}{4} \cos 2t]_0^{\pi/2} = 1/2$.

Question 6

Suppose a differentiable vector function $\mathbf{r}(t)$ satisfies

$$\|\mathbf{r}(t)\| = 7$$

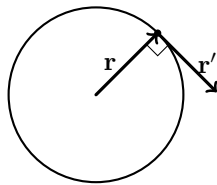
for every t in an interval.

Tasks

1. Prove that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ throughout the interval.
2. Explain the geometric meaning when $\mathbf{r}'(t) \neq \mathbf{0}$.
3. Show that $\mathbf{r} \cdot \mathbf{r}' = 0$ on an interval implies constant length; explain what extra datum fixes that length at 7.

Question 6 – Solution

Strategy. Square the norm to avoid differentiating a square root, then use the dot-product rule.



constant distance $\Rightarrow$ perpendicular change

Step 1: Differentiate the invariant The hypothesis gives

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|^2 = 49.$$

Differentiating both sides,

$$\mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 0.$$

Because the dot product is commutative, the two terms are equal. Hence

$$2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \quad \Rightarrow \quad \boxed{\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0}.$$

Step 2: Geometry The position vector points radially from the origin. When $\mathbf{r}'(t) \neq \mathbf{0}$, it points along the tangent to the curve. Their zero dot product means the tangent is perpendicular to the radius, as expected for motion constrained to a sphere.

Step 3: Converse Now assume $\mathbf{r} \cdot \mathbf{r}' = 0$. Then

$$\frac{d}{dt} \|\mathbf{r}(t)\|^2 = \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r}) = 2\mathbf{r} \cdot \mathbf{r}' = 0.$$

Therefore $\|\mathbf{r}(t)\|^2$ is constant on the interval, so its nonnegative square root $\|\mathbf{r}(t)\|$ is also constant. The constant is 7 if $\|\mathbf{r}(t_0)\| = 7$ at one point of the interval.

Question 7

Define

$$\mathbf{r}(t) = \begin{cases} \langle t, t^2, t^3 \rangle, & t \leq 0, \\ \langle t, at, t^2 \rangle, & t > 0, \end{cases}$$

where a is a real constant.

Tasks

1. Determine all a for which $\mathbf{r}$ is continuous at 0.
2. Determine all a for which $\mathbf{r}$ is differentiable at 0.
3. For the differentiable case, find $\mathbf{r}'(0)$.

Question 7 – Solution

Strategy. Continuity compares one-sided function values; differentiability compares one-sided vector difference quotients.

Step 1: Continuity The defined value from the first branch is

$$\mathbf{r}(0) = \langle 0, 0, 0 \rangle.$$

From the left, $\langle t, t^2, t^3 \rangle \rightarrow \langle 0, 0, 0 \rangle$. From the right,

$$\langle t, at, t^2 \rangle \rightarrow \langle 0, 0, 0 \rangle$$

for every fixed real a . Therefore $\mathbf{r}$ is continuous at 0 for every real value of a .

Step 2: Left derivative As h approaches zero through negative values,

$$\frac{\mathbf{r}(h) - \mathbf{r}(0)}{h} = \langle 1, h, h^2 \rangle \longrightarrow \langle 1, 0, 0 \rangle.$$

Step 3: Right derivative As h approaches zero through positive values,

$$\frac{\mathbf{r}(h) - \mathbf{r}(0)}{h} = \langle 1, a, h \rangle \longrightarrow \langle 1, a, 0 \rangle.$$

The one-sided derivatives agree exactly when $a = 0$. Hence

$$\boxed{a = 0}, \quad \boxed{\mathbf{r}'(0) = \langle 1, 0, 0 \rangle}.$$

This also shows explicitly why continuity alone does not guarantee differentiability.

Question 8

Let

$$\mathbf{F}(t) = \int_1^{t^2} \left\langle e^u, \frac{1}{u}, \cos(\pi u) \right\rangle du, \quad t > 0.$$

Tasks

1. Find $\mathbf{F}'(t)$ without first evaluating the integral.
2. Evaluate $\mathbf{F}'(1)$ and $\mathbf{F}(1)$.
3. Explain the roles of the Fundamental Theorem of Calculus and the chain rule.

Question 8 – Solution

Strategy. Differentiate the vector integral componentwise. The moving upper limit contributes its derivative by the chain rule.

Step 1: General rule If

$$\mathbf{F}(t) = \int_a^{g(t)} \mathbf{f}(u) du,$$

then the Fundamental Theorem gives the derivative with respect to the upper limit, while the chain rule multiplies by $g'(t)$:

$$\mathbf{F}'(t) = \mathbf{f}(g(t))g'(t).$$

Step 2: Apply the rule Here $g(t) = t^2$, so $g'(t) = 2t$. Therefore

$$\begin{aligned}\mathbf{F}'(t) &= 2t \left\langle e^{t^2}, \frac{1}{t^2}, \cos(\pi t^2) \right\rangle \\ &= \left\langle 2te^{t^2}, \frac{2}{t}, 2t \cos(\pi t^2) \right\rangle.\end{aligned}$$

The restriction $t > 0$ ensures the interval between 1 and t^2 never crosses the singularity $u = 0$.

Step 3: Values at $t = 1$

$$\mathbf{F}'(1) = \langle 2e, 2, -2 \rangle.$$

Also, an integral with identical limits is the zero vector:

$$\mathbf{F}(1) = \int_1^1 \mathbf{f}(u) du = \langle 0, 0, 0 \rangle.$$

These results distinguish accumulated value from its instantaneous rate of change.

Question 9

Let

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \quad t \geq 0,$$

and reparametrize it by $t = s^2 + 1$.

Tasks

1. Write the composite vector function $\mathbf{R}(s) = \mathbf{r}(s^2 + 1)$.
2. Compute $\mathbf{R}'(s)$ directly.
3. Compute it again using the vector chain rule and reconcile the fact that $\mathbf{R}'(0) = \mathbf{0}$ while $\mathbf{r}'(1) \neq \mathbf{0}$.

Question 9 – Solution

Strategy. Compare direct componentwise differentiation with $(\mathbf{r} \circ g)' = \mathbf{r}'(g)g'$.

Step 1: Composition

$$\mathbf{R}(s) = \langle s^2 + 1, (s^2 + 1)^2, (s^2 + 1)^3 \rangle.$$

Step 2: Direct differentiation

$$\begin{aligned}\mathbf{R}'(s) &= \langle 2s, 2(s^2 + 1)(2s), 3(s^2 + 1)^2(2s) \rangle \\ &= \langle 2s, 4s(s^2 + 1), 6s(s^2 + 1)^2 \rangle.\end{aligned}$$

Step 3: Chain-rule check Since

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle, \quad g(s) = s^2 + 1,$$

we obtain

$$\begin{aligned}\mathbf{R}'(s) &= \mathbf{r}'(g(s))g'(s) \\ &= \langle 1, 2(s^2 + 1), 3(s^2 + 1)^2 \rangle (2s),\end{aligned}$$

which is identical to the direct result.

At $s = 0$, $g'(0) = 0$, so $\mathbf{R}'(0) = \mathbf{0}$ even though

$$\mathbf{r}'(1) = \langle 1, 2, 3 \rangle \neq \mathbf{0}.$$

There is no contradiction: the new parameter momentarily stops changing the old parameter. The geometric curve is regular at $t = 1$, but this particular parametrization is not regular at $s = 0$.

Question 10

Find every differentiable vector function $\mathbf{r}(t)$ satisfying

$$\mathbf{r}'(t) = 2t\mathbf{r}(t), \quad \mathbf{r}(0) = \langle a, b, c \rangle,$$

where a, b, c are constants. Then prove directly that any two solutions with the same initial vector must be identical.

Tasks

1. Solve the vector initial-value problem component by component.
2. Verify the resulting formula by differentiation and substitution.
3. Prove uniqueness using an integrating factor applied to the difference of two solutions.

Question 10 – Solution

Strategy. The vector equation is three identical scalar equations. Solve them componentwise, then prove uniqueness with an integrating factor.

Step 1: Split into components Write $\mathbf{r} = \langle x, y, z \rangle$. Then

$$x' = 2tx, \quad y' = 2ty, \quad z' = 2tz.$$

For a nonzero component w , separation gives

$$\frac{w'}{w} = 2t \implies \ln |w| = t^2 + C \implies w = Ke^{t^2}.$$

The formula also includes the zero solution when $K = 0$.

Step 2: Apply the initial vector Since $e^0 = 1$, the constants are a, b, c . Thus

$$\boxed{\mathbf{r}(t) = e^{t^2} \langle a, b, c \rangle}.$$

Indeed,

$$\mathbf{r}'(t) = 2te^{t^2} \langle a, b, c \rangle = 2t\mathbf{r}(t),$$

and $\mathbf{r}(0) = \langle a, b, c \rangle$.

Step 3: Direct uniqueness proof Suppose $\mathbf{r}_1$ and $\mathbf{r}_2$ share the initial value, and let $\mathbf{d} = \mathbf{r}_1 - \mathbf{r}_2$. Then

$$\mathbf{d}' = 2t\mathbf{d}, \quad \mathbf{d}(0) = \mathbf{0}.$$

Multiply by e^{-t^2} and use the product rule:

$$\frac{d}{dt}(e^{-t^2} \mathbf{d}(t)) = -2te^{-t^2} \mathbf{d} + e^{-t^2} \mathbf{d}' = \mathbf{0}.$$

Hence $e^{-t^2} \mathbf{d}$ is constant and equals its value $\mathbf{0}$ at $t = 0$. Therefore $\mathbf{d} = \mathbf{0}$, proving $\boxed{\mathbf{r}_1 = \mathbf{r}_2}$.

Question 1

For

$$\mathbf{r}(t) = \langle t^2, 2t, t^2 - 2t \rangle,$$

find the unit tangent vector at $t = 1$ and write the tangent line there.

Tasks

1. Compute the point and velocity vector.
2. Normalize the velocity to obtain $\mathbf{T}(1)$.
3. Write and verify a tangent-line parametrization.

Question 1 – Solution

Strategy. At a regular point, the unit tangent is $\mathbf{T} = \mathbf{r}' / \|\mathbf{r}'\|$.

Step 1: Point and velocity

$$\mathbf{r}(1) = \langle 1, 2, -1 \rangle, \quad \mathbf{r}'(t) = \langle 2t, 2, 2t - 2 \rangle,$$

so $\mathbf{r}'(1) = \langle 2, 2, 0 \rangle \neq \mathbf{0}$.

Step 2: Normalize

$$\|\mathbf{r}'(1)\| = \sqrt{2^2 + 2^2} = 2\sqrt{2}.$$

Therefore

$$\mathbf{T}(1) = \frac{\langle 2, 2, 0 \rangle}{2\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle.$$

Step 3: Tangent line Either $\mathbf{r}'(1)$ or the parallel unit vector may be used:

$$\mathbf{L}(s) = \langle 1, 2, -1 \rangle + s \langle 1, 1, 0 \rangle.$$

At $s = 0$ the line passes through the curve point, and its direction $\langle 1, 1, 0 \rangle$ is one-half of $\mathbf{r}'(1)$.

Verification. The squared length of $\mathbf{T}(1)$ is $1/2 + 1/2 = 1$, as required.

Question 2

For the circular helix

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle,$$

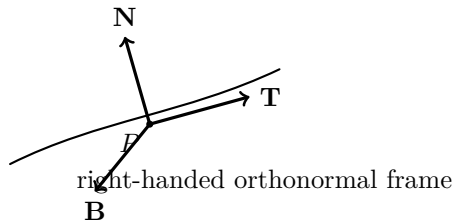
find the complete Frenet frame $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$.

Tasks

1. Normalize $\mathbf{r}'(t)$.
2. Normalize $\mathbf{T}'(t)$.
3. Compute $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ and verify orientation.

Question 2 – Solution

Strategy. The helix has constant speed, which makes each normalization especially transparent.

**Step 1: Unit tangent**

$$\mathbf{r}'(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{9 \sin^2 t + 9 \cos^2 t + 16} = 5.$$

Thus

$$\mathbf{T}(t) = \left\langle -\frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5} \right\rangle.$$

Step 2: Principal unit normal

$$\mathbf{T}'(t) = \left\langle -\frac{3}{5} \cos t, -\frac{3}{5} \sin t, 0 \right\rangle, \quad \|\mathbf{T}'(t)\| = \frac{3}{5}.$$

Therefore

$$\mathbf{N}(t) = \langle -\cos t, -\sin t, 0 \rangle.$$

Step 3: Binormal

$$\begin{aligned} \mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\ &= \left\langle \frac{4}{5} \sin t, -\frac{4}{5} \cos t, \frac{3}{5} \right\rangle. \end{aligned}$$

Its norm is 1, and direct dot products give $\mathbf{T} \cdot \mathbf{N} = \mathbf{T} \cdot \mathbf{B} = \mathbf{N} \cdot \mathbf{B} = 0$. The defining cross product confirms the right-handed orientation.

Question 3

Consider the plane curve

$$\mathbf{r}(t) = \langle t, t^2, 0 \rangle.$$

At $t = 1$, determine $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

Tasks

1. Differentiate and normalize to find $\mathbf{T}(1)$.
2. Differentiate the general $\mathbf{T}(t)$ before evaluating to find $\mathbf{N}(1)$.
3. Compute $\mathbf{B}(1)$ and interpret its direction.

Question 3 – Solution

Strategy. Computing $\mathbf{T}'(1)$ requires differentiating the normalized tangent, not merely normalizing $\mathbf{r}''(1)$.

Step 1: Unit tangent

$$\mathbf{r}'(t) = \langle 1, 2t, 0 \rangle, \quad \mathbf{T}(t) = \frac{\langle 1, 2t, 0 \rangle}{\sqrt{1 + 4t^2}}.$$

Hence

$$\mathbf{T}(1) = \frac{1}{\sqrt{5}} \langle 1, 2, 0 \rangle.$$

Step 2: Differentiate $\mathbf{T}$ Componentwise differentiation gives

$$\mathbf{T}'(t) = \frac{1}{(1 + 4t^2)^{3/2}} \langle -4t, 2, 0 \rangle.$$

At $t = 1$ this vector is a positive multiple of $\langle -2, 1, 0 \rangle$, whose length is $\sqrt{5}$. Therefore

$$\mathbf{N}(1) = \frac{1}{\sqrt{5}} \langle -2, 1, 0 \rangle.$$

Step 3: Binormal

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \frac{1}{5} \langle 1, 2, 0 \rangle \times \langle -2, 1, 0 \rangle = \langle 0, 0, 1 \rangle.$$

The curve lies in the xy -plane, so its binormal points perpendicular to that plane. The three vectors are unit length and mutually perpendicular.

Question 4

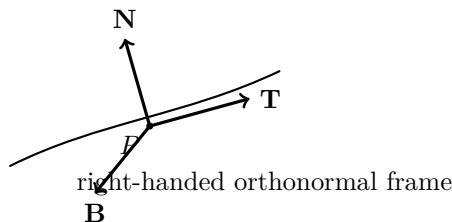
Let $\mathbf{r}$ be a regular curve for which $\mathbf{T}'(t) \neq \mathbf{0}$.

Tasks

1. Prove that $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$.
2. Prove that $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector.
3. Explain why $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is a right-handed orthonormal frame.

Question 4 – Solution

Strategy. Differentiate the invariant $\mathbf{T} \cdot \mathbf{T} = 1$, then use standard cross-product identities.



Step 1: Tangent–normal orthogonality Since $\mathbf{T}$ is a unit vector,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = 1.$$

Differentiate using the dot-product rule:

$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 0 \implies 2\mathbf{T} \cdot \mathbf{T}' = 0.$$

Because $\mathbf{N} = \mathbf{T}' / \|\mathbf{T}'\|$ and the denominator is nonzero,

$$\boxed{\mathbf{T} \cdot \mathbf{N} = 0}.$$

Step 2: Length of B For vectors separated by angle θ ,

$$\|\mathbf{T} \times \mathbf{N}\| = \|\mathbf{T}\| \|\mathbf{N}\| \sin \theta.$$

Here both lengths are 1 and $\theta = \pi/2$, so

$$\boxed{\|\mathbf{B}\| = 1}.$$

The cross product is perpendicular to both factors; hence $\mathbf{B}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

Step 3: Orientation The ordered relation $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ is exactly the right-hand rule. Thus the three mutually perpendicular unit vectors form a right-handed orthonormal frame.

Question 5

The curve

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$$

is reparametrized by $t = 2u + 1$. Let $\mathbf{R}(u) = \mathbf{r}(2u + 1)$.

Tasks

1. Compare the unit tangents of $\mathbf{r}$ at $t = 2u + 1$ and $\mathbf{R}$ at u .
2. Compare their principal normals.
3. State what changes if the reparametrization is instead $t = 1 - 2u$.

Question 5 – Solution

Strategy. Apply the chain rule and keep track of the sign of the parameter derivative during normalization.

Step 1: Increasing reparametrization The chain rule gives

$$\mathbf{R}'(u) = 2\mathbf{r}'(2u + 1).$$

Since the factor 2 is positive,

$$\frac{\mathbf{R}'(u)}{\|\mathbf{R}'(u)\|} = \frac{2\mathbf{r}'(2u + 1)}{2\|\mathbf{r}'(2u + 1)\|} = \boxed{\mathbf{T}_r(2u + 1)}.$$

Thus an increasing reparametrization preserves the unit tangent.

Step 2: Principal normal Differentiating $\mathbf{T}_R(u) = \mathbf{T}_r(2u + 1)$ gives

$$\mathbf{T}'_R(u) = 2\mathbf{T}'_r(2u + 1).$$

Again the positive factor cancels upon normalization, so

$$\boxed{\mathbf{N}_R(u) = \mathbf{N}_r(2u + 1)}.$$

Step 3: Reversed parameter If $t = 1 - 2u$, then $dt/du = -2$. Consequently

$$\boxed{\mathbf{T}_R = -\mathbf{T}_r},$$

at corresponding points. Differentiating this relation with respect to u introduces a second negative factor, so $\mathbf{T}'_R = 2\mathbf{T}'_r$ and

$$\boxed{\mathbf{N}_R = \mathbf{N}_r}.$$

Therefore $\mathbf{B}_R = \mathbf{T}_R \times \mathbf{N}_R = -\mathbf{B}_r$. Reversing orientation changes $\mathbf{T}$ and $\mathbf{B}$, but not the principal normal.

Question 6

At a regular point of a curve, suppose

$$\mathbf{r}'(t_0) = \langle 1, 2, 2 \rangle, \quad \mathbf{r}''(t_0) = \langle 2, 0, 1 \rangle.$$

Tasks

1. Find the unit tangent $\mathbf{T}(t_0)$.
2. Find the binormal using $\mathbf{r}' \times \mathbf{r}''$.
3. Obtain the principal normal from $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and verify the frame.

Question 6 – Solution

Strategy. The cross product $\mathbf{r}' \times \mathbf{r}''$ gives the binormal direction without differentiating a normalized vector.

Step 1: Tangent

$$\|\mathbf{r}'(t_0)\| = \sqrt{1+4+4} = 3, \quad \boxed{\mathbf{T}(t_0) = \frac{1}{3} \langle 1, 2, 2 \rangle}.$$

Step 2: Binormal direction

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \langle 2, 3, -4 \rangle.$$

Its length is $\sqrt{4+9+16} = \sqrt{29}$, so

$$\boxed{\mathbf{B}(t_0) = \frac{1}{\sqrt{29}} \langle 2, 3, -4 \rangle}.$$

Step 3: Principal normal In a right-handed frame, $\mathbf{B} \times \mathbf{T} = \mathbf{N}$. Thus

$$\begin{aligned} \mathbf{N}(t_0) &= \frac{1}{3\sqrt{29}} \langle 2, 3, -4 \rangle \times \langle 1, 2, 2 \rangle \\ &= \boxed{\frac{1}{3\sqrt{29}} \langle 14, -8, 1 \rangle}. \end{aligned}$$

Its squared numerator length is $196 + 64 + 1 = 261 = 9(29)$, so it has unit length. Direct dot products with $\mathbf{T}$ and $\mathbf{B}$ are zero, verifying the frame.

Question 7

For

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle,$$

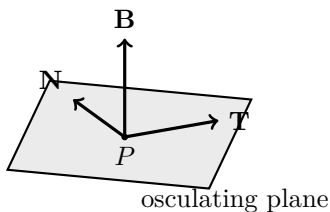
find equations of the osculating plane and normal plane at $t = 1$.

Tasks

1. Compute suitable directions for $\mathbf{T}$ and $\mathbf{B}$.
2. Use $\mathbf{B}$ as the osculating-plane normal.
3. Use $\mathbf{T}$ as the normal-plane normal and verify both planes pass through the curve point.

Question 7 – Solution

Strategy. The osculating plane is spanned by $\mathbf{T}$ and $\mathbf{N}$, so its normal is $\mathbf{B}$. The normal plane is perpendicular to $\mathbf{T}$.

**Step 1: Point and tangent direction**

$$P = \mathbf{r}(1) = (1, 1, 1), \quad \mathbf{r}'(1) = \langle 1, 2, 3 \rangle.$$

We may use $\langle 1, 2, 3 \rangle$ as the normal vector of the normal plane.

Step 2: Binormal direction

$$\mathbf{r}''(t) = \langle 0, 2, 6t \rangle, \quad \mathbf{r}''(1) = \langle 0, 2, 6 \rangle.$$

Then

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 3 \rangle \times \langle 0, 2, 6 \rangle = \langle 6, -6, 2 \rangle = 2 \langle 3, -3, 1 \rangle.$$

Hence an osculating-plane normal is $\langle 3, -3, 1 \rangle$.

Step 3: Plane equations Through $P = (1, 1, 1)$, the osculating plane is

$$3(x - 1) - 3(y - 1) + (z - 1) = 0 \implies \boxed{3x - 3y + z = 1}.$$

The normal plane is

$$(x - 1) + 2(y - 1) + 3(z - 1) = 0 \implies \boxed{x + 2y + 3z = 6}.$$

Substitution of $(1, 1, 1)$ verifies both equations, and their normals have the required Frenet directions.

Question 8

Consider

$$\mathbf{r}(t) = \langle t, t^3, 0 \rangle.$$

Tasks

1. Find $\mathbf{T}(0)$.
2. Determine whether the principal normal and binormal are defined at $t = 0$ by the standard Frenet formulas.
3. Compute the one-sided limiting normal directions and explain the geometric issue.

Question 8 – Solution

Strategy. Regularity of $\mathbf{r}$ guarantees $\mathbf{T}$, but defining $\mathbf{N}$ additionally requires $\mathbf{T}' \neq \mathbf{0}$.

Step 1: Unit tangent

$$\mathbf{r}'(t) = \langle 1, 3t^2, 0 \rangle, \quad \mathbf{T}(t) = \frac{\langle 1, 3t^2, 0 \rangle}{\sqrt{1 + 9t^4}}.$$

Thus $\boxed{\mathbf{T}(0) = \langle 1, 0, 0 \rangle}$.

Step 2: Differentiate $\mathbf{T}$ Differentiation and simplification give

$$\mathbf{T}'(t) = \frac{1}{(1 + 9t^4)^{3/2}} \langle -18t^3, 6t, 0 \rangle = \frac{6t}{(1 + 9t^4)^{3/2}} \langle -3t^2, 1, 0 \rangle.$$

Therefore $\mathbf{T}'(0) = \mathbf{0}$. The expression

$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|}$$

would divide by zero, so $\boxed{\mathbf{N}(0) \text{ and } \mathbf{B}(0) \text{ are not defined by the Frenet formulas}}$.

Step 3: One-sided behavior For $t > 0$, normalization yields a vector tending to $\langle 0, 1, 0 \rangle$. For $t < 0$, the scalar factor $6t$ reverses direction, giving the limit $\langle 0, -1, 0 \rangle$. The unequal limits show that the preferred bending direction flips at the inflection point. Although the curve has a tangent there, it has no single principal-normal direction.

Question 9

Let

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle.$$

Find all $t \in [0, 2\pi)$ for which the unit tangent is perpendicular to the fixed vector $\mathbf{a} = \langle 1, 1, 0 \rangle$, and find the corresponding points.

Tasks

1. Explain why it is enough to test $\mathbf{r}'(t) \cdot \mathbf{a} = 0$.
2. Solve the resulting trigonometric equation exactly.
3. Evaluate the curve at every admissible parameter.

Question 9 – Solution

Strategy. Normalizing a nonzero vector multiplies it by a positive scalar, which does not change whether its dot product with $\mathbf{a}$ is zero.

Step 1: Tangent direction

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 1 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{5}.$$

Thus $\mathbf{T} \cdot \mathbf{a} = 0$ exactly when $\mathbf{r}' \cdot \mathbf{a} = 0$.

Step 2: Solve

$$\begin{aligned} \mathbf{r}'(t) \cdot \mathbf{a} &= -2 \sin t + 2 \cos t = 0 \\ \implies \sin t &= \cos t \\ \implies \tan t &= 1. \end{aligned}$$

On $[0, 2\pi)$,

$$t = \frac{\pi}{4}, \frac{5\pi}{4}.$$

Neither solution is lost by using tangent because $\cos t \neq 0$ at both values.

Step 3: Points

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left\langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4} \right\rangle,$$

and

$$\mathbf{r}\left(\frac{5\pi}{4}\right) = \left\langle -\sqrt{2}, -\sqrt{2}, \frac{5\pi}{4} \right\rangle.$$

At either parameter, substituting into $-2 \sin t + 2 \cos t$ gives zero, verifying perpendicularity.

Question 10

At a point $P = (1, -1, 2)$ on a regular curve, a unit tangent and a candidate principal normal are

$$\mathbf{T} = \frac{1}{3} \langle 2, 1, 2 \rangle, \quad \mathbf{N} = \frac{1}{3\sqrt{5}} \langle -4, 7, \frac{1}{2} \rangle.$$

Tasks

1. Determine whether the supplied vectors can belong to a Frenet frame, and diagnose any defect.
2. Replace $\mathbf{N}$ by the unit vector in the direction of the component of $\langle -4, 7, 1 \rangle$ perpendicular to $\mathbf{T}$.
3. Find the corrected $\mathbf{B}$ and the osculating plane at P .

Question 10 – Solution

Strategy. First audit unit length and orthogonality. Then use vector projection to remove the tangential component from the proposed bending direction.

Step 1: Diagnose the supplied normal The tangent is unit because $(4 + 1 + 4)/9 = 1$. But

$$\mathbf{T} \cdot \mathbf{N} = \frac{2(-4) + 1(7) + 2(1/2)}{9\sqrt{5}} = 0,$$

while

$$\|\mathbf{N}\|^2 = \frac{16 + 49 + 1/4}{45} = \frac{29}{20} \neq 1.$$

Thus it is perpendicular to $\mathbf{T}$ but is not a unit vector, so the data are not a Frenet frame.

Step 2: Correct the bending direction Let $\mathbf{w} = \langle -4, 7, 1 \rangle$. Its tangential scalar component is

$$\mathbf{w} \cdot \mathbf{T} = \frac{-8 + 7 + 2}{3} = \frac{1}{3}.$$

Therefore

$$\mathbf{w}_{\perp} = \mathbf{w} - (\mathbf{w} \cdot \mathbf{T})\mathbf{T} = \langle -4, 7, 1 \rangle - \frac{1}{9} \langle 2, 1, 2 \rangle = \frac{1}{9} \langle -38, 62, 7 \rangle.$$

Since $\sqrt{(-38)^2 + 62^2 + 7^2} = \sqrt{5337} = 3\sqrt{593}$,

$$\hat{\mathbf{N}} = \frac{1}{3\sqrt{593}} \langle -38, 62, 7 \rangle.$$

Step 3: Binormal and osculating plane

$$\mathbf{B} = \mathbf{T} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{593}} \langle -13, -10, 18 \rangle.$$

The numerator has length $\sqrt{169 + 100 + 324} = \sqrt{593}$. Since $\mathbf{B}$ is normal to the osculating plane through P ,

$$-13(x - 1) - 10(y + 1) + 18(z - 2) = 0,$$

or

$$13x + 10y - 18z + 33 = 0.$$

Substitution of P gives zero, and the construction guarantees a right-handed orthonormal frame.

Question 1

Find the exact length of

$$\mathbf{r}(t) = \left\langle 3t, 4t, \frac{1}{2}t^2 \right\rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Compute the speed and simplify it completely.
2. Set up and evaluate the arc-length integral.
3. Check that the result exceeds the straight-line distance between the endpoints.

Question 1 – Solution

Strategy. Arc length is the integral of speed: $L = \int_a^b \|\mathbf{r}'(t)\| dt$.

Step 1: Velocity and speed

$$\mathbf{r}'(t) = \langle 3, 4, t \rangle.$$

Therefore

$$\|\mathbf{r}'(t)\| = \sqrt{3^2 + 4^2 + t^2} = \sqrt{25 + t^2}.$$

Step 2: Integrate Use

$$\int \sqrt{t^2 + a^2} dt = \frac{1}{2} \left(t\sqrt{t^2 + a^2} + a^2 \ln \left| t + \sqrt{t^2 + a^2} \right| \right).$$

With $a = 5$,

$$\begin{aligned} L &= \int_0^2 \sqrt{25 + t^2} dt \\ &= \frac{1}{2} \left[t\sqrt{25 + t^2} + 25 \ln \left(t + \sqrt{25 + t^2} \right) \right]_0^2 \\ &= \boxed{\sqrt{29} + \frac{25}{2} \ln \left(\frac{2 + \sqrt{29}}{5} \right)}. \end{aligned}$$

Step 3: Endpoint check The displacement is

$$\mathbf{r}(2) - \mathbf{r}(0) = \langle 6, 8, 2 \rangle,$$

whose length is $\sqrt{36 + 64 + 4} = 2\sqrt{26} \approx 10.198$. The arc length is approximately 10.261, which is slightly larger, as a nonstraight path requires.

Question 2

A helix is parametrized by

$$\mathbf{r}(t) = \langle 5 \cos t, 5 \sin t, 12t \rangle, \quad 0 \leq t \leq 3\pi.$$

Tasks

1. Find its exact length.
2. Determine the length traveled during one complete revolution.
3. Explain why equal parameter intervals produce equal arc lengths here.

Question 2 – Solution

Strategy. Differentiate and look for constant speed.

Step 1: Speed

$$\mathbf{r}'(t) = \langle -5 \sin t, 5 \cos t, 12 \rangle.$$

Hence

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{25 \sin^2 t + 25 \cos^2 t + 144} \\ &= \sqrt{25 + 144} = 13.\end{aligned}$$

Step 2: Total length

$$L = \int_0^{3\pi} 13 \, dt = 13[t]_0^{3\pi} = \boxed{39\pi}.$$

Step 3: One revolution A complete revolution changes t by 2π . Therefore

$$L_{\text{rev}} = \int_t^{t+2\pi} 13 \, du = \boxed{26\pi}.$$

Because the speed is the same constant 13 at every parameter value, an interval of width Δt always contributes length $13\Delta t$. Thus equal parameter intervals have equal arc lengths.

Verification. The interval $[0, 3\pi]$ contains $3/2$ revolutions, and $(3/2)(26\pi) = 39\pi$, confirming the total.

Question 3

Determine the exact length of

$$\mathbf{r}(t) = \left\langle t - \frac{t^3}{3}, t^2, t + \frac{t^3}{3} \right\rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Compute the speed without prematurely expanding every square.
2. Identify and justify the perfect-square simplification.
3. Evaluate the length and verify the speed is never negative.

Question 3 – Solution

Strategy. Differentiate first and group symmetric components; the radicals collapse to a polynomial.

Step 1: Differentiate

$$\mathbf{r}'(t) = \langle 1 - t^2, 2t, 1 + t^2 \rangle.$$

Thus

$$\begin{aligned}\|\mathbf{r}'(t)\|^2 &= (1 - t^2)^2 + (2t)^2 + (1 + t^2)^2 \\ &= (1 - 2t^2 + t^4) + 4t^2 + (1 + 2t^2 + t^4) \\ &= 2 + 4t^2 + 2t^4 \\ &= 2(1 + t^2)^2.\end{aligned}$$

Step 2: Take the square root

$$\|\mathbf{r}'(t)\| = \sqrt{2}|1 + t^2|.$$

Since $1 + t^2 > 0$ for every real t , this becomes

$$\|\mathbf{r}'(t)\| = \sqrt{2}(1 + t^2).$$

Step 3: Integrate

$$\begin{aligned}L &= \sqrt{2} \int_0^2 (1 + t^2) dt \\ &= \sqrt{2} \left[t + \frac{t^3}{3} \right]_0^2 \\ &= \boxed{\frac{14\sqrt{2}}{3}}.\end{aligned}$$

The speed is strictly positive because $1 + t^2 > 0$, so the curve is regular throughout the interval.

Question 4

For

$$\mathbf{r}(t) = \langle 3t, 4t, 0 \rangle, \quad t \geq 1,$$

let $s(t)$ denote distance traveled from the starting parameter $t = 1$.

Tasks

1. Find $s(t)$ and its inverse $t(s)$.
2. Reparametrize the curve by arc length measured from $t = 1$.
3. Verify directly that the new parametrization has unit speed.

Question 4 – Solution

Strategy. Integrate speed from the specified base parameter, invert the resulting monotone function, and substitute.

Step 1: Arc-length function

$$\mathbf{r}'(t) = \langle 3, 4, 0 \rangle, \quad \|\mathbf{r}'(t)\| = 5.$$

Therefore

$$s(t) = \int_1^t 5 \, du = 5(t - 1).$$

Since $t \geq 1$, we have $s \geq 0$, and solving for t gives

$$\boxed{t(s) = 1 + \frac{s}{5}}.$$

Step 2: Substitute

$$\begin{aligned} \mathbf{R}(s) &= \mathbf{r}\left(1 + \frac{s}{5}\right) \\ &= \boxed{\left\langle 3 + \frac{3}{5}s, 4 + \frac{4}{5}s, 0 \right\rangle}, \quad s \geq 0. \end{aligned}$$

Step 3: Unit-speed verification

$$\mathbf{R}'(s) = \left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle,$$

so

$$\|\mathbf{R}'(s)\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1.$$

Thus an increase of one unit in s corresponds to exactly one unit of distance along the curve.

Question 5

The upper semicircle $x^2 + y^2 = 4$ is represented by

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 0 \rangle, \quad 0 \leq t \leq \pi,$$

and also by

$$\mathbf{R}(u) = \langle 2 \cos(u^3), 2 \sin(u^3), 0 \rangle, \quad 0 \leq u \leq \sqrt[3]{\pi}.$$

Tasks

1. Compute the length using each parametrization.
2. Explain why the nonconstant speed of $\mathbf{R}$ does not change the result.
3. Identify the substitution connecting the two integrals.

Question 5 – Solution

Strategy. Calculate both speed integrals, taking care that $u \geq 0$ when simplifying an absolute value.

Step 1: Standard parametrization

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 0 \rangle, \quad \|\mathbf{r}'(t)\| = 2.$$

Hence

$$L_r = \int_0^\pi 2 \, dt = \boxed{2\pi}.$$

Step 2: Cubic reparametrization

$$\mathbf{R}'(u) = \langle -6u^2 \sin(u^3), 6u^2 \cos(u^3), 0 \rangle.$$

Because $u \geq 0$,

$$\|\mathbf{R}'(u)\| = 6u^2.$$

Therefore

$$L_R = \int_0^{\sqrt[3]{\pi}} 6u^2 \, du = [2u^3]_0^{\sqrt[3]{\pi}} = \boxed{2\pi}.$$

Step 3: Invariance The substitution $t = u^3$ gives $dt = 3u^2 du$, so

$$2 \, dt = 6u^2 \, du.$$

The second parametrization changes the rate at which points are visited, but it is increasing and covers the same semicircle once. The speed factor exactly compensates for the parameter distortion.

Question 6

A path travels from $A = (-2, 0, 0)$ to $C = (0, 2, 0)$ and then to $B = (2, 0, 0)$ according to

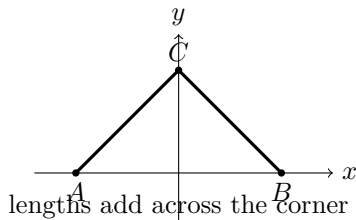
$$\mathbf{r}(t) = \begin{cases} \langle t, t+2, 0 \rangle, & -2 \leq t \leq 0, \\ \langle t, 2-t, 0 \rangle, & 0 < t \leq 2. \end{cases}$$

Tasks

1. Verify continuity at the joining parameter.
2. Compute the total arc length by splitting the integral.
3. Compare it with $\|B - A\|$ and explain the difference.

Question 6 – Solution

Strategy. Arc length is additive across a corner, even though the derivative changes abruptly there.



Step 1: Join check Both formulas give

$$\langle 0, 2, 0 \rangle$$

at $t = 0$ (the second by its right-hand limit), so the path is continuous at C .

Step 2: Length of each segment On the first branch,

$$\mathbf{r}'(t) = \langle 1, 1, 0 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{2}.$$

On the second branch,

$$\mathbf{r}'(t) = \langle 1, -1, 0 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{2}.$$

Therefore

$$\begin{aligned} L &= \int_{-2}^0 \sqrt{2} \, dt + \int_0^2 \sqrt{2} \, dt \\ &= 2\sqrt{2} + 2\sqrt{2} = \boxed{4\sqrt{2}}. \end{aligned}$$

Step 3: Chord comparison The endpoint distance is

$$\|B - A\| = \|\langle 4, 0, 0 \rangle\| = 4.$$

Since $4\sqrt{2} > 4$, the broken path is longer. Equality would require travel directly along the segment from A to B without the detour through C .

Question 7

For

$$\mathbf{r}(t) = \left\langle t^2, \frac{1}{6}(4t+1)^{3/2}, 0 \right\rangle, \quad t \geq 0,$$

find the parameter value $T > 0$ at which the distance traveled from $t = 0$ first equals $35/4$.

Tasks

1. Derive the arc-length function $s(T)$.
2. Solve the resulting equation exactly.
3. Explain why the solution is unique.

Question 7 – Solution

Strategy. Differentiate carefully; the squared speed is a perfect square on the stated domain.

Step 1: Speed

$$\mathbf{r}'(t) = \langle 2t, \sqrt{4t+1}, 0 \rangle.$$

For $t \geq 0$,

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{4t^2 + 4t + 1} \\ &= \sqrt{(2t+1)^2} = |2t+1| = 2t+1.\end{aligned}$$

Step 2: Arc-length function

$$\begin{aligned}s(T) &= \int_0^T (2t+1) dt \\ &= [t^2 + t]_0^T = T^2 + T.\end{aligned}$$

Set this equal to the prescribed distance:

$$T^2 + T = \frac{35}{4}.$$

Multiplying by 4 and factoring,

$$4T^2 + 4T - 35 = (2T-5)(2T+7) = 0.$$

The candidates are $T = 5/2$ and $T = -7/2$; only the first lies in the domain. Hence

$$\boxed{T = \frac{5}{2}}.$$

Step 3: Uniqueness For $T > 0$, $s'(T) = 2T + 1 > 0$, so s is strictly increasing. Therefore the displayed equation has at most one positive solution. Since $s(0) = 0$ and $s(T) \rightarrow \infty$, it has exactly one. Direct substitution gives $s(5/2) = 25/4 + 10/4 = 35/4$, verifying the result.

Question 8

Find the exact length of the semicubical parabola

$$\mathbf{r}(t) = \langle t^2, t^3, 0 \rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Factor the speed using the sign of t .
2. Evaluate the arc-length integral by substitution.
3. Explain why the zero velocity at $t = 0$ does not make the length integral improper.

Question 8 – Solution

Strategy. Factor t^2 from the squared speed and use $t \geq 0$ before integrating.

Step 1: Speed

$$\mathbf{r}'(t) = \langle 2t, 3t^2, 0 \rangle.$$

Hence

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{4t^2 + 9t^4} \\ &= \sqrt{t^2(4 + 9t^2)} \\ &= |t|\sqrt{4 + 9t^2}.\end{aligned}$$

On $[0, 2]$, $|t| = t$, so the speed is $t\sqrt{4 + 9t^2}$.

Step 2: Integrate Let $u = 4 + 9t^2$, so $du = 18t \, dt$:

$$\begin{aligned}L &= \int_0^2 t\sqrt{4 + 9t^2} \, dt \\ &= \frac{1}{18} \int_4^{40} u^{1/2} \, du \\ &= \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_4^{40} \\ &= \frac{1}{27} (40^{3/2} - 4^{3/2}) \\ &= \boxed{\frac{80\sqrt{10} - 8}{27}}.\end{aligned}$$

Step 3: Endpoint regularity Although $\mathbf{r}'(0) = \mathbf{0}$, the speed $t\sqrt{4 + 9t^2}$ is continuous at 0 and equals zero there. A continuous integrand on a closed interval defines an ordinary proper integral; no singularity is present.

Question 9

A wire is wound exactly N times around a cylinder of radius R , rising uniformly through total height H . Model its centerline by

$$\mathbf{r}(t) = \left\langle R \cos t, R \sin t, \frac{H}{2\pi N} t \right\rangle, \quad 0 \leq t \leq 2\pi N.$$

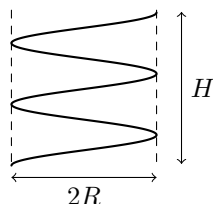
Tasks

1. Derive a formula for the wire length in terms of R, H, N .
2. For $R = 3$, $H = 24$, and length 30π , determine $N > 0$.
3. Verify whether the resulting value represents a whole number of turns.

Question 9 – Solution

Strategy. The model has constant speed; after integration, solve the design equation by squaring positive quantities.

wire wound around a cylinder



Step 1: General formula

$$\mathbf{r}'(t) = \left\langle -R \sin t, R \cos t, \frac{H}{2\pi N} \right\rangle,$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{R^2 + \left(\frac{H}{2\pi N}\right)^2}.$$

Multiplying this constant speed by the parameter length $2\pi N$ gives

$$L = \sqrt{(2\pi NR)^2 + H^2}.$$

Step 2: Apply the data With $R = 3$, $H = 24$, and $L = 30\pi$,

$$(30\pi)^2 = (6\pi N)^2 + 24^2.$$

Therefore

$$\begin{aligned} 36\pi^2 N^2 &= 900\pi^2 - 576, \\ N^2 &= 25 - \frac{16}{\pi^2}. \end{aligned}$$

Since $N > 0$,

$$N = \sqrt{25 - \frac{16}{\pi^2}}.$$

Step 3: Feasibility Numerically, $N \approx 4.835$, which is not an integer. Thus the given continuous model has a positive solution, but the specifications are incompatible with a whole number of complete turns. This is a design-consistency conclusion, not an algebraic failure.

Question 10

Let $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ be continuously differentiable.

Tasks

1. Prove the chord–arc inequality

$$\|\mathbf{r}(b) - \mathbf{r}(a)\| \leq \int_a^b \|\mathbf{r}'(t)\| dt.$$

2. Give a clear geometric interpretation.
3. Show that equality holds if $\mathbf{r}'(t) = \lambda(t)\mathbf{u}$, where $\mathbf{u}$ is fixed unit vector and $\lambda(t) \geq 0$.

Question 10 – Solution

Strategy. Express displacement as the integral of the velocity vector and apply a dot-product estimate in the displacement direction.

Step 1: Zero-displacement case If $\mathbf{r}(b) = \mathbf{r}(a)$, the left side is zero and the inequality follows because the integral of speed is nonnegative.

Now suppose $\mathbf{d} = \mathbf{r}(b) - \mathbf{r}(a) \neq \mathbf{0}$ and define the unit vector $\mathbf{u} = \mathbf{d}/\|\mathbf{d}\|$.

Step 2: Use the Fundamental Theorem

$$\mathbf{d} = \int_a^b \mathbf{r}'(t) dt.$$

Dot both sides with $\mathbf{u}$:

$$\begin{aligned}\|\mathbf{d}\| &= \mathbf{d} \cdot \mathbf{u} \\ &= \int_a^b \mathbf{r}'(t) \cdot \mathbf{u} dt \\ &\leq \int_a^b |\mathbf{r}'(t) \cdot \mathbf{u}| dt \\ &\leq \int_a^b \|\mathbf{r}'(t)\| \|\mathbf{u}\| dt \\ &= \int_a^b \|\mathbf{r}'(t)\| dt.\end{aligned}$$

Therefore

$$\|\mathbf{r}(b) - \mathbf{r}(a)\| \leq \int_a^b \|\mathbf{r}'(t)\| dt.$$

Step 3: Geometry and equality The straight chord is the shortest connection between the endpoints; following a curved or reversing path cannot be shorter. If $\mathbf{r}' = \lambda \mathbf{u}$ with $\lambda \geq 0$, then

$$\mathbf{r}(b) - \mathbf{r}(a) = \mathbf{u} \int_a^b \lambda(t) dt,$$

so its norm is $\int_a^b \lambda(t) dt$. Meanwhile $\|\mathbf{r}'\| = \lambda$, giving equality. The condition describes travel in one fixed direction without backtracking.

Question 1

For the plane curve $\mathbf{r}(t) = \langle t, t^2, 0 \rangle$, find the curvature at $t = 1$.

Tasks

1. Compute $\mathbf{r}'$ and $\mathbf{r}''$.
2. Use $\kappa = \|\mathbf{r}' \times \mathbf{r}''\| / \|\mathbf{r}'\|^3$.
3. State the radius of curvature at the point.

Question 1 – Solution

Strategy. The cross-product formula avoids differentiating the unit tangent.

Step 1: Derivatives

$$\mathbf{r}'(t) = \langle 1, 2t, 0 \rangle, \quad \mathbf{r}''(t) = \langle 0, 2, 0 \rangle.$$

At $t = 1$, $\mathbf{r}'(1) = \langle 1, 2, 0 \rangle$ and $\|\mathbf{r}'(1)\| = \sqrt{5}$.

Step 2: Cross product

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 0 \rangle \times \langle 0, 2, 0 \rangle = \langle 0, 0, 2 \rangle,$$

so its magnitude is 2. Therefore

$$\kappa(1) = \frac{2}{(\sqrt{5})^3} = \frac{2}{5\sqrt{5}}.$$

Step 3: Radius of curvature Since $\rho = 1/\kappa$,

$$\rho(1) = \frac{5\sqrt{5}}{2}.$$

Both values are positive, and their product is 1, verifying the reciprocal relationship.

Question 2

Find the curvature of the circular helix

$$\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle, \quad a > 0,$$

where b is constant.

Tasks

1. Derive $\kappa(t)$ symbolically.
2. Explain why it is constant.
3. Check the special cases $b = 0$ and increasing $|b|$.

Question 2 – Solution

Strategy. Compute the speed and cross product in symbolic form before simplifying.

Step 1: Derivatives and speed

$$\mathbf{r}' = \langle -a \sin t, a \cos t, b \rangle, \quad \mathbf{r}'' = \langle -a \cos t, -a \sin t, 0 \rangle,$$

and

$$\|\mathbf{r}'\| = \sqrt{a^2 + b^2}.$$

Step 2: Cross product

$$\mathbf{r}' \times \mathbf{r}'' = \langle ab \sin t, -ab \cos t, a^2 \rangle.$$

Its magnitude is

$$\sqrt{a^2 b^2 (\sin^2 t + \cos^2 t) + a^4} = a \sqrt{a^2 + b^2}.$$

Thus

$$\kappa(t) = \frac{a \sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}.$$

Step 3: Interpretation No t remains, so curvature is constant. When $b = 0$, the helix becomes a circle of radius a and $\kappa = 1/a$. As $|b|$ grows, $a/(a^2 + b^2)$ decreases: the helix stretches vertically and bends less sharply.

Question 3

For the graph $y = \ln x$, $x > 0$, determine where its curvature is greatest.

Tasks

1. Derive $\kappa(x)$ using the graph formula.
2. Maximize it exactly.
3. Give the point and maximum curvature.

Question 3 – Solution

Strategy. Use $\kappa = |y''|/[1 + (y')^2]^{3/2}$, then simplify before differentiating.

Step 1: Curvature function

$$y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}.$$

Therefore

$$\begin{aligned}\kappa(x) &= \frac{1/x^2}{(1 + 1/x^2)^{3/2}} \\ &= \frac{x}{(x^2 + 1)^{3/2}}, \quad x > 0.\end{aligned}$$

Step 2: Critical point

$$\begin{aligned}\kappa'(x) &= (x^2 + 1)^{-3/2} - 3x^2(x^2 + 1)^{-5/2} \\ &= \frac{1 - 2x^2}{(x^2 + 1)^{5/2}}.\end{aligned}$$

Thus $\kappa'(x) = 0$ when $x = 1/\sqrt{2}$. The numerator is positive before this point and negative after it, so this is the unique maximum.

Step 3: Point and value The point is

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2} \ln 2 \right).$$

At that value,

$$\kappa_{\max} = \frac{1/\sqrt{2}}{(3/2)^{3/2}} = \frac{2}{3\sqrt{3}}.$$

Also $\kappa \rightarrow 0$ as $x \rightarrow 0^+$ or $x \rightarrow \infty$, confirming the interior maximum.

Question 4

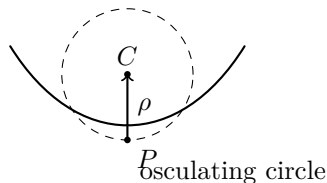
Find the osculating circle of $y = x^2$ at the vertex.

Tasks

1. Compute the curvature and principal-normal direction at $x = 0$.
2. Find the center and radius of the osculating circle.
3. Write its Cartesian equation and verify tangency.

Question 4 – Solution

Strategy. The center lies one radius of curvature from the point in the principal-normal direction.



Step 1: Curvature For $y = x^2$,

$$y' = 2x, \quad y'' = 2,$$

so

$$\kappa(0) = \frac{|2|}{(1+0)^{3/2}} = 2, \quad \rho(0) = \frac{1}{2}.$$

At the vertex the curve bends upward, so $\mathbf{N} = \langle 0, 1 \rangle$.

Step 2: Center With $P = (0, 0)$,

$$C = P + \rho \mathbf{N} = (0, 0) + \frac{1}{2}(0, 1) = \left(0, \frac{1}{2}\right).$$

Step 3: Circle equation Radius $1/2$ and center $(0, 1/2)$ give

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

The circle contains $(0, 0)$. Implicit differentiation gives

$$2x + 2\left(y - \frac{1}{2}\right)y' = 0,$$

so at $(0, 0)$, $y' = 0$, matching the parabola's horizontal tangent. The common curvature at contact gives second-order agreement.

Question 5

The same parabola is parametrized by

$$\mathbf{r}(t) = \langle t, t^2, 0 \rangle \quad \text{and} \quad \mathbf{R}(u) = \langle u^3, u^6, 0 \rangle, \quad u > 0.$$

Tasks

1. Compute curvature from each parametrization.
2. Compare the answers at corresponding points $t = u^3$.
3. Explain why the restriction $u > 0$ is useful.

Question 5 – Solution

Strategy. Apply the cross-product formula twice and simplify all powers before comparing.

Step 1: First parametrization

$$\mathbf{r}' = \langle 1, 2t, 0 \rangle, \quad \mathbf{r}'' = \langle 0, 2, 0 \rangle.$$

Thus

$$\kappa_r(t) = \frac{2}{(1 + 4t^2)^{3/2}}.$$

Step 2: Second parametrization

$$\mathbf{R}' = \langle 3u^2, 6u^5, 0 \rangle, \quad \mathbf{R}'' = \langle 6u, 30u^4, 0 \rangle.$$

The cross product has k -component

$$(3u^2)(30u^4) - (6u^5)(6u) = 54u^6.$$

Also

$$\|\mathbf{R}'\| = 3u^2\sqrt{1 + 4u^6}$$

because $u > 0$. Hence

$$\kappa_R(u) = \frac{54u^6}{27u^6(1 + 4u^6)^{3/2}} = \frac{2}{(1 + 4u^6)^{3/2}}.$$

Step 3: Compare Substituting $t = u^3$ into κ_r gives exactly κ_R . Curvature depends on the geometric curve, not its regular parametrization. The restriction $u > 0$ ensures $\mathbf{R}' \neq \mathbf{0}$ and selects the positive-parameter branch. The speed simplification uses $u^2 \geq 0$, which also holds for negative u .

Question 6

At a point of a regular curve,

$$\mathbf{r}' = \langle 2, -1, 2 \rangle, \quad \mathbf{r}'' = \langle 1, 2, 0 \rangle.$$

Tasks

1. Compute the curvature.
2. Decompose $\mathbf{r}''$ into components parallel and perpendicular to $\mathbf{r}'$.
3. Verify curvature using $\kappa = \|\mathbf{r}''_{\perp}\|/\|\mathbf{r}'\|^2$.

Question 6 – Solution

Strategy. Compare the cross-product formula with the geometric perpendicular-acceleration formula.

Step 1: Cross-product computation

$$\mathbf{r}' \times \mathbf{r}'' = \langle 2, -1, 2 \rangle \times \langle 1, 2, 0 \rangle = \langle -4, 2, 5 \rangle.$$

Its length is $\sqrt{16 + 4 + 25} = 3\sqrt{5}$, while $\|\mathbf{r}'\| = 3$. Therefore

$$\kappa = \frac{3\sqrt{5}}{3^3} = \frac{\sqrt{5}}{9}.$$

Step 2: Parallel component Since

$$\mathbf{r}'' \cdot \mathbf{r}' = 2 - 2 + 0 = 0,$$

the projection of $\mathbf{r}''$ onto $\mathbf{r}'$ is

$$\mathbf{r}''_{\parallel} = \frac{\mathbf{r}'' \cdot \mathbf{r}'}{\|\mathbf{r}'\|^2} \mathbf{r}' = \mathbf{0}.$$

Thus $\mathbf{r}''_{\perp} = \mathbf{r}'' = \langle 1, 2, 0 \rangle$ and $\|\mathbf{r}''_{\perp}\| = \sqrt{5}$.

Step 3: Verification

$$\frac{\|\mathbf{r}''_{\perp}\|}{\|\mathbf{r}'\|^2} = \frac{\sqrt{5}}{9},$$

which agrees exactly with the cross-product result.

Question 7

For the space curve

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle,$$

find all real parameters at which the curvature is zero.

Tasks

1. Compute $\mathbf{r}' \times \mathbf{r}''$.
2. Decide whether its magnitude can vanish.
3. Interpret the conclusion geometrically.

Question 7 – Solution

Strategy. Since the curve is regular, curvature is zero exactly when $\mathbf{r}'$ and $\mathbf{r}''$ are parallel.

Step 1: Derivatives

$$\mathbf{r}' = \langle 1, 2t, 3t^2 \rangle, \quad \mathbf{r}'' = \langle 0, 2, 6t \rangle.$$

The first component of $\mathbf{r}'$ is always 1, so the curve is regular.

Step 2: Cross product

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= \langle 1, 2t, 3t^2 \rangle \times \langle 0, 2, 6t \rangle \\ &= \langle 12t^2 - 6t^2, -6t, 2 \rangle \\ &= \langle 6t^2, -6t, 2 \rangle.\end{aligned}$$

Its squared magnitude is

$$36t^4 + 36t^2 + 4.$$

Every term is nonnegative and the constant term is $4 > 0$. Therefore the cross product never vanishes.

Step 3: Conclusion

$\kappa(t) > 0$ for every real t ; there are no zero-curvature points.

Geometrically, the tangent direction is changing everywhere. In particular, no point of this twisted cubic is locally straight in the curvature sense.

Question 8

Consider $\mathbf{r}(t) = \langle t, t^3, 0 \rangle$.

Tasks

1. Derive the curvature function.
2. Explain why curvature vanishes at the origin even though the curve is regular there.
3. Determine the two points where curvature is greatest.

Question 8 – Solution

Strategy. Use the planar graph formula and maximize the even function by working with $u = t^2$.

Step 1: Curvature

$$y' = 3t^2, \quad y'' = 6t,$$

so

$$\kappa(t) = \frac{6|t|}{(1 + 9t^4)^{3/2}}.$$

At $t = 0$, $\mathbf{r}'(0) = \langle 1, 0, 0 \rangle \neq \mathbf{0}$, so the curve is regular, but $y''(0) = 0$ and therefore $\kappa(0) = 0$. This is an inflection point.

Step 2: Maximize for $t > 0$ There $|t| = t$. Logarithmic differentiation gives

$$\frac{\kappa'}{\kappa} = \frac{1}{t} - \frac{54t^3}{1 + 9t^4}.$$

Setting this equal to zero,

$$1 + 9t^4 = 54t^4 \implies t^4 = \frac{1}{45}.$$

Thus $t = 45^{-1/4}$ on the positive side. By even symmetry, the other maximizer is its negative.

Step 3: Points With $\alpha = 45^{-1/4}$, the two curve points are

$$(\alpha, \alpha^3, 0) \quad \text{and} \quad (-\alpha, -\alpha^3, 0).$$

Curvature tends to zero as $|t| \rightarrow \infty$ and is zero at $t = 0$, so these critical points are global maxima.

Question 9

A helix must have radius 3 and constant curvature $1/5$. It is modeled by

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, bt \rangle, \quad b > 0.$$

Tasks

1. Determine b exactly.
2. Find the vertical rise per complete revolution.
3. Verify the designed curvature directly.

Question 9 – Solution

Strategy. Use the helix curvature formula and then translate parameter change into vertical rise.

Step 1: Solve the design equation For a helix of radius a ,

$$\kappa = \frac{a}{a^2 + b^2}.$$

With $a = 3$ and $\kappa = 1/5$,

$$\begin{aligned}\frac{3}{9 + b^2} &= \frac{1}{5}, \\ 15 &= 9 + b^2, \\ b^2 &= 6.\end{aligned}$$

Since $b > 0$, $\boxed{b = \sqrt{6}}$.

Step 2: Rise per revolution A full revolution changes t by 2π . Since $z = bt$,

$$\Delta z = b(2\pi) = \boxed{2\pi\sqrt{6}}.$$

Step 3: Verification Substituting $b^2 = 6$ into the curvature formula gives

$$\kappa = \frac{3}{3^2 + (\sqrt{6})^2} = \frac{3}{9 + 6} = \boxed{\frac{1}{5}}.$$

The positive root also makes the helix rise as t increases, satisfying the stated orientation.

Question 10

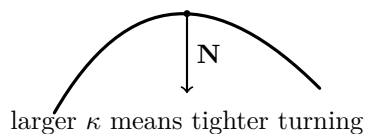
Let $\mathbf{r}(s)$ be a twice-differentiable curve parametrized by arc length, so $\|\mathbf{r}'(s)\| = 1$.

Tasks

1. Prove that $\mathbf{r}''(s)$ is perpendicular to $\mathbf{r}'(s)$.
2. Prove that $\kappa(s) = \|\mathbf{r}''(s)\|$.
3. Explain what $\mathbf{r}''(s) = \mathbf{0}$ means locally.

Question 10 – Solution

Strategy. Differentiate the unit-speed invariant, then compare the definition $\kappa = \|d\mathbf{T}/ds\|$ with $\mathbf{T} = \mathbf{r}'$.



Step 1: Orthogonality Unit speed means

$$\mathbf{r}'(s) \cdot \mathbf{r}'(s) = 1.$$

Differentiate both sides:

$$2\mathbf{r}'(s) \cdot \mathbf{r}''(s) = 0.$$

Therefore

$$\boxed{\mathbf{r}'(s) \cdot \mathbf{r}''(s) = 0}.$$

Thus the change of the unit tangent has no tangential component.

Step 2: Curvature formula Because the parameter is arc length,

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s).$$

Consequently

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''(s)\|,$$

so

$$\boxed{\kappa(s) = \|\mathbf{r}''(s)\|}.$$

Step 3: Zero second derivative If $\mathbf{r}'' = \mathbf{0}$ throughout a neighborhood, then $\mathbf{r}'$ is a constant unit vector there. Integrating gives $\mathbf{r}(s) = \mathbf{p} + s\mathbf{u}$ locally, a straight line traversed at unit speed. At a single point, $\mathbf{r}''(s_0) = \mathbf{0}$ means zero curvature at that point, but does not by itself force an entire neighboring segment to be straight.

Question 1

A particle has position

$$\mathbf{r}(t) = \langle t^2 - 2t, t^3 - 3t, 4t + 1 \rangle, \quad t \geq 0.$$

Tasks

1. Find velocity, acceleration, and speed.
2. Evaluate them at $t = 1$.
3. Find the tangent line to the trajectory at that time.

Question 1 – Solution

Strategy. Differentiate position componentwise; speed is the magnitude of velocity.

Step 1: Velocity and acceleration

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t - 2, 3t^2 - 3, 4 \rangle,$$

and

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 6t, 0 \rangle.$$

Therefore

$$\text{speed} = \|\mathbf{v}(t)\| = \sqrt{(2t - 2)^2 + (3t^2 - 3)^2 + 16}.$$

Step 2: Values at $t = 1$

$$\mathbf{r}(1) = \langle -1, -2, 5 \rangle, \quad \mathbf{v}(1) = \langle 0, 0, 4 \rangle, \quad \mathbf{a}(1) = \langle 2, 6, 0 \rangle.$$

The speed is $\boxed{4}$.

Step 3: Tangent line Since $\mathbf{v}(1) \neq \mathbf{0}$, it gives the tangent direction:

$$\mathbf{L}(s) = \langle -1, -2, 5 \rangle + s \langle 0, 0, 1 \rangle.$$

The simplified direction is parallel to velocity. At $s = 0$ the line passes through the correct position.

Question 2

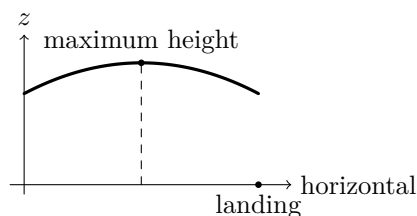
A ball is launched from $(0, 0, 5)$ with initial velocity $\langle 30, 40, 50 \rangle$ ft/s. Gravity is $\langle 0, 0, -32 \rangle$ ft/s² and air resistance is neglected.

Tasks

1. Find position and velocity for $t \geq 0$.
2. Find the maximum height and when it occurs.
3. Find the landing time and horizontal landing point.

Question 2 – Solution

Strategy. Integrate constant acceleration twice and apply the initial data; landing occurs when the vertical coordinate is zero.

**Step 1: Motion functions**

$$\mathbf{v}(t) = \langle 30, 40, 50 - 32t \rangle.$$

Integrating again and using $\mathbf{r}(0) = \langle 0, 0, 5 \rangle$,

$$\mathbf{r}(t) = \langle 30t, 40t, 5 + 50t - 16t^2 \rangle.$$

Step 2: Maximum height The vertical velocity is zero when

$$50 - 32t = 0 \implies t = \frac{25}{16}.$$

Then

$$z = 5 + 50\left(\frac{25}{16}\right) - 16\left(\frac{25}{16}\right)^2 = \boxed{\frac{705}{16} \text{ ft}}.$$

Step 3: Landing Set $z = 0$:

$$5 + 50t - 16t^2 = 0 \implies t = \frac{25 \pm \sqrt{705}}{16}.$$

Only the positive root is physical:

$$t_L = \frac{25 + \sqrt{705}}{16}.$$

Thus the horizontal landing point is

$$(x, y) = (30t_L, 40t_L).$$

Substitution into the vertical equation verifies height zero.

Question 3

A particle satisfies

$$\mathbf{a}(t) = \langle 6t, 4e^{2t}, -2 \rangle, \quad \mathbf{v}(0) = \langle 1, -1, 3 \rangle, \quad \mathbf{r}(0) = \langle 2, 0, -4 \rangle.$$

Tasks

1. Recover $\mathbf{v}(t)$.
2. Recover $\mathbf{r}(t)$.
3. Verify both initial conditions and the acceleration.

Question 3 – Solution

Strategy. Integrate componentwise twice, determining a new constant vector at each stage.

Step 1: Velocity

$$\mathbf{v}(t) = \langle 3t^2 + C_1, 2e^{2t} + C_2, -2t + C_3 \rangle.$$

At $t = 0$, comparison with $\langle 1, -1, 3 \rangle$ gives

$$C_1 = 1, \quad 2 + C_2 = -1, \quad C_3 = 3.$$

Therefore

$$\boxed{\mathbf{v}(t) = \langle 3t^2 + 1, 2e^{2t} - 3, 3 - 2t \rangle}.$$

Step 2: Position Integrating velocity,

$$\mathbf{r}(t) = \langle t^3 + t + D_1, e^{2t} - 3t + D_2, 3t - t^2 + D_3 \rangle.$$

Using $\mathbf{r}(0) = \langle 2, 0, -4 \rangle$ gives $D_1 = 2$, $1 + D_2 = 0$, and $D_3 = -4$. Hence

$$\boxed{\mathbf{r}(t) = \langle t^3 + t + 2, e^{2t} - 3t - 1, 3t - t^2 - 4 \rangle}.$$

Step 3: Verification Differentiating $\mathbf{r}$ gives the displayed $\mathbf{v}$; differentiating again gives $\langle 6t, 4e^{2t}, -2 \rangle$. Direct substitution at $t = 0$ returns both required initial vectors.

Question 4

A particle moves on a circle according to

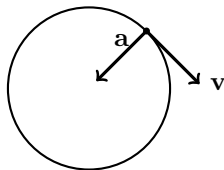
$$\mathbf{r}(t) = \langle R \cos(\omega t), R \sin(\omega t), 0 \rangle, \quad R > 0, \omega > 0.$$

Tasks

1. Find velocity, speed, and acceleration.
2. Prove velocity is perpendicular to acceleration.
3. Express acceleration in terms of position and identify its magnitude.

Question 4 – Solution

Strategy. Differentiate symbolically and use trigonometric identities and dot products.



velocity tangent; acceleration inward

Step 1: Differentiate

$$\mathbf{v} = \langle -R\omega \sin(\omega t), R\omega \cos(\omega t), 0 \rangle,$$

so

$$\|\mathbf{v}\| = R\omega.$$

Differentiating again,

$$\mathbf{a} = \langle -R\omega^2 \cos(\omega t), -R\omega^2 \sin(\omega t), 0 \rangle.$$

Step 2: Orthogonality

$$\mathbf{v} \cdot \mathbf{a} = R^2\omega^3 \sin(\omega t) \cos(\omega t) - R^2\omega^3 \cos(\omega t) \sin(\omega t) = 0.$$

Thus $\mathbf{v} \perp \mathbf{a}$.

Step 3: Inward acceleration Comparing with $\mathbf{r}$ gives

$$\mathbf{a}(t) = -\omega^2 \mathbf{r}(t).$$

It points toward the origin, and

$$\|\mathbf{a}\| = R\omega^2 = \frac{(R\omega)^2}{R} = \frac{v^2}{R}.$$

This is purely normal acceleration; the constant speed confirms there is no tangential component.

Question 5

For the helical motion

$$\mathbf{r}(t) = \langle 3 \cos 2t, 3 \sin 2t, 4t \rangle,$$

determine the velocity, acceleration, speed, and angle between velocity and the positive z -axis.

Tasks

1. Compute $\mathbf{v}$ and $\mathbf{a}$.
2. Show speed and the requested angle are constant.
3. Prove acceleration points horizontally toward the helix axis.

Question 5 – Solution

Strategy. Differentiate, then use the dot-product angle formula with $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Step 1: Motion quantities

$$\mathbf{v}(t) = \langle -6 \sin 2t, 6 \cos 2t, 4 \rangle,$$

and

$$\mathbf{a}(t) = \langle -12 \cos 2t, -12 \sin 2t, 0 \rangle.$$

The speed is

$$\|\mathbf{v}\| = \sqrt{36(\sin^2 2t + \cos^2 2t) + 16} = 2\sqrt{13}.$$

Step 2: Direction angle If θ is the angle with the positive z -axis,

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{k}}{\|\mathbf{v}\| \|\mathbf{k}\|} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}.$$

Therefore

$$\theta = \cos^{-1} \left(\frac{2}{\sqrt{13}} \right).$$

No t occurs, so both speed and angle are constant.

Step 3: Acceleration geometry Its horizontal component is

$$-4 \langle 3 \cos 2t, 3 \sin 2t, 0 \rangle,$$

which is opposite the radial position from the z -axis. Its z -component is zero, so acceleration points horizontally inward. Also $\mathbf{v} \cdot \mathbf{a} = 0$, consistent with constant speed.

Question 6

Two objects move according to

$$\mathbf{r}_1(t) = \langle t, 2t, 3 - t \rangle, \quad \mathbf{r}_2(t) = \langle 4 - t, t + 1, 2t - 2 \rangle, \quad t \geq 0.$$

Tasks

1. Determine whether they collide.
2. Find their velocities and relative velocity.
3. Verify your collision conclusion using relative position.

Question 6 – Solution

Strategy. A collision requires equal position vectors at the same time, not merely intersecting geometric paths.

Step 1: Match coordinates Equality requires

$$t = 4 - t, \quad 2t = t + 1, \quad 3 - t = 2t - 2.$$

The first gives $t = 2$, while the second gives $t = 1$. Since no single time satisfies even these two equations, the objects do not collide:

$$\boxed{\text{no collision for } t \geq 0}.$$

Step 2: Velocities

$$\mathbf{v}_1 = \langle 1, 2, -1 \rangle, \quad \mathbf{v}_2 = \langle -1, 1, 2 \rangle.$$

The velocity of object 1 relative to object 2 is

$$\boxed{\mathbf{v}_{1/2} = \mathbf{v}_1 - \mathbf{v}_2 = \langle 2, 1, -3 \rangle}.$$

Step 3: Relative-position check

$$\mathbf{r}_1 - \mathbf{r}_2 = \langle 2t - 4, t - 1, 5 - 3t \rangle.$$

For this vector to be zero, its first, second, and third components would require respectively $t = 2$, $t = 1$, and $t = 5/3$. The inconsistent times independently confirm that relative position never vanishes.

Question 7

A particle has velocity

$$\mathbf{v}(t) = \langle t - 2, 2t + 1, 2 \rangle, \quad t \geq 0.$$

Tasks

1. Find when its speed is smallest.
2. Find the minimum speed.
3. Show that velocity is perpendicular to acceleration at that instant and explain why.

Question 7 – Solution

Strategy. Minimize squared speed, which has the same minimizers as speed and avoids a square root.

Step 1: Squared speed

$$\begin{aligned} q(t) = \|\mathbf{v}(t)\|^2 &= (t-2)^2 + (2t+1)^2 + 4 \\ &= 5t^2 + 9. \end{aligned}$$

On $t \geq 0$, this is minimized at $t = 0$.

Step 2: Minimum speed

$$\mathbf{v}(0) = \langle -2, 1, 2 \rangle, \quad \|\mathbf{v}(0)\| = 3.$$

Step 3: Orthogonality Acceleration is

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 1, 2, 0 \rangle.$$

At $t = 0$,

$$\mathbf{v}(0) \cdot \mathbf{a}(0) = -2 + 2 + 0 = 0.$$

Also $q'(t) = 2\mathbf{v} \cdot \mathbf{a} = 10t$, so the vanishing dot product is exactly the zero derivative of squared speed. Here the minimum occurs at the endpoint, and the right-hand derivative is nonnegative immediately afterward.

Question 8

A particle starts at $P = (1, 0, 2)$ with velocity $\langle 2, -1, 0 \rangle$ and constant acceleration $\langle -1, 2, 3 \rangle$.

Tasks

1. Find position and velocity at time t .
2. Determine when the particle first reaches the plane $z = 8$.
3. Find its position and speed then.

Question 8 – Solution

Strategy. For constant acceleration, use $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}_0 + \frac{1}{2}t^2\mathbf{a}$.

Step 1: Motion functions

$$\mathbf{v}(t) = \langle 2 - t, -1 + 2t, 3t \rangle,$$

and

$$\mathbf{r}(t) = \left\langle 1 + 2t - \frac{1}{2}t^2, -t + t^2, 2 + \frac{3}{2}t^2 \right\rangle.$$

At $t = 0$, these reproduce both initial vectors.

Step 2: Reach $z = 8$

$$2 + \frac{3}{2}t^2 = 8 \quad \implies \quad t^2 = 4.$$

For forward time, $t = 2$.

Step 3: Position and speed

$$\mathbf{r}(2) = \langle 1 + 4 - 2, -2 + 4, 8 \rangle = \langle 3, 2, 8 \rangle.$$

Also

$$\mathbf{v}(2) = \langle 0, 3, 6 \rangle,$$

so

$$\text{speed} = \sqrt{0^2 + 3^2 + 6^2} = 3\sqrt{5}.$$

The vertical coordinate and positive time verify that this is the first forward-time crossing.

Question 9

A particle follows

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle, \quad 0 \leq t \leq 2\pi.$$

Tasks

1. Find its displacement and average velocity over the interval.
2. Find total distance traveled and average speed.
3. Explain why the horizontal motion contributes to distance but not displacement.

Question 9 – Solution

Strategy. Displacement uses endpoints; distance integrates speed. These measure different features of motion.

Step 1: Displacement

$$\mathbf{r}(0) = \langle 3, 0, 0 \rangle, \quad \mathbf{r}(2\pi) = \langle 3, 0, 8\pi \rangle.$$

Thus

$$\Delta \mathbf{r} = \langle 0, 0, 8\pi \rangle.$$

The elapsed time is 2π , so

$$\mathbf{v}_{\text{avg}} = \frac{\Delta \mathbf{r}}{2\pi} = \langle 0, 0, 4 \rangle.$$

Step 2: Distance and average speed

$$\mathbf{v}(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle,$$

so $\|\mathbf{v}(t)\| = \sqrt{9 + 16} = 5$. Hence

$$D = \int_0^{2\pi} 5 \, dt = 10\pi,$$

and

$$\text{average speed} = \frac{10\pi}{2\pi} = 5.$$

Step 3: Interpretation During one revolution, the horizontal coordinates return to their starting values, so their net changes cancel in displacement. Nevertheless the particle continuously moves horizontally around the circle, adding positive length to the traveled path. Therefore distance 10π exceeds displacement magnitude 8π .

Question 10

Let a regular motion have velocity $\mathbf{v}$, speed $v = \|\mathbf{v}\|$, unit tangent $\mathbf{T}$, curvature κ , and principal normal $\mathbf{N}$.

Tasks

1. Derive $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$.
2. Derive formulas for the tangential and normal scalar components using only $\mathbf{v}$ and $\mathbf{a}$.
3. Explain the physical meaning of each component.

Question 10 – Solution

Strategy. Begin with $\mathbf{v} = v\mathbf{T}$, differentiate, and convert $d\mathbf{T}/dt$ through arc length.

Step 1: Differentiate velocity

$$\mathbf{a} = \frac{d}{dt}(v\mathbf{T}) = v'\mathbf{T} + v\frac{d\mathbf{T}}{dt}.$$

By the chain rule,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} = (\kappa\mathbf{N})v.$$

Therefore

$$\boxed{\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}}.$$

Step 2: Tangential component Dot with $\mathbf{T} = \mathbf{v}/v$:

$$a_T = \mathbf{a} \cdot \mathbf{T} = \boxed{\frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}}.$$

It also equals v' , the rate of change of speed.

Step 3: Normal component Orthogonality gives

$$\|\mathbf{a}\|^2 = a_T^2 + a_N^2.$$

Thus

$$\boxed{a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}} = \boxed{\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}} = \kappa v^2.$$

The tangential component changes speed; the normal component changes direction. A motion can therefore accelerate even at constant speed if its path bends.

Question 1

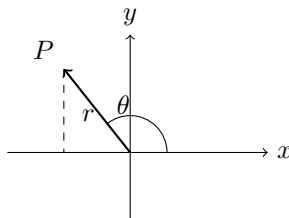
Convert $P = (-3, 3\sqrt{3}, 4)$ from Cartesian to cylindrical coordinates.

Tasks

1. Find r , a principal angle $0 \leq \theta < 2\pi$, and z .
2. Give two other equivalent cylindrical representations.
3. Verify by converting back.

Question 1 – Solution

Strategy. Use $r = \sqrt{x^2 + y^2}$ and determine the quadrant before choosing θ .



Step 1: Radius $r = \sqrt{9 + 27} = 6$.

Step 2: Angle Since $x < 0$ and $y > 0$, the point is in quadrant II. Also $\tan \theta = -\sqrt{3}$, so $\theta = 2\pi/3$. The height is unchanged. Thus

$$(r, \theta, z) = (6, 2\pi/3, 4).$$

Step 3: Equivalent forms Adding 2π gives $(6, 8\pi/3, 4)$. Reversing the radial sign and adding π gives $(-6, 5\pi/3, 4)$.

Verification. $6 \cos(2\pi/3) = -3$, $6 \sin(2\pi/3) = 3\sqrt{3}$, and $z = 4$.

Question 2

Convert $(r, \theta, z) = (4, 7\pi/6, -2)$ to Cartesian coordinates and determine its distance from the z -axis and from the origin.

Tasks

1. Find (x, y, z) .
2. Find both distances.
3. Verify the radial identity.

Question 2 – Solution

Strategy. Use $x = r \cos \theta$, $y = r \sin \theta$; radial distance is r when $r \geq 0$.

Step 1: Convert Since $\cos(7\pi/6) = -\sqrt{3}/2$ and $\sin(7\pi/6) = -1/2$,

$$\boxed{(x, y, z) = (-2\sqrt{3}, -2, -2)}.$$

Step 2: Distances Distance to the z -axis is $\sqrt{x^2 + y^2} = 4$. Distance to the origin is

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{16 + 4} = \boxed{2\sqrt{5}}.$$

Verification. $x^2 + y^2 = 12 + 4 = 16 = r^2$, confirming the conversion and axis distance.

Question 3

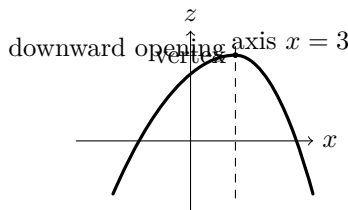
Rewrite $x^2 + y^2 - 6x + 4z = 3$ in cylindrical coordinates and identify the surface.

Tasks

1. Convert and solve for z .
2. Describe horizontal traces.
3. Locate the vertex and axis.

Question 3 – Solution

Strategy. Replace $x^2 + y^2$ by r^2 and x by $r \cos \theta$.



Step 1: Convert

$$r^2 - 6r \cos \theta + 4z = 3, \quad \boxed{z = \frac{3 - r^2 + 6r \cos \theta}{4}}.$$

For geometric identification, complete the Cartesian square:

$$(x - 3)^2 + y^2 = 12 - 4z.$$

Step 2: Identify Equivalently,

$$z = 3 - \frac{(x - 3)^2 + y^2}{4}.$$

This is a downward-opening circular paraboloid with vertical axis $x = 3, y = 0$ and vertex $\boxed{(3, 0, 3)}$.

Step 3: Traces At height $z = c$, the trace is a circle centered at $(3, 0, c)$ with radius $2\sqrt{3 - c}$, present for $c \leq 3$.

Question 4

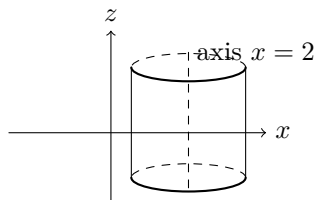
Describe the surface $r = 4 \cos \theta$ in Cartesian coordinates.

Tasks

1. Convert without dividing by r .
2. Identify the surface and its axis.
3. State its intersections with the coordinate axes.

Question 4 – Solution

Strategy. Multiply by r and use $r^2 = x^2 + y^2$ and $r \cos \theta = x$.



Step 1: Convert From $r = 4 \cos \theta$,

$$r^2 = 4r \cos \theta \implies x^2 + y^2 = 4x.$$

Completing the square,

$$\boxed{(x - 2)^2 + y^2 = 4}.$$

Step 2: Identify Since z is unrestricted, this is a circular cylinder of radius 2 whose axis is the vertical line $x = 2, y = 0$.

Step 3: Axis intersections In the xy -plane it meets the x -axis at $(0, 0, 0)$ and $(4, 0, 0)$ and the y -axis only at the origin. More generally, the cylinder contains the vertical lines $(0, 0, z)$ and $(4, 0, z)$.

Question 5

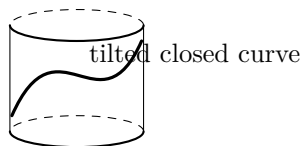
Find a cylindrical parametrization of the curve where $x^2 + y^2 = 9$ and $z = x + y$.

Tasks

1. Express both surfaces cylindrically.
2. Parametrize the entire intersection once.
3. Find its highest and lowest points.

Question 5 – Solution

Strategy. The cylinder fixes r ; let θ be the curve parameter and use the plane for z .



Step 1: Constraints The cylinder gives $r = 3$. The plane becomes

$$z = r \cos \theta + r \sin \theta = 3(\cos \theta + \sin \theta).$$

Thus

$$\mathbf{r}(\theta) = \langle 3 \cos \theta, 3 \sin \theta, 3(\cos \theta + \sin \theta) \rangle, \quad 0 \leq \theta < 2\pi.$$

Step 2: Extrema Use $\cos \theta + \sin \theta = \sqrt{2} \cos(\theta - \pi/4)$. Hence $z_{\max} = 3\sqrt{2}$ at $\theta = \pi/4$, and $z_{\min} = -3\sqrt{2}$ at $\theta = 5\pi/4$.

Step 3: Points

$$P_{\max} = (3/\sqrt{2}, 3/\sqrt{2}, 3\sqrt{2}), \quad P_{\min} = (-3/\sqrt{2}, -3/\sqrt{2}, -3\sqrt{2}).$$

Both satisfy the original cylinder and plane equations.

Question 6

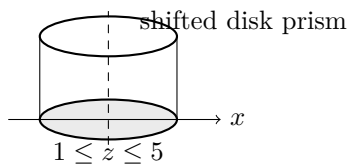
Convert the region $x^2 + y^2 \leq 4x$, $1 \leq z \leq 5$ to cylindrical inequalities.

Tasks

1. Determine the correct angular interval.
2. Give radial and vertical bounds.
3. Explain the geometry and why negative radial bounds are unnecessary.

Question 6 – Solution

Strategy. Convert the disk inequality and enforce the convention $r \geq 0$.



Step 1: Convert The inequality becomes

$$r^2 \leq 4r \cos \theta.$$

With $r \geq 0$, this is $0 \leq r \leq 4 \cos \theta$. The upper bound must be nonnegative, so $\cos \theta \geq 0$.

Step 2: Bounds One nonoverlapping description is

$$\boxed{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 4 \cos \theta, \quad 1 \leq z \leq 5}.$$

Step 3: Geometry Completing the square gives $(x - 2)^2 + y^2 \leq 4$: a radius-2 disk centered at $(2, 0)$, extended from height 1 to 5. Negative r is unnecessary because directions with $\cos \theta < 0$ would duplicate points already represented using nonnegative r and opposite angles.

Question 7

Find the distance between cylindrical points $P = (2, \pi/6, 1)$ and $Q = (5, 5\pi/6, 4)$ without first listing both Cartesian points.

Tasks

1. Derive a cylindrical distance formula.
2. Evaluate it exactly.
3. Check the horizontal contribution geometrically.

Question 7 – Solution

Strategy. Expand $(x_1 - x_2)^2 + (y_1 - y_2)^2$ and use the cosine difference identity.

Step 1: Formula For cylindrical points,

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2) + (z_1 - z_2)^2.$$

This follows from the planar law of cosines plus the vertical difference.

Step 2: Substitute Here $\theta_1 - \theta_2 = -2\pi/3$, so its cosine is $-1/2$. Thus

$$\begin{aligned} d^2 &= 2^2 + 5^2 - 2(2)(5)(-1/2) + (1 - 4)^2 \\ &= 4 + 25 + 10 + 9 = 48. \end{aligned}$$

Therefore $\boxed{d = 4\sqrt{3}}$.

Step 3: Check The horizontal squared distance is $4 + 25 + 10 = 39$; adding vertical squared distance 9 gives 48, consistent with perpendicular horizontal and vertical components.

Question 8

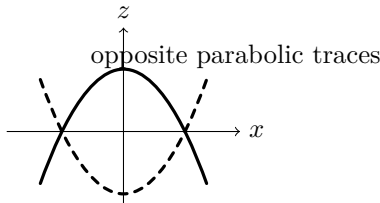
Analyze the surface $z = r^2 \cos 2\theta$.

Tasks

1. Convert it to Cartesian form.
2. Identify its principal vertical traces.
3. Describe its signs by angular sector and its symmetry.

Question 8 – Solution

Strategy. Use $r^2 \cos 2\theta = x^2 - y^2$.



Step 1: Convert Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$,

$$r^2 \cos 2\theta = r^2 \cos^2 \theta - r^2 \sin^2 \theta = x^2 - y^2.$$

Thus $\boxed{z = x^2 - y^2}$, a hyperbolic paraboloid.

Step 2: Traces In $y = 0$, $z = x^2$ opens upward. In $x = 0$, $z = -y^2$ opens downward. On $y = \pm x$, $z = 0$.

Step 3: Angular behavior The height is positive where $\cos 2\theta > 0$, negative where it is negative, and zero at $\theta = \pi/4 + k\pi/2$. The equation is unchanged by $\theta \mapsto \theta + \pi$ and by reflections across either coordinate axis, matching the Cartesian symmetries.

Question 9

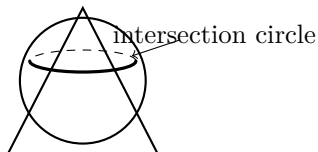
A point has cylindrical coordinates (r, θ, z) with $r = 2z$, $z \geq 0$, and lies on the sphere $x^2 + y^2 + z^2 = 45$.

Tasks

1. Determine r and z .
2. Describe all possible points.
3. Find the horizontal circle containing them.

Question 9 – Solution

Strategy. Convert the sphere to $r^2 + z^2 = 45$ and combine it with the cone relation.



Step 1: Solve Substitute $r = 2z$:

$$(2z)^2 + z^2 = 45 \implies 5z^2 = 45.$$

Since $z \geq 0$, $\boxed{z = 3}$ and $\boxed{r = 6}$.

Step 2: Angular freedom No condition restricts θ , so

$$\boxed{(r, \theta, z) = (6, \theta, 3), \quad 0 \leq \theta < 2\pi}.$$

In Cartesian form the points are $(6 \cos \theta, 6 \sin \theta, 3)$.

Step 3: Locus They form the horizontal circle

$$\boxed{x^2 + y^2 = 36, \quad z = 3}.$$

Verification gives $36 + 9 = 45$ and $r = 6 = 2(3)$.

Question 10

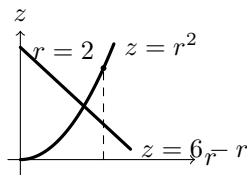
A solid is described cylindrically by $0 \leq r \leq 2 + \cos \theta$, $0 \leq \theta < 2\pi$, and $r^2 \leq z \leq 6 - r$.

Tasks

1. Determine whether the vertical bounds are consistent everywhere in the stated radial range.
2. Correct the radial upper bound if necessary.
3. Give a precise final description and explain which constraint is active.

Question 10 – Solution

Strategy. Vertical consistency requires lower height not exceed upper height.

**Step 1: Compare heights**

$$r^2 \leq 6 - r \iff r^2 + r - 6 \leq 0 \iff (r + 3)(r - 2) \leq 0.$$

With $r \geq 0$, this requires $0 \leq r \leq 2$.

Step 2: Combine radial restrictions The stated planar bound is $r \leq 2 + \cos \theta$, which ranges from 1 to 3. Both conditions must hold, so

$$0 \leq r \leq \min\{2, 2 + \cos \theta\}.$$

Equivalently, the upper bound is 2 when $\cos \theta \geq 0$ and $2 + \cos \theta$ when $\cos \theta < 0$.

Step 3: Final description

$$0 \leq \theta < 2\pi, \quad 0 \leq r \leq \min\{2, 2 + \cos \theta\}, \quad r^2 \leq z \leq 6 - r.$$

The height surfaces meet at $r = 2$. For angles on the right half-plane the height condition truncates the planar radius; on the left half-plane the limaçon-type boundary is already smaller than 2.

Question 1

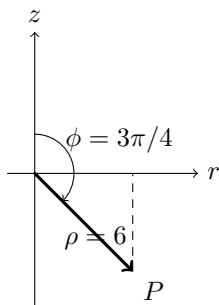
Convert the Cartesian point $P = (-3, 3, -3\sqrt{2})$ to spherical coordinates using $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$.

Tasks

1. Find ρ , θ , and ϕ .
2. Give one equivalent spherical representation outside the principal angular range.
3. Verify by converting back.

Question 1 – Solution

Strategy. Find radial distance first, use the quadrant of (x, y) for θ , and use $\cos \phi = z/\rho$.



Step 1: Radius

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{9 + 9 + 18} = 6.$$

Step 2: Angles The projection $(-3, 3)$ lies in quadrant II, so $\theta = 3\pi/4$. Also

$$\cos \phi = \frac{-3\sqrt{2}}{6} = -\frac{\sqrt{2}}{2}, \quad 0 \leq \phi \leq \pi,$$

giving $\phi = 3\pi/4$. Hence

$$\boxed{(\rho, \theta, \phi) = (6, 3\pi/4, 3\pi/4)}.$$

Step 3: Equivalence and verification Adding 2π to the azimuth gives $(6, 11\pi/4, 3\pi/4)$. Moreover,

$$6 \sin(3\pi/4) \cos(3\pi/4) = -3, \quad 6 \sin(3\pi/4) \sin(3\pi/4) = 3, \quad 6 \cos(3\pi/4) = -3\sqrt{2}.$$

Question 2

Convert $(\rho, \theta, \phi) = (8, 5\pi/6, \pi/3)$ to Cartesian and cylindrical coordinates.

Tasks

1. Find (x, y, z) exactly.
2. Find the corresponding principal cylindrical coordinates.
3. Verify both radial identities.

Question 2 – Solution

Strategy. Use $r = \rho \sin \phi$ and $z = \rho \cos \phi$, then resolve r using θ .

Step 1: Cylindrical data

$$r = 8 \sin(\pi/3) = 4\sqrt{3}, \quad z = 8 \cos(\pi/3) = 4.$$

Thus the principal cylindrical coordinates are

$$(r, \theta, z) = (4\sqrt{3}, 5\pi/6, 4).$$

Step 2: Cartesian data

$$x = r \cos \theta = (4\sqrt{3})(-\sqrt{3}/2) = -6, \quad y = r \sin \theta = (4\sqrt{3})(1/2) = 2\sqrt{3}.$$

Therefore

$$(x, y, z) = (-6, 2\sqrt{3}, 4).$$

Verification. $x^2 + y^2 = 36 + 12 = 48 = r^2$, and $x^2 + y^2 + z^2 = 48 + 16 = 64 = \rho^2$.

Question 3

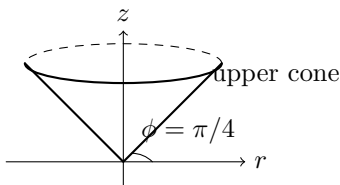
Describe geometrically the coordinate surfaces $\rho = 5$, $\theta = 2\pi/3$, and $\phi = \pi/4$.

Tasks

1. Identify each full coordinate surface.
2. Describe the pairwise intersections.
3. Find their common point.

Question 3 – Solution

Strategy. Interpret ρ as distance, θ as azimuth, and ϕ as inclination from the positive z -axis.



Step 1: Surfaces $\rho = 5$ is the sphere of radius 5. The equation $\theta = 2\pi/3$ is a vertical half-plane from the z -axis in azimuth $2\pi/3$. The equation $\phi = \pi/4$ is the upper nappe of the cone making angle $\pi/4$ with the positive z -axis.

Step 2: Pairwise intersections Sphere–half-plane and sphere–cone intersections are semicircle and horizontal circle, respectively. Half-plane–cone is a ray from the origin.

Step 3: Common point Substitution gives

$$x = 5\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) = -\frac{5\sqrt{2}}{4}, \quad y = 5\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{6}}{4}, \quad z = \frac{5\sqrt{2}}{2}.$$

Thus

$$P = \left(-5\sqrt{2}/4, 5\sqrt{6}/4, 5\sqrt{2}/2\right).$$

Question 4

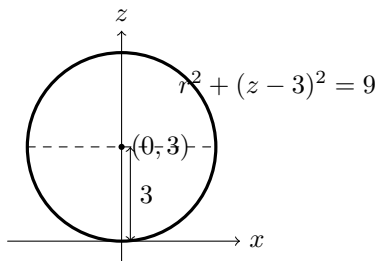
Rewrite the sphere $x^2 + y^2 + z^2 = 6z$ in spherical coordinates and identify its geometry.

Tasks

1. Obtain the nontrivial spherical equation.
2. Find its Cartesian center and radius.
3. Explain the role of $\rho = 0$ when simplifying.

Question 4 – Solution

Strategy. Substitute $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$, while preserving any solution lost by division.

**Step 1: Convert**

$$\rho^2 = 6\rho \cos \phi \iff \rho(\rho - 6 \cos \phi) = 0.$$

The usual surface equation is

$$\boxed{\rho = 6 \cos \phi}, \quad 0 \leq \phi \leq \frac{\pi}{2},$$

and it includes the origin at $\phi = \pi/2$.

Step 2: Identify Completing the square in Cartesian form gives

$$x^2 + y^2 + (z - 3)^2 = 9.$$

It is the sphere centered at $\boxed{(0, 0, 3)}$ with radius $\boxed{3}$.

Step 3: Origin check Division by ρ would temporarily discard $\rho = 0$. The origin satisfies the original equation and is recovered by $\rho = 6 \cos(\pi/2) = 0$; thus no point is absent from the final parametrized surface.

Question 5

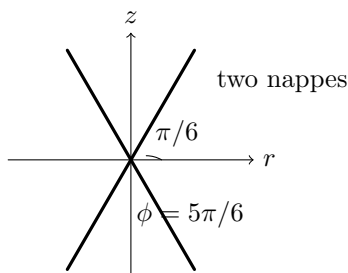
Convert the surface $z^2 = 3(x^2 + y^2)$ to spherical coordinates and determine exactly which cone it represents.

Tasks

1. Derive all permitted constant values of ϕ .
2. Identify both nappes.
3. State the status of the origin.

Question 5 – Solution

Strategy. Substitute $z = \rho \cos \phi$ and $x^2 + y^2 = \rho^2 \sin^2 \phi$, treating $\rho = 0$ separately.



Step 1: Convert For $\rho > 0$,

$$\rho^2 \cos^2 \phi = 3\rho^2 \sin^2 \phi \implies \tan^2 \phi = \frac{1}{3}.$$

On $0 \leq \phi \leq \pi$, the solutions are

$$\boxed{\phi = \frac{\pi}{6} \quad \text{or} \quad \phi = \frac{5\pi}{6}}.$$

Step 2: Geometry The first value is the upper nappe, $z \geq 0$; the second is the lower nappe, $z \leq 0$. Each makes an acute angle $\pi/6$ with its adjacent half of the z -axis.

Step 3: Origin At $\rho = 0$, every angle represents the origin, which satisfies the Cartesian equation. Therefore the two nappes meet at the included vertex $\boxed{(0, 0, 0)}$.

Question 6

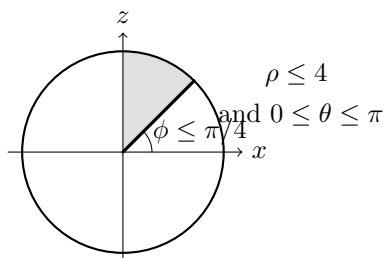
Describe in spherical inequalities the solid inside $x^2 + y^2 + z^2 \leq 16$, above the cone $z = \sqrt{x^2 + y^2}$, and in the half-space $y \geq 0$.

Tasks

1. Determine bounds for ρ , θ , and ϕ .
2. Explain each angular bound geometrically.
3. State which boundary pieces meet at the origin.

Question 6 – Solution

Strategy. Translate the sphere to a radial bound, the half-space to azimuth, and compare z with the cylindrical radius to bound inclination.



Step 1: Radial bound The sphere gives $0 \leq \rho \leq 4$.

Step 2: Azimuth Since $y = \rho \sin \phi \sin \theta$ and $\sin \phi \geq 0$, $y \geq 0$ is represented without overlap by $0 \leq \theta \leq \pi$.

Step 3: Inclination Above the cone means

$$\rho \cos \phi \geq \rho \sin \phi.$$

For $\rho > 0$ this gives $0 \leq \phi \leq \pi/4$. Consequently

$$\boxed{0 \leq \rho \leq 4, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq \pi/4}.$$

The sphere is the outer boundary; the cone and the two half-plane faces $\theta = 0, \pi$ all meet at the origin. Angular multiplicity there does not duplicate physical volume.

Question 7

Find the distance between $P = (3, \pi/6, \pi/3)$ and $Q = (5, 5\pi/6, 2\pi/3)$ in spherical coordinates without first converting both points to Cartesian form.

Tasks

1. Derive the angle between their radial directions.
2. Compute the distance exactly.
3. Check that the result obeys the triangle bounds.

Question 7 – Solution

Strategy. Take the dot product of the two unit radial directions and apply the three-dimensional law of cosines.

Step 1: Direction angle If γ is the angle between the radial directions, then

$$\cos \gamma = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2).$$

Here $\cos \phi_1 \cos \phi_2 = -1/4$, $\sin \phi_1 \sin \phi_2 = 3/4$, and $\cos(-2\pi/3) = -1/2$, so $\cos \gamma = -5/8$.

Step 2: Distance Therefore

$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos \gamma = 9 + 25 - 30(-5/8) = \frac{211}{4}.$$

Thus

$$d = \frac{\sqrt{211}}{2}.$$

Verification. The triangle inequality requires $|5-3| \leq d \leq 5+3$. Numerically $d \approx 7.26$, which lies between 2 and 8.

Question 8

Analyze all spherical-coordinate representations of points on the z -axis under the conventions $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$.

Tasks

1. Characterize a point $(0, 0, a)$ for $a > 0$, $a < 0$, and $a = 0$.
2. Identify which coordinates are nonunique.
3. Explain why the conversion formulas remain consistent.

Question 8 – Solution

Strategy. On the axis, the horizontal projection has length $\rho \sin \phi = 0$, so analyze the poles and origin separately.

Step 1: Positive axis If $a > 0$, then $\rho = a$, $\phi = 0$, and θ is arbitrary:

$$\boxed{(a, \theta, 0), \quad 0 \leq \theta < 2\pi}.$$

Step 2: Negative axis If $a < 0$, then $\rho = |a|$, $\phi = \pi$, and again θ is arbitrary:

$$\boxed{(|a|, \theta, \pi), \quad 0 \leq \theta < 2\pi}.$$

At the origin, $\rho = 0$ and both angles are arbitrary in their stated ranges.

Step 3: Consistency At $\phi = 0$ or π , $\sin \phi = 0$, so $x = y = 0$ regardless of θ , while $z = \rho \cos \phi$ has the correct sign. At $\rho = 0$, all three Cartesian coordinates vanish regardless of either angle. Thus the nonuniqueness is coordinate degeneracy, not geometric ambiguity.

Question 9

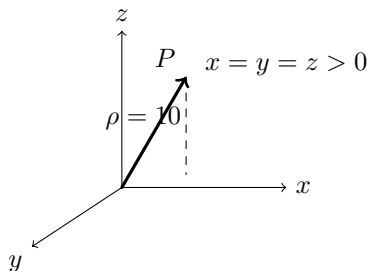
A point P has spherical coordinates (ρ, θ, ϕ) with $\rho = 10$ and is equally distant from the positive x -, y -, and z -axes. Assume P lies in the first octant.

Tasks

1. Determine the Cartesian coordinates of P .
2. Find its principal spherical angles.
3. Verify the three axis distances directly.

Question 9 – Solution

Strategy. Express squared distance to each coordinate axis and use first-octant signs.



Step 1: Equal-distance conditions Squared distances to the x -, y -, and z -axes are $y^2 + z^2$, $x^2 + z^2$, and $x^2 + y^2$. Equality implies $x^2 = y^2 = z^2$. In the first octant, $x = y = z$.

Step 2: Radius condition Since $x^2 + y^2 + z^2 = 100$, we obtain $3x^2 = 100$ and

$$P = (10/\sqrt{3}, 10/\sqrt{3}, 10/\sqrt{3}).$$

Then $\theta = \pi/4$, and $\cos \phi = z/\rho = 1/\sqrt{3}$, so

$$(\rho, \theta, \phi) = (10, \pi/4, \arccos(1/\sqrt{3})).$$

Verification. Every squared axis distance is $100/3 + 100/3 = 200/3$, so every distance is $10\sqrt{2/3}$.

Question 10

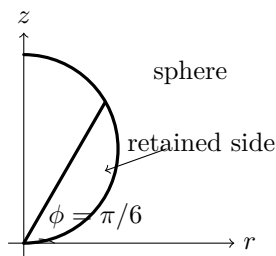
A solid is proposed in spherical coordinates by $0 \leq \rho \leq 6 \cos \phi$, $\pi/6 \leq \phi \leq 2\pi/3$, and $0 \leq \theta < 2\pi$.

Tasks

1. Determine where the radial bounds are consistent.
2. Correct the angular interval while retaining all actual points described.
3. Identify the solid and describe its boundary circle.

Question 10 – Solution

Strategy. A nonnegative radial variable can lie below $6 \cos \phi$ only when that upper bound is nonnegative.



Step 1: Consistency Since $\rho \geq 0$, we require $6 \cos \phi \geq 0$. On $0 \leq \phi \leq \pi$, this means $0 \leq \phi \leq \pi/2$. Intersecting with the proposed range gives

$$\boxed{\pi/6 \leq \phi \leq \pi/2}.$$

Angles $\pi/2 < \phi \leq 2\pi/3$ contribute no points because their proposed upper radial bound is negative.

Step 2: Correct description

$$\boxed{0 \leq \theta < 2\pi, \quad \pi/6 \leq \phi \leq \pi/2, \quad 0 \leq \rho \leq 6 \cos \phi}.$$

The radial surface is the sphere $x^2 + y^2 + (z - 3)^2 = 9$. The lower angular cutoff removes the portion closer than $\pi/6$ to the positive z -axis, leaving the part of the ball outside that narrow cone.

Step 3: Boundary circle On $\phi = \pi/6$ and the sphere, $\rho = 3\sqrt{3}$. Hence

$$z = \rho \cos \phi = \frac{9}{2}, \quad r = \rho \sin \phi = \frac{3\sqrt{3}}{2}.$$

The surfaces meet in

$$\boxed{x^2 + y^2 = 27/4, \quad z = 9/2}.$$

Question 1

Evaluate

$$\lim_{(x,y) \rightarrow (2,-1)} (x^2y + 3xy^2 - 2y + 5).$$

Tasks

1. State why direct substitution is valid.
2. Evaluate the limit exactly.
3. Verify the arithmetic term by term.

Question 1 – Solution

Strategy. A polynomial in two variables is continuous everywhere, so its limit equals its value at the target point.

Step 1: Continuity Each monomial and constant is continuous, and finite sums of continuous functions are continuous. Direct substitution is therefore valid.

Step 2: Substitute

$$\begin{aligned}x^2y + 3xy^2 - 2y + 5 &= (2)^2(-1) + 3(2)(-1)^2 - 2(-1) + 5 \\&= -4 + 6 + 2 + 5 = 9.\end{aligned}$$

Thus

$$\boxed{\lim_{(x,y) \rightarrow (2,-1)} (x^2y + 3xy^2 - 2y + 5) = 9}.$$

Verification. The four contributions are -4 , 6 , 2 , and 5 ; their sum is 9 . No denominator or restricted operation can fail at $(2, -1)$.

Question 2

Evaluate

$$\lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + xy - 3x - y + 2}{x - 1}.$$

Tasks

1. Explain why direct substitution is inconclusive.
2. Factor and simplify on the punctured domain.
3. Evaluate the limit and distinguish it from the function value.

Question 2 – Solution

Strategy. Factor the numerator to expose a common factor with the vanishing denominator.

Step 1: Indeterminate form At $(1, 2)$, numerator and denominator are both 0, so substitution gives $0/0$, not a limit value.

Step 2: Factor Treat the numerator as a polynomial in y :

$$x^2 - 3x + 2 + y(x - 1) = (x - 1)(x - 2) + y(x - 1) = (x - 1)(x + y - 2).$$

Whenever $x \neq 1$,

$$\frac{(x - 1)(x + y - 2)}{x - 1} = x + y - 2.$$

Step 3: Limit The simplified function is continuous, so

$$\lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + xy - 3x - y + 2}{x - 1} = 1.$$

The original quotient is undefined on the entire line $x = 1$, including the target, but points of its domain approach $(1, 2)$ and the missing line does not change the limit.

Question 3

Determine whether

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$$

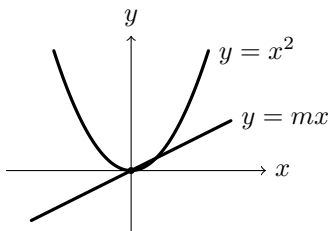
exists.

Tasks

1. Test every straight-line path $y = mx$.
2. Test a curved path suggested by the denominator.
3. Explain why the first test alone cannot prove existence.

Question 3 – Solution

Strategy. Compare straight lines with the balancing path $y = kx^2$, where x^4 and y^2 have the same order.



Step 1: Lines Along $y = mx$,

$$\frac{x^2(mx)}{x^4 + m^2x^2} = \frac{mx}{x^2 + m^2} \rightarrow 0$$

for $m \neq 0$; the path $y = 0$ also gives 0.

Step 2: Parabolas Along $y = kx^2$,

$$\frac{x^2(kx^2)}{x^4 + k^2x^4} = \frac{k}{1 + k^2}.$$

For example, $k = 1$ gives $1/2$, whereas $k = 0$ gives 0.

Conclusion. Since two paths produce different values,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4 + y^2} \text{ does not exist.}$$

Agreement along all lines is necessary but not sufficient because nonlinearly approaching paths remain possible.

Question 4

Evaluate

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2}.$$

Tasks

1. Establish a useful bound without selecting paths.
2. Apply the squeeze theorem.
3. Give an independent polar-coordinate check.

Question 4 – Solution

Strategy. Bound the numerator using $x^2 \leq x^2 + y^2$ and then compare with a quantity tending to zero.

Step 1: Bound For $(x, y) \neq (0, 0)$,

$$0 \leq \frac{x^2 y^2}{x^2 + y^2} \leq \frac{(x^2 + y^2) y^2}{x^2 + y^2} = y^2.$$

As $(x, y) \rightarrow (0, 0)$, $y^2 \rightarrow 0$.

Step 2: Squeeze Both bounding functions tend to 0, hence

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2} = 0}.$$

Step 3: Polar check With $x = r \cos \theta$ and $y = r \sin \theta$,

$$\frac{x^2 y^2}{x^2 + y^2} = r^2 \cos^2 \theta \sin^2 \theta.$$

Because $0 \leq \cos^2 \theta \sin^2 \theta \leq 1/4$, its absolute value is at most $r^2/4 \rightarrow 0$, uniformly in direction.

Question 5

For

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq (0, 0),$$

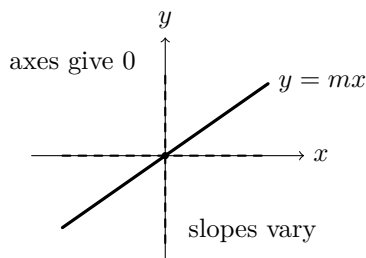
compare its two iterated limits with its joint limit at the origin.

Tasks

1. Compute both iterated limits.
2. Test the joint limit along $y = mx$.
3. Explain why the conclusions differ.

Question 5 – Solution

Strategy. Iterated limits approach along coordinate-aligned stages; the joint limit must agree along every path.



Step 1: Iterated limits For fixed $y \neq 0$, $\lim_{x \rightarrow 0} xy/(x^2 + y^2) = 0$, so

$$\lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) = 0.$$

Similarly, for fixed $x \neq 0$, the inner y -limit is 0, giving

$$\lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = 0.$$

Step 2: Joint paths Along $y = mx$,

$$f(x, mx) = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}.$$

This is 0 for $m = 0$ but $1/2$ for $m = 1$.

Conclusion. Both iterated limits equal 0, yet the joint limit does not exist. Iterated limits inspect only two staged approaches and cannot control arbitrary simultaneous approaches.

Question 6

Let

$$D = \{(x, y) : x \geq 0, 0 \leq y \leq \sqrt{3}x\}, \quad f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

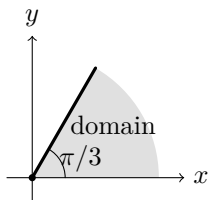
Find the limit of f as $(x, y) \rightarrow (0, 0)$ within D , or prove it does not exist.

Tasks

1. Describe the allowed directions.
2. Compare two valid boundary paths.
3. State the relative-limit conclusion.

Question 6 – Solution

Strategy. A relative limit uses only paths in the domain, but all such paths must still agree.



Step 1: Directions The wedge is $0 \leq \theta \leq \pi/3$. In polar form,

$$f = \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2} = \cos 2\theta.$$

Step 2: Boundary paths Along $y = 0$ ($\theta = 0$), $f = 1$. Along $y = \sqrt{3}x$ ($\theta = \pi/3$),

$$f = \cos(2\pi/3) = -\frac{1}{2}.$$

Both paths stay in D and approach the origin.

Conclusion. Because the two permitted approaches disagree,

$\lim_{\substack{(x,y) \rightarrow (0,0) \\ (x,y) \in D}} f(x,y) \text{ does not exist.}$

Question 7

Evaluate

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right).$$

Tasks

1. Explain why the oscillatory factor has no limit.
2. Bound the full expression.
3. Prove the limit despite the oscillation.

Question 7 – Solution

Strategy. The sine factor is bounded even though it oscillates; the radial factor forces the product to zero.

Step 1: Absolute bound Since $|\sin u| \leq 1$,

$$\left| (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) \right| \leq x^2 + y^2.$$

Step 2: Squeeze As $(x, y) \rightarrow (0, 0)$, the right side tends to 0. Therefore

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) = 0.$$

Step 3: Why oscillation is harmless Along radii for which $1/(x^2 + y^2)$ hits alternating sine peaks, the sine values do not settle. Nevertheless, the product's magnitude never exceeds the shrinking radius squared, so every path is controlled by the same bound.

Question 8

Prove directly from the ε - δ definition that

$$\lim_{(x,y) \rightarrow (1,-2)} (3x - 4y) = 11.$$

Tasks

1. Rewrite the output error.
2. Bound it by the Euclidean input distance.
3. Give an explicit valid choice of δ .

Question 8 – Solution

Strategy. Use Cauchy–Schwarz to compare the linear output change with the distance to $(1, -2)$.

Step 1: Error Let $h = x - 1$ and $k = y + 2$. Then

$$|(3x - 4y) - 11| = |3h - 4k|.$$

Step 2: Cauchy–Schwarz

$$|3h - 4k| \leq \sqrt{3^2 + (-4)^2} \sqrt{h^2 + k^2} = 5\sqrt{(x - 1)^2 + (y + 2)^2}.$$

Step 3: Choose δ Given $\varepsilon > 0$, set

$$\boxed{\delta = \varepsilon/5}.$$

If $0 < \sqrt{(x - 1)^2 + (y + 2)^2} < \delta$, then

$$|(3x - 4y) - 11| < 5\delta = \varepsilon.$$

This is exactly the two-variable limit definition, so the claimed limit is proved.

Question 9

Suppose (a, b) lies strictly inside the disk $x^2 + y^2 < 25$. Prove

$$\lim_{(x,y) \rightarrow (a,b)} \sqrt{25 - x^2 - y^2} = \sqrt{25 - a^2 - b^2},$$

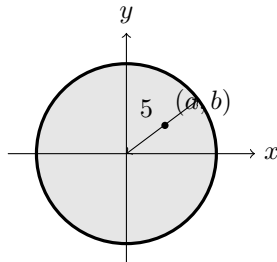
and give a concrete neighborhood that remains inside the domain.

Tasks

1. Find a positive safety margin to the boundary.
2. Choose a disk around (a, b) contained in the domain.
3. Justify the limit by composition.

Question 9 – Solution

Strategy. Use the triangle inequality to keep nearby points inside the radius-5 disk, then apply continuity.



Step 1: Margin Let $s = \sqrt{a^2 + b^2} < 5$. The radial margin is $5 - s > 0$.

Step 2: Neighborhood Choose

$$\delta_0 = \frac{5 - s}{2}.$$

If $\sqrt{(x - a)^2 + (y - b)^2} < \delta_0$, then by the triangle inequality

$$\sqrt{x^2 + y^2} \leq s + \sqrt{(x - a)^2 + (y - b)^2} < s + \frac{5 - s}{2} < 5.$$

Thus this entire neighborhood lies in the square-root domain.

Step 3: Composition The polynomial $25 - x^2 - y^2$ is continuous and remains positive there; $\sqrt{\cdot}$ is continuous on positive inputs. Their composition is continuous at (a, b) , proving the boxed limit stated in the question.

Question 10

Define

$$F(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ c, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Determine $\lim_{(x,y) \rightarrow (0,0)} F(x, y)$ using polar coordinates.
2. Find the unique value of c making F continuous at the origin.
3. Verify the estimate uniformly in angle.

Question 10 – Solution

Strategy. Recognize the cubic numerator's polar form and isolate one factor of radial distance.

Step 1: Convert With $x = r \cos \theta$ and $y = r \sin \theta$,

$$x^3 - 3xy^2 = r^3(\cos^3 \theta - 3 \cos \theta \sin^2 \theta) = r^3 \cos 3\theta.$$

Since $x^2 + y^2 = r^2$, for $r > 0$,

$$F(x, y) = r \cos 3\theta.$$

Step 2: Uniform estimate For every angle,

$$|F(x, y)| = r |\cos 3\theta| \leq r = \sqrt{x^2 + y^2} \rightarrow 0.$$

Hence

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} F(x, y) = 0}.$$

Step 3: Continuity value Continuity requires the assigned value to equal the limit, so the unique choice is

$$\boxed{c = 0}.$$

The bound is independent of θ , verifying that no directional or curved approach can change the result.

Question 1

Let

$$f(x, y) = 4x^3y^2 - 5x^2y + 7y^3 - 6.$$

Tasks

1. Find f_x and f_y .
2. Evaluate both at $(1, -2)$.
3. Verify that each differentiation treats the other variable as constant.

Question 1 – Solution

Strategy. Differentiate term by term, holding the non-differentiated variable fixed.

Step 1: Differentiate with respect to x

$$f_x = 12x^2y^2 - 10xy.$$

The terms $7y^3$ and -6 are constant with respect to x .

Step 2: Differentiate with respect to y

$$f_y = 8x^3y - 5x^2 + 21y^2.$$

Step 3: Evaluate

$$f_x(1, -2) = 12(1)(4) - 10(1)(-2) = \boxed{68},$$

$$f_y(1, -2) = 8(1)(-2) - 5(1) + 21(4) = \boxed{63}.$$

Verification. The x -derivative lowers only powers of x ; the y -derivative lowers only powers of y . Substitution gives $48 + 20 = 68$ and $-16 - 5 + 84 = 63$.

Question 2

For

$$g(x, y) = \frac{x^2 + 3y}{2x - y},$$

find its first partial derivatives on its domain.

Tasks

1. State the domain restriction.
2. Compute g_x and g_y using the quotient rule.
3. Simplify and check the denominators.

Question 2 – Solution

Strategy. Apply the one-variable quotient rule separately in each variable and preserve the original domain.

Step 1: Domain The function and its partial derivatives are defined where

$$\boxed{2x - y \neq 0}.$$

Step 2: x -partial Holding y fixed,

$$\begin{aligned} g_x &= \frac{(2x)(2x - y) - 2(x^2 + 3y)}{(2x - y)^2} \\ &= \boxed{\frac{2x^2 - 2xy - 6y}{(2x - y)^2}}. \end{aligned}$$

Step 3: y -partial Holding x fixed,

$$\begin{aligned} g_y &= \frac{3(2x - y) - (x^2 + 3y)(-1)}{(2x - y)^2} \\ &= \boxed{\frac{x^2 + 6x}{(2x - y)^2}}. \end{aligned}$$

Verification. Both results have the squared quotient-rule denominator and remain restricted to $2x - y \neq 0$; expanding each unsimplified numerator reproduces the displayed form.

Question 3

Let

$$h(x, y) = e^{xy} \sin(x + y).$$

Tasks

1. Compute h_x and h_y .
2. Evaluate both at $(0, \pi/2)$.
3. Identify the product-rule contribution that vanishes at the point.

Question 3 – Solution

Strategy. Use the product rule, differentiating each composite factor with only one variable active.

Step 1: x -partial

$$h_x = ye^{xy} \sin(x+y) + e^{xy} \cos(x+y) = e^{xy} [y \sin(x+y) + \cos(x+y)].$$

Step 2: y -partial

$$h_y = xe^{xy} \sin(x+y) + e^{xy} \cos(x+y) = e^{xy} [x \sin(x+y) + \cos(x+y)].$$

Step 3: Evaluate At $(0, \pi/2)$, $e^{xy} = 1$, $\sin(x+y) = 1$, and $\cos(x+y) = 0$. Therefore

$$\boxed{h_x(0, \pi/2) = \pi/2}, \quad \boxed{h_y(0, \pi/2) = 0}.$$

The cosine contribution vanishes in both partials; the $xe^{xy} \sin(x+y)$ contribution in h_y also vanishes because $x = 0$.

Question 4

On the domain $x > 0$, define

$$p(x, y) = x^y.$$

Tasks

1. Find p_x by treating y as constant.
2. Find p_y using an exponential representation.
3. Evaluate both partial derivatives at $(e, 2)$ and verify their forms.

Question 4 – Solution

Strategy. Use the ordinary power rule for x and write $x^y = e^{y \ln x}$ when differentiating with respect to the exponent.

Step 1: x -partial With y fixed,

$$p_x(x, y) = yx^{y-1}.$$

Step 2: y -partial Since $p = e^{y \ln x}$ and $\ln x$ is constant with respect to y ,

$$p_y(x, y) = x^y \ln x.$$

Step 3: Evaluate

$$p_x(e, 2) = 2e, \quad p_y(e, 2) = e^2 \ln e = e^2.$$

Thus

$$p_x(e, 2) = 2e, \quad p_y(e, 2) = e^2.$$

Verification. For fixed $y = 2$, $p = x^2$ has derivative $2x$. For fixed $x = e$, $p = e^y$ has derivative e^y , confirming both values.

Question 5

Let

$$F(x, y, z) = x^2yz + \ln(x + z) - ye^z, \quad x + z > 0.$$

Tasks

1. Compute F_x , F_y , and F_z .
2. Evaluate all three at $(1, 2, 0)$.
3. Check which terms are constant in each differentiation.

Question 5 – Solution

Strategy. Differentiate with one variable active at a time and keep the logarithm's domain restriction.

Step 1: Partial derivatives

$$F_x = 2xyz + \frac{1}{x+z},$$

$$F_y = x^2z - e^z,$$

$$F_z = x^2y + \frac{1}{x+z} - ye^z.$$

Step 2: Evaluate At $(1, 2, 0)$,

$$F_x = 0 + 1 = \boxed{1}, \quad F_y = 0 - 1 = \boxed{-1},$$

$$F_z = 2 + 1 - 2 = \boxed{1}.$$

Verification. In F_x , $-ye^z$ is constant; in F_y , $\ln(x+z)$ is constant; in F_z , every term varies with z . The point satisfies $x+z=1>0$.

Question 6

The equation

$$x^2 + y^2 + z^2 + xyz = 4$$

defines z locally as a function of x and y near $P = (1, 1, 1)$.

Tasks

1. Find z_x and z_y by partial implicit differentiation.
2. Evaluate them at P .
3. State the nonzero factor that permits solving for the derivatives.

Question 6 – Solution

Strategy. Differentiate the equation while treating $z = z(x, y)$, then collect the terms containing the requested derivative.

Step 1: Differentiate in x

$$2x + 2zz_x + yz + xyz_x = 0,$$

so

$$z_x = -\frac{2x + yz}{2z + xy}.$$

Step 2: Differentiate in y

$$2y + 2zz_y + xz + xyz_y = 0,$$

so

$$z_y = -\frac{2y + xz}{2z + xy}.$$

Step 3: Evaluate At $(1, 1, 1)$, the common denominator is $2z + xy = 3 \neq 0$. Hence

$$z_x(1, 1) = -1, \quad z_y(1, 1) = -1.$$

Substitution into the differentiated equations gives $2 + 2(-1) + 1 + (-1) = 0$ in each case.

Question 7

For the surface

$$z = f(x, y) = x^2 + xy + y^2,$$

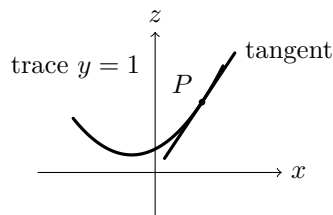
hold $y = 1$ fixed and examine the point $P = (1, 1, 3)$.

Tasks

1. Find $f_x(1, 1)$.
2. Obtain the tangent line to the trace $y = 1$ at P .
3. Verify the slope by differentiating the trace directly.

Question 7 – Solution

Strategy. Restrict the surface to the vertical plane $y = 1$; the resulting one-variable curve has slope $f_x(1, 1)$.



Step 1: Partial derivative

$$f_x = 2x + y, \quad \boxed{f_x(1, 1) = 3}.$$

Step 2: Trace and tangent At $y = 1$,

$$z = x^2 + x + 1.$$

The tangent line in the plane $y = 1$ passes through $(x, z) = (1, 3)$ with slope 3:

$$\boxed{y = 1, \quad z - 3 = 3(x - 1)}.$$

Step 3: Direct check Differentiating the trace gives $dz/dx = 2x + 1$, which equals 3 at $x = 1$, matching the partial derivative.

Question 8

Determine whether constants a , b , and c exist such that

$$q(x, y) = ax^2 + bxy + cy^2$$

satisfies

$$q_x(1, 1) = 7, \quad q_y(1, 1) = 8, \quad q(1, 1) = 6.$$

Tasks

1. Translate the data into a linear system.
2. Solve for a, b, c or prove inconsistency.
3. Verify your conclusion against all three conditions.

Question 8 – Solution

Strategy. Compute the partial derivatives symbolically, evaluate at $(1, 1)$, and solve the resulting three equations.

Step 1: Equations

$$q_x = 2ax + by, \quad q_y = bx + 2cy.$$

Thus

$$2a + b = 7, \quad b + 2c = 8, \quad a + b + c = 6.$$

Step 2: Solve From the first two equations,

$$a = \frac{7-b}{2}, \quad c = \frac{8-b}{2}.$$

Substitution into $a + b + c = 6$ gives

$$\frac{7-b}{2} + b + \frac{8-b}{2} = \frac{15}{2},$$

which cannot equal 6. Therefore the supplied data are inconsistent.

Conclusion.

No constants a, b, c satisfy all three conditions.

Verification. In fact, adding the two derivative equations gives $2(a + b + c) = 15$, forcing $q(1, 1) = 15/2$, not 6.

Question 9

Define

$$u(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $u_x(0, 0)$ from the definition.
2. Compute $u_y(0, 0)$ from the definition.
3. Compute formulas for u_x, u_y away from the origin and explain why they do not determine the values at the origin.

Question 9 – Solution

Strategy. At the exceptional point, use one-variable difference quotients along the coordinate axes.

Step 1: x -partial at the origin

$$u_x(0,0) = \lim_{h \rightarrow 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \boxed{0}.$$

Step 2: y -partial at the origin

$$u_y(0,0) = \lim_{k \rightarrow 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \rightarrow 0} \frac{0}{k} = \boxed{0}.$$

Step 3: Away from the origin Quotient differentiation gives

$$\boxed{u_x = \frac{2xy^3}{(x^2 + y^2)^2}}, \quad \boxed{u_y = \frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2}} \quad ((x,y) \neq (0,0)).$$

These formulas were derived where the denominator is nonzero, so substituting $(0,0)$ would be invalid. The definition supplies the missing partial values.

Question 10

Define

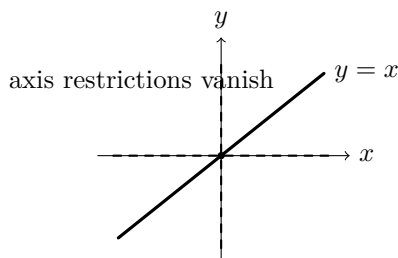
$$v(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Show that both first partial derivatives exist at the origin.
2. Test continuity along $y = x$.
3. Explain what this example proves about partial derivatives and continuity.

Question 10 – Solution

Strategy. Partial derivatives inspect axis restrictions only; compare those restrictions with a diagonal approach.



Step 1: Axis difference quotients Since $v(h, 0) = 0$ and $v(0, k) = 0$,

$$v_x(0, 0) = \lim_{h \rightarrow 0} \frac{v(h, 0) - v(0, 0)}{h} = 0,$$

$$v_y(0, 0) = \lim_{k \rightarrow 0} \frac{v(0, k) - v(0, 0)}{k} = 0.$$

Thus

$$\boxed{v_x(0, 0) = v_y(0, 0) = 0}.$$

Step 2: Diagonal path Along $y = x$ with $x \neq 0$,

$$v(x, x) = \frac{x^2}{2x^2} = \frac{1}{2}.$$

Therefore $\lim_{x \rightarrow 0} v(x, x) = 1/2 \neq v(0, 0) = 0$.

Conclusion. The function is not continuous at the origin even though both first partial derivatives exist there. Hence

$$\boxed{\text{existence of } v_x, v_y \text{ at a point does not imply continuity there.}}$$

Question 1

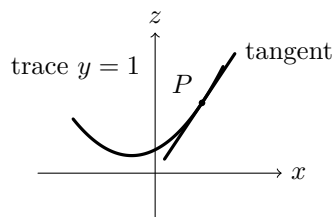
For $f(x, y) = x^2 + 3xy - y^2$ at $P = (1, 2, 3)$:

Tasks

1. Find f_x, f_y at $(1, 2)$.
2. Interpret them as trace slopes.
3. Write both trace tangent lines.

Question 1 – Solution

Strategy. Hold one input fixed at a time.



Step 1: Rates $f_x = 2x + 3y$, $f_y = 3x - 2y$, hence $\boxed{f_x(1, 2) = 8}$ and $\boxed{f_y(1, 2) = -1}$.

Step 2: Meaning With $y = 2$, height rises 8 units per x unit; with $x = 1$, it falls 1 unit per y unit.

Step 3: Lines $\boxed{y = 2, z - 3 = 8(x - 1)}$ and $\boxed{x = 1, z - 3 = -(y - 2)}$.

Question 2

Temperature is $T(x, y) = 80 - 2x^2 - y^2$ degrees Celsius, with distance in centimeters.

Tasks

1. Find $T_x(3, 4), T_y(3, 4)$ with units.
2. Interpret signs and magnitudes.
3. Estimate separate changes for 0.05 cm moves in $+x$ and $+y$.

Question 2 – Solution

Strategy. Read each partial as temperature change per centimeter with the other coordinate fixed.

Step 1: Rates $T_x = -4x$, $T_y = -2y$, so $T_x = -12\text{ }^\circ\text{C/cm}$ and $T_y = -8\text{ }^\circ\text{C/cm}$ at $(3, 4)$.

Step 2: Meaning Both directions cool; the x trace is locally steeper.

Step 3: Estimates $\Delta T_x \approx -12(.05) = -0.60^\circ\text{C}$ and $\Delta T_y \approx -8(.05) = -0.40^\circ\text{C}$.

Question 3

Revenue is $R(p, a) = 1200p - 30p^2 + 50a - 2a^2 + 4pa$, where p is dollars and a is hundreds of advertising dollars.

Tasks

1. Compute $R_p(10, 5), R_a(10, 5)$.
2. State units.
3. Interpret both values.

Question 3 – Solution

Strategy. Attach output units per input unit.

Step 1: Compute $R_p = 1200 - 60p + 4a$ and $R_a = 50 - 4a + 4p$, so $\boxed{R_p(10, 5) = 620}$ and $\boxed{R_a(10, 5) = 70}$.

Step 2: Units These are revenue dollars per price dollar and revenue dollars per hundred advertising dollars.

Step 3: Meaning Locally, a \$1 price rise predicts \$620 more revenue; an added \$100 advertising predicts \$70 more, with the other input fixed.

Question 4

Pressure is $P(V, T) = 8T/V$ kPa, with V in liters and T in kelvins.

Tasks

1. Find $P_V(4, 300)$, $P_T(4, 300)$.
2. Supply units.
3. Interpret each sign.

Question 4 – Solution

Strategy. Vary one physical input while fixing the other.

Step 1: Volume $P_V = -8T/V^2$, so $P_V(4, 300) = -150 \text{ kPa/L}$.

Step 2: Temperature $P_T = 8/V$, so $P_T(4, 300) = 2 \text{ kPa/K}$.

Step 3: Meaning Added volume lowers pressure locally; added temperature raises it. The statements are instantaneous and hold the other variable fixed.

Question 5

At a terrain point, $f_x = -0.25$ and $f_y = 0.60$, with all distances in meters.

Tasks

1. Interpret both as rise/run.
2. Find signed trace angles.
3. Identify the steeper trace.

Question 5 – Solution

Strategy. A trace slope is the tangent of its signed inclination.

Step 1: Rise/run A one-meter positive x move predicts -0.25 m elevation change; a positive y move predicts 0.60 m, separately.

Step 2: Angles $\alpha_x = \arctan(-.25) \approx -14.0^\circ$ and $\alpha_y = \arctan(.60) \approx 31.0^\circ$.

Step 3: Compare Since $.60 > .25$, the y trace is steeper; its positive sign indicates ascent.

Question 6

Near $(2, 3)$, the columns give x and the rows give y :

	1.9	2.0	2.1
2.9	7.62	7.80	7.98
3.0	7.71	7.90	8.09
3.1	7.80	8.00	8.20

Tasks

1. Estimate $f_x(2, 3)$ centrally.
2. Estimate $f_y(2, 3)$ centrally.
3. Interpret both.

Question 6 – Solution

Strategy. Use the center row and column.

Step 1: x rate $f_x \approx (8.09 - 7.71)/.2 = \boxed{1.90}$.

Step 2: y rate $f_y \approx (8.00 - 7.80)/.2 = \boxed{1.00}$.

Step 3: Meaning Output rises about 1.90 per x unit with y fixed and 1.00 per y unit with x fixed. Symmetric differences reduce one-sided bias.

Question 7

Adjacent contours differ by 6, with labels increasing in both positive coordinate directions. Along $+x$ they are 2 units apart; along $+y$ they are 3 units apart.

Tasks

1. Estimate both partial derivatives.
2. Relate spacing to magnitude.
3. Explain how decreasing labels change signs.

Question 7 – Solution

Strategy. Divide contour-value change by coordinate distance.

Step 1: Estimates $f_x \approx 6/2 = 3$ and $f_y \approx 6/3 = 2$.

Step 2: Spacing Equal output change over less distance gives larger slope, so the x trace is steeper.

Step 3: Sign If labels decrease in a positive direction, the corresponding partial is negative. Spacing controls magnitude; label order controls sign.

Question 8

Cylinder volume is $V(r, h) = \pi r^2 h$.

Tasks

1. Interpret V_r, V_h geometrically.
2. Evaluate at $(3, 10)$ cm.
3. Estimate separate effects of 0.1 cm increases.

Question 8 – Solution

Strategy. Connect derivatives to familiar areas.

Step 1: Geometry $V_r = 2\pi rh$ is circumference times height; $V_h = \pi r^2$ is base area.

Step 2: Values $V_r = 60\pi \text{ cm}^2$, $V_h = 9\pi \text{ cm}^2$.

Step 3: Estimates $\Delta V_r \approx 6\pi \text{ cm}^3$ and $\Delta V_h \approx 0.9\pi \text{ cm}^3$. The radial change has the larger first-order effect.

Question 9

At one point a concentration field has $C_x = 4$, $C_y = 0$, $C_t = -1.5$ in appropriate mg/L per meter or minute units.

Tasks

1. Interpret each.
2. Explain what $C_y = 0$ does not imply.
3. Predict separate changes for .2 m in $+x$, .2 m in $+y$, and 2 minutes.

Question 9 – Solution

Strategy. Preserve held-fixed conditions and units.

Step 1: Meaning Concentration rises 4 mg/L per x meter, has zero instantaneous y slope, and falls 1.5 mg/L per minute.

Step 2: Caution $C_y = 0$ at one point does not mean independence from y nearby.

Step 3: Predictions $\Delta C_x \approx .8 \text{ mg/L}$, $\Delta C_y \approx 0$, and $\Delta C_t \approx -3.0 \text{ mg/L}$.

Question 10

At the same input, models satisfy $A_x = 5$, $A_y = -2$ and $B_x = -1$, $B_y = -2$.

Tasks

1. Compare positive- x responses.
2. Compare positive- y responses.
3. State what cannot be concluded from these data.

Question 10 – Solution

Strategy. Partial derivatives give local coordinate-specific rates.

Step 1: x A locally rises at 5 per unit; B falls at 1 per unit. Their signs oppose and A has greater absolute sensitivity.

Step 2: y Both locally fall at 2 per positive y unit.

Step 3: Limits We cannot infer $A > B$, global monotonicity, or exact finite changes without more information.

The data justify local rate comparisons only.

Question 1

For

$$f(x, y) = x^4y^2 - 3x^2y^3 + 2xy,$$

Tasks

1. Find $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.
2. Evaluate them at $(1, -1)$.
3. Verify the mixed-partial equality algebraically.

Question 1 – Solution

Strategy. Compute both first partials before differentiating again.

Step 1: First partials

$$f_x = 4x^3y^2 - 6xy^3 + 2y, \quad f_y = 2x^4y - 9x^2y^2 + 2x.$$

Step 2: Second partials

$$\boxed{f_{xx} = 12x^2y^2 - 6y^3}, \quad \boxed{f_{xy} = 8x^3y - 18xy^2 + 2},$$
$$\boxed{f_{yx} = 8x^3y - 18xy^2 + 2}, \quad \boxed{f_{yy} = 2x^4 - 18x^2y}.$$

Step 3: Evaluate

$$\boxed{f_{xx}(1, -1) = 18}, \quad \boxed{f_{xy}(1, -1) = f_{yx}(1, -1) = -24}, \quad \boxed{f_{yy}(1, -1) = 20}.$$

The two mixed formulas coincide term by term, as expected for a polynomial.

Question 2

Let $g(x, y) = e^{xy} \cos y$.

Tasks

1. Compute g_{xx} and g_{xy} .
2. Compute g_{yx} independently from g_y .
3. Evaluate them at $(0, \pi)$ and verify agreement.

Question 2 – Solution

Strategy. Keep product-rule terms grouped by their exponential factor.

Step 1: Differentiate from g_x Since $g_x = ye^{xy} \cos y$,

$$\boxed{g_{xx} = y^2 e^{xy} \cos y},$$

$$\boxed{g_{xy} = e^{xy} [(1 + xy) \cos y - y \sin y]}.$$

Step 2: Reverse order

$$g_y = e^{xy} (x \cos y - \sin y),$$

$$g_{yx} = ye^{xy} (x \cos y - \sin y) + e^{xy} \cos y = \boxed{e^{xy} [(1 + xy) \cos y - y \sin y]}.$$

Step 3: Evaluate At $(0, \pi)$, $e^{xy} = 1$, $\cos \pi = -1$, and $\sin \pi = 0$:

$$\boxed{g_{xx} = -\pi^2}, \quad \boxed{g_{xy} = g_{yx} = -1}.$$

Question 3

For $u(x, y) = \ln(x^2 + y^2)$ on its natural domain:

Tasks

1. Find all second partial derivatives.
2. Show that $u_{xx} + u_{yy} = 0$.
3. Explain why the result cannot be extended to the origin by substitution.

Question 3 – Solution

Strategy. Differentiate the rational first partials and preserve the punctured-plane domain.

Step 1: First partials

$$u_x = \frac{2x}{x^2 + y^2}, \quad u_y = \frac{2y}{x^2 + y^2}.$$

Step 2: Second partials

$$\boxed{u_{xx} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}}, \quad \boxed{u_{yy} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}},$$
$$\boxed{u_{xy} = u_{yx} = -\frac{4xy}{(x^2 + y^2)^2}}.$$

Step 3: Cancellation

$$u_{xx} + u_{yy} = \frac{2(y^2 - x^2 + x^2 - y^2)}{(x^2 + y^2)^2} = \boxed{0}.$$

The origin is excluded from u , and every denominator above vanishes there; substitution cannot create a derivative at a point outside the domain.

Question 4

Find the third-order derivative f_{xxy} for

$$f(x, y) = x^2 e^{xy}.$$

Tasks

1. Compute it in the encoded order.
2. Compute f_{yxx} independently.
3. Explain the agreement and evaluate at $(1, 0)$.

Question 4 – Solution

Strategy. Follow each subscript successively and retain all product-rule terms.

Step 1: Route x, x, y

$$\begin{aligned}f_x &= e^{xy}(2x + x^2y), \\f_{xx} &= e^{xy}(2 + 4xy + x^2y^2), \\f_{xxy} &= e^{xy}(6x + 6x^2y + x^3y^2).\end{aligned}$$

Step 2: Route y, x, x

$$\begin{aligned}f_y &= x^3e^{xy}, \quad f_{yx} = e^{xy}(3x^2 + x^3y), \\f_{yxx} &= e^{xy}(6x + 6x^2y + x^3y^2).\end{aligned}$$

Step 3: Verify Polynomial and exponential factors are smooth, so the derivative order may be permuted. At $(1, 0)$ both expressions equal $\boxed{6}$.

Question 5

A proposed Hessian is

$$H_f(x, y) = \begin{pmatrix} 6x - 2y & -2x + 4y \\ -2x + 4y & 4x + 6y \end{pmatrix}.$$

Tasks

1. Check its compatibility conditions.
2. Recover one possible $f(x, y)$.
3. Describe every function with this Hessian.

Question 5 – Solution

Strategy. Integrate f_{xx} and use the other entries to determine the unknown functions of integration.

Step 1: Compatibility The off-diagonal entries agree, and

$$\partial_y f_{xx} = -2 = \partial_x f_{xy}, \quad \partial_y f_{xy} = 4 = \partial_x f_{yy}.$$

Step 2: Reconstruct

$$f_x = 3x^2 - 2xy + A(y).$$

Since $f_{xy} = -2x + A'(y) = -2x + 4y$, $A(y) = 2y^2 + C$. Integrating in x ,

$$f = x^3 - x^2y + 2xy^2 + Cx + B(y).$$

The condition $f_{yy} = 4x + B''(y) = 4x + 6y$ gives $B = y^3 + Dy + E$.

Result

$$\boxed{f = x^3 - x^2y + 2xy^2 + y^3 + Cx + Dy + E}.$$

Conversely, direct differentiation shows every choice of C, D, E has exactly the proposed Hessian.

Question 6

Assume the mixed partials of f are continuous near (a, b) . Define

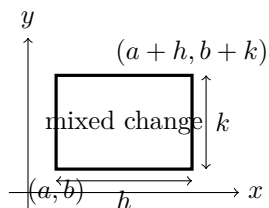
$$D(h, k) = \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}.$$

Tasks

1. Interpret the numerator as a rectangular mixed change.
2. Apply the mean value theorem in each variable.
3. Find $\lim_{(h,k) \rightarrow (0,0)} D(h, k)$.

Question 6 – Solution

Strategy. Apply one-variable mean value arguments successively across the small rectangle.



Step 1: Difference structure Let

$$G(y) = f(a + h, y) - f(a, y).$$

The numerator is $G(b + k) - G(b)$.

Step 2: Mean value steps For some η between b and $b + k$,

$$G(b + k) - G(b) = kG'(\eta) = k[f_y(a + h, \eta) - f_y(a, \eta)].$$

For some ξ between a and $a + h$, the bracket equals $hf_{yx}(\xi, \eta)$. Hence

$$D(h, k) = f_{yx}(\xi, \eta).$$

Step 3: Limit As $(h, k) \rightarrow (0, 0)$, $(\xi, \eta) \rightarrow (a, b)$. Continuity and equality of mixed partials give

$$\boxed{\lim D(h, k) = f_{yx}(a, b) = f_{xy}(a, b)}.$$

Question 7

For $w(x, y, z) = x^2yz^3 + \sin(xz)$:

Tasks

1. Find w_{xyz} .
2. Find w_{zzx} .
3. Evaluate both at $(1, 2, 0)$ and discuss derivative order.

Question 7 – Solution

Strategy. Follow the subscripts and keep polynomial and trigonometric contributions separate.

Step 1: w_{xyz}

$$w_x = 2xyz^3 + z \cos(xz), \quad w_{xy} = 2xz^3, \quad \boxed{w_{xyz} = 6xz^2}.$$

Step 2: w_{zzx}

$$\begin{aligned} w_z &= 3x^2yz^2 + x \cos(xz), \\ w_{zz} &= 6x^2yz - x^2 \sin(xz), \\ \boxed{w_{zzx} &= 12xyz - 2x \sin(xz) - x^2 z \cos(xz)}. \end{aligned}$$

Step 3: Evaluate Both happen to equal $\boxed{0}$ at $(1, 2, 0)$. Smoothness permits reordering derivatives that contain the same multiset of variables; xyz and zzx are different derivative types and need not agree generally.

Question 8

Let $f(x, y) = x^2y^2/(x^2 + y^2)$ away from $(0, 0)$ and $f(0, 0) = 0$.

Tasks

1. Find $f_x(0, y)$ and $f_y(x, 0)$ from definitions.
2. Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.
3. Explain why axis data suffice here.

Question 8 – Solution

Strategy. Restrict first derivatives to an axis, then differentiate that restriction at zero.

Step 1: Axis first partials For $y \neq 0$,

$$f_x(0, y) = \lim_{h \rightarrow 0} \frac{h^2 y^2 / (h^2 + y^2)}{h} = \lim_{h \rightarrow 0} \frac{h y^2}{h^2 + y^2} = 0.$$

At $y = 0$ it is also zero. By symmetry, $f_y(x, 0) = 0$ for every x .

Step 2: Mixed values

$$f_{xy}(0, 0) = \lim_{k \rightarrow 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = 0,$$

$$f_{yx}(0, 0) = \lim_{h \rightarrow 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = 0.$$

Thus $\boxed{f_{xy}(0, 0) = f_{yx}(0, 0) = 0}$.

Verification. Each definition uses only the relevant axis restriction, so off-axis values do not enter these two particular quotients.

Question 9

Suppose u satisfies $u_{tt} = 9u_{xx}$ and

$$u(x, t) = A \cos(2x) \cos(\omega t).$$

Tasks

1. Compute u_{tt} and u_{xx} .
2. Determine all real ω for nonzero A .
3. Verify the result and analyze the $A = 0$ edge case.

Question 9 – Solution

Strategy. Substitute the separated trial form and compare its common factors.

Step 1: Derivatives

$$u_{tt} = -A\omega^2 \cos(2x) \cos(\omega t), \quad u_{xx} = -4A \cos(2x) \cos(\omega t).$$

Step 2: Match For $A \neq 0$, equality for every (x, t) requires

$$-\omega^2 = 9(-4), \quad \omega^2 = 36,$$

so $\boxed{\omega = \pm 6}$.

Step 3: Verify Directly, $u_{tt} = -36u = 9(-4u) = 9u_{xx}$. Because $\cos(-6t) = \cos(6t)$, the signs produce the same function. If $A = 0$, the zero function satisfies the equation for every ω .

Question 10

Find all real constants a, b, c for which

$$u(x, y) = e^{ax+by} \cos(cx)$$

satisfies $u_{xx} + u_{yy} = 0$ everywhere.

Tasks

1. Compute the pure second partials.
2. Separate sine and cosine coefficients.
3. Solve all conditions, including $c = 0$.

Question 10 – Solution

Strategy. Equality for every x forces independent trigonometric coefficients to vanish.

Step 1: Differentiate

$$\begin{aligned}u_{xx} &= e^{ax+by}[(a^2 - c^2)\cos(cx) - 2ac\sin(cx)], \\u_{yy} &= b^2 e^{ax+by}\cos(cx).\end{aligned}$$

Thus the sum is

$$e^{ax+by}[(a^2 + b^2 - c^2)\cos(cx) - 2ac\sin(cx)].$$

Step 2: Nonzero c If $c \neq 0$, independence of sine and cosine gives

$$a^2 + b^2 - c^2 = 0, \quad ac = 0.$$

Thus $a = 0$ and $b = \pm c$.

Step 3: Edge case If $c = 0$, the condition is $a^2 + b^2 = 0$, hence $a = b = 0$. This is already included below:

$$(a, b, c) = (0, c, c) \text{ or } (0, -c, c), \quad c \in \mathbb{R}.$$

Substitution makes both coefficients vanish, verifying sufficiency.

Question 1

For

$$z = x^2 e^y + \ln(x + y), \quad x + y > 0,$$

Tasks

1. Find the total differential dz .
2. Evaluate it at $(1, 1)$ for $dx = -0.02$, $dy = 0.03$.
3. Identify the contribution from each input change.

Question 1 – Solution

Strategy. Use $dz = f_x dx + f_y dy$ and evaluate coefficients before applying the increments.

Step 1: Partial derivatives

$$f_x = 2xe^y + \frac{1}{x+y}, \quad f_y = x^2e^y + \frac{1}{x+y}.$$

Thus

$$dz = \left(2xe^y + \frac{1}{x+y} \right) dx + \left(x^2e^y + \frac{1}{x+y} \right) dy.$$

Step 2: Evaluate At $(1, 1)$,

$$dz = (2e + \tfrac{1}{2})(-0.02) + (e + \tfrac{1}{2})(0.03).$$

Combining terms,

$$dz = -0.04e - 0.01 + 0.03e + 0.015 = \boxed{0.005 - 0.01e \approx -0.02218}.$$

Step 3: Contributions The x change contributes $-(0.04e + 0.01)$; the y change contributes $0.03e + 0.015$. Their signs oppose, and the x contribution has larger magnitude.

Question 2

Use differentials to estimate

$$(4.02)^2\sqrt{8.97}.$$

Tasks

1. Choose a two-variable function and nearby convenient base point.
2. Compute the differential estimate.
3. State the estimated value and identify the neglected order of error.

Question 2 – Solution

Strategy. Let $f(x, y) = x^2\sqrt{y}$ and linearize its change from $(4, 9)$.

Step 1: Base data

$$f(4, 9) = 16(3) = 48, \quad dx = 0.02, \quad dy = -0.03.$$

$$f_x = 2x\sqrt{y}, \quad f_y = \frac{x^2}{2\sqrt{y}}.$$

At $(4, 9)$, $f_x = 24$ and $f_y = 8/3$.

Step 2: Differential

$$df = 24(0.02) + \frac{8}{3}(-0.03) = 0.48 - 0.08 = 0.40.$$

Step 3: Estimate

$$\boxed{(4.02)^2\sqrt{8.97} \approx 48 + 0.40 = 48.40}.$$

The differential keeps terms linear in dx, dy and neglects quadratic and higher-order increment effects.

Question 3

For $f(x, y) = x^2 + y^2$, start at $(3, 4)$ and use $dx = 0.1$, $dy = -0.2$.

Tasks

1. Compute df .
2. Compute the exact change Δf .
3. Derive the discrepancy $\Delta f - df$ and explain its sign.

Question 3 – Solution

Strategy. Expand the exact finite change so the first-order and second-order parts separate visibly.

Step 1: Differential

$$df = 2x \, dx + 2y \, dy = 6(0.1) + 8(-0.2) = \boxed{-1.0}.$$

Step 2: Exact change

$$\Delta f = (3.1)^2 + (3.8)^2 - (3^2 + 4^2) = 9.61 + 14.44 - 25 = \boxed{-0.95}.$$

Step 3: Discrepancy Algebraically,

$$\Delta f = 2x \, dx + 2y \, dy + (dx)^2 + (dy)^2 = df + (dx)^2 + (dy)^2.$$

Therefore

$$\boxed{\Delta f - df = 0.01 + 0.04 = 0.05}.$$

It is nonnegative because the omitted terms are squares; the upward curvature makes the exact change exceed the linear prediction.

Question 4

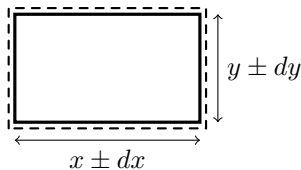
A rectangular plate has measured sides $x = 30.0$ cm and $y = 20.0$ cm, each accurate to within 0.1 cm. Its area is $A = xy$.

Tasks

1. Use differentials to estimate the maximum absolute area error.
2. Estimate the maximum relative and percentage errors.
3. Compare with the exact worst-case outward error.

Question 4 – Solution

Strategy. For a worst-case differential bound, replace signed increments by their maximum absolute values.



Step 1: Absolute error

$$dA = y \, dx + x \, dy.$$

Thus

$$|dA| \leq 20(0.1) + 30(0.1) = \boxed{5.0 \text{ cm}^2}.$$

Step 2: Relative error Since $A = 600 \text{ cm}^2$,

$$\frac{|dA|}{A} \approx \frac{5}{600} = \boxed{\frac{1}{120} \approx 0.00833},$$

or $\boxed{0.833\%}$.

Step 3: Exact comparison The largest outward area is

$$(30.1)(20.1) = 605.01,$$

so the exact positive error is 5.01 cm^2 . The differential misses only the product $dx \, dy = 0.01 \text{ cm}^2$.

Question 5

A cone has volume $V(r, h) = \frac{1}{3}\pi r^2 h$. At $r = 5$ cm, $h = 12$ cm, suppose $dr = 0.04$ cm and $dh = -0.10$ cm.

Tasks

1. Find dV exactly in terms of π .
2. Determine which measurement change dominates the estimate.
3. Compute the relative differential dV/V in two ways.

Question 5 – Solution

Strategy. Compute component contributions, then confirm the relative formula by logarithmic structure.

Step 1: Differential

$$dV = \frac{2\pi r h}{3} dr + \frac{\pi r^2}{3} dh.$$

At the given dimensions,

$$dV = 40\pi(0.04) + \frac{25\pi}{3}(-0.10) = 1.6\pi - \frac{5}{6}\pi = \boxed{\frac{23\pi}{30} \text{ cm}^3}.$$

Step 2: Compare The radial contribution is 1.6π ; the height contribution is $-5\pi/6$. The radial contribution dominates, so the net predicted change is positive.

Step 3: Relative form

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h} = 2\frac{0.04}{5} - \frac{0.10}{12} = \boxed{\frac{23}{3000}}.$$

Also $V = 100\pi$, and $(23\pi/30)/(100\pi) = 23/3000$, verifying the result.

Question 6

Points constrained to the sphere

$$x^2 + y^2 + z^2 = 49$$

undergo small changes near $(2, 3, 6)$.

Tasks

1. Derive the differential constraint relating dx, dy, dz .
2. Estimate dz when $dx = 0.03$ and $dy = -0.02$.
3. Explain geometrically why the first-order change satisfies this relation.

Question 6 – Solution

Strategy. Differentiate the constant constraint; its total differential must vanish along allowed infinitesimal changes.

Step 1: Constraint

$$2x \, dx + 2y \, dy + 2z \, dz = 0,$$

so, because $z = 6 \neq 0$,

$$\boxed{dz = -\frac{x \, dx + y \, dy}{z}}.$$

Step 2: Estimate

$$dz = -\frac{2(0.03) + 3(-0.02)}{6} = -\frac{0.06 - 0.06}{6} = \boxed{0}.$$

Step 3: Geometry The radius vector $\langle x, y, z \rangle$ is normal to the sphere. The relation

$$\langle x, y, z \rangle \cdot \langle dx, dy, dz \rangle = 0$$

says the first-order displacement is perpendicular to that normal, hence tangent to the sphere. Here the chosen horizontal change is already orthogonal to the radius projection, so no first-order vertical correction is needed.

Question 7

Determine whether

$$\omega = (2xy + 3) dx + (x^2 + 4y) dy$$

is the differential of a scalar function $f(x, y)$.

Tasks

1. Test the necessary compatibility condition.
2. Recover every possible f if the test passes.
3. Verify by redifferentiating.

Question 7 – Solution

Strategy. If $df = M dx + N dy$, then $M_y = N_x$ on a simply connected domain.

Step 1: Compatibility Here

$$M = 2xy + 3, \quad N = x^2 + 4y.$$

Then

$$M_y = 2x = N_x,$$

so the form is compatible throughout $\mathbb{R}^2$.

Step 2: Recover f Integrate $f_x = M$ in x :

$$f = x^2y + 3x + C(y).$$

Then

$$f_y = x^2 + C'(y) = x^2 + 4y,$$

so $C'(y) = 4y$ and $C(y) = 2y^2 + C_0$.

Result and verification

$$\boxed{f(x, y) = x^2y + 3x + 2y^2 + C_0}.$$

Indeed, differentiating gives $df = (2xy + 3) dx + (x^2 + 4y) dy$. The additive constant is invisible to the differential.

Question 8

A quantity is

$$Q = \frac{x^3 \sqrt{y}}{z^2}, \quad x, y, z > 0.$$

Tasks

1. Derive a relative-differential formula for dQ/Q .
2. Estimate the signed percentage change if x rises 1%, y falls 4%, and z rises 0.5%.
3. Identify the dominant contribution.

Question 8 – Solution

Strategy. Divide the total differential by Q , or equivalently differentiate the logarithmic power structure.

Step 1: Relative differential

$$\ln Q = 3 \ln x + \frac{1}{2} \ln y - 2 \ln z.$$

Differentiating gives

$$\boxed{\frac{dQ}{Q} = 3 \frac{dx}{x} + \frac{1}{2} \frac{dy}{y} - 2 \frac{dz}{z}}.$$

Step 2: Percentage estimate

$$\frac{dQ}{Q} \approx 3(0.01) + \frac{1}{2}(-0.04) - 2(0.005) = 0.03 - 0.02 - 0.01 = \boxed{0}.$$

Thus the predicted first-order percentage change is $\boxed{0\%}$.

Step 3: Contributions The x increase contributes $+3\%$, the y decrease -2% , and the z increase -1% . The x term is individually largest, but the three first-order effects cancel exactly.

Question 9

A differential field is proposed as

$$df = (y^2 + 2x) dx + (2xy + e^y) dy, \quad f(0, 0) = 5.$$

Tasks

1. Reconstruct f .
2. Use df to estimate $f(0.02, -0.01)$ from the origin.
3. Compare the estimate with the exact value through second order.

Question 9 – Solution

Strategy. Reconstruct the function first, then separate its linear and nonlinear increment terms.

Step 1: Reconstruct Integrating $f_x = y^2 + 2x$ gives

$$f = xy^2 + x^2 + C(y).$$

Since $f_y = 2xy + C'(y) = 2xy + e^y$, $C = e^y + C_0$. The initial value gives $1 + C_0 = 5$, so

$$\boxed{f = xy^2 + x^2 + e^y + 4}.$$

Step 2: Differential estimate At $(0, 0)$,

$$f_x = 0, \quad f_y = 1.$$

For $dx = 0.02$, $dy = -0.01$,

$$f(0.02, -0.01) \approx 5 + df = 5 - 0.01 = \boxed{4.99}.$$

Step 3: Exact comparison Exactly,

$$f = 0.02(0.01)^2 + (0.02)^2 + e^{-0.01} + 4 \approx 4.99045183.$$

The estimate error is about $\boxed{0.00045183}$, while the second-order correction is $dx^2 + dy^2/2 = 0.00045$. Thus the second-order estimate is 4.99045, consistent with the exact value.

Question 10

Define

$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $f_x(0, 0)$ and $f_y(0, 0)$.
2. Test whether a differential $df = f_x dx + f_y dy$ gives a valid first-order approximation at the origin.
3. Identify a path that proves or disproves differentiability.

Question 10 – Solution

Strategy. Existing partial derivatives propose a linear map, but differentiability requires the remainder divided by distance to vanish along every path.

Step 1: Partial

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0)}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1,$$

$$f_y(0, 0) = \lim_{k \rightarrow 0} \frac{f(0, k)}{k} = 0.$$

The candidate differential is therefore $L(dx, dy) = dx$.

Step 2: Remainder test Differentiability would require

$$\frac{f(x, y) - x}{\sqrt{x^2 + y^2}} \rightarrow 0.$$

Along $y = x$,

$$f(x, x) = \frac{x^3}{2x^2} = \frac{x}{2},$$

so

$$\frac{f(x, x) - x}{\sqrt{2x^2}} = \frac{-x/2}{\sqrt{2}|x|} = -\frac{\text{sgn}(x)}{2\sqrt{2}},$$

which does not tend to zero.

Conclusion Although both partials exist, f is not differentiable at the origin, so

$df = dx$ is not a valid total first-order approximation there.

Question 1

Let $z = x^2y + \sin y$, where $x = t^2 - 1$ and $y = e^t$.

Tasks

1. Find dz/dt using the multivariable chain rule.
2. Evaluate it at $t = 0$.
3. Verify by differentiating the explicit composite function.

Question 1 – Solution

Strategy. Combine the two routes $t \rightarrow x \rightarrow z$ and $t \rightarrow y \rightarrow z$.

Step 1: Ingredients

$$z_x = 2xy, \quad z_y = x^2 + \cos y, \quad x' = 2t, \quad y' = e^t.$$

Therefore

$$\frac{dz}{dt} = 2xy(2t) + (x^2 + \cos y)e^t.$$

Step 2: Evaluate At $t = 0$, $x = -1$ and $y = 1$, giving

$$\left. \frac{dz}{dt} \right|_{t=0} = 1 + \cos 1.$$

Step 3: Direct check The composite is

$$z(t) = (t^2 - 1)^2 e^t + \sin(e^t).$$

Its derivative is $4t(t^2 - 1)e^t + (t^2 - 1)^2 e^t + \cos(e^t)e^t$, which at zero is $1 + \cos 1$.

Question 2

Suppose $F(x, y) = \ln(x^2 + y^2)$ and a curve is $x = 3 \cos t$, $y = 3 \sin t$.

Tasks

1. Compute dF/dt by the chain rule.
2. Explain the cancellation geometrically.
3. Verify using the explicit composite.

Question 2 – Solution

Strategy. Evaluate the two chain-rule contributions before simplifying.

Step 1: Chain rule

$$F_x = \frac{2x}{x^2 + y^2}, \quad F_y = \frac{2y}{x^2 + y^2},$$
$$x' = -3 \sin t, \quad y' = 3 \cos t.$$

Thus

$$\frac{dF}{dt} = \frac{2x(-3 \sin t) + 2y(3 \cos t)}{x^2 + y^2} = 0,$$

after substituting $x = 3 \cos t$, $y = 3 \sin t$.

Step 2: Meaning The curve remains on the circle $x^2 + y^2 = 9$, and F depends only on that squared radius. Hence $\boxed{dF/dt = 0}$.

Step 3: Direct check $F(3 \cos t, 3 \sin t) = \ln 9$, whose derivative is zero.

Question 3

Let

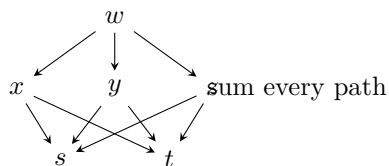
$$w = x^2 + yz, \quad x = s + t, \quad y = st, \quad z = s^2 - t^2.$$

Tasks

1. Draw or use the dependency structure to find w_s, w_t .
2. Evaluate both at $(s, t) = (1, 2)$.
3. Verify one result by explicit expansion.

Question 3 – Solution

Strategy. Sum the contribution along every dependency path.

**Step 1: Derivatives**

$$w_x = 2x, \quad w_y = z, \quad w_z = y.$$

$$x_s = 1, \quad y_s = t, \quad z_s = 2s; \quad x_t = 1, \quad y_t = s, \quad z_t = -2t.$$

Therefore

$$\boxed{w_s = 2x + zt + 2sy}, \quad \boxed{w_t = 2x + zs - 2ty}.$$

Step 2: Evaluate At $(1, 2)$, $(x, y, z) = (3, 2, -3)$:

$$\boxed{w_s = 6 - 6 + 4 = 4}, \quad \boxed{w_t = 6 - 3 - 8 = -5}.$$

Step 3: Check Expanding gives $w = (s+t)^2 + st(s^2 - t^2)$. Differentiating in s yields $2(s+t) + t(3s^2 - t^2)$, which equals 4 at $(1, 2)$.

Question 4

A surface satisfies $x^2 + y^2 + z^2 = 14$. Along a path on the surface, $x = t$, $y = t^2$, and $z > 0$ near $t = 1$.

Tasks

1. Find dz/dt implicitly without solving for $z(t)$ first.
2. Evaluate at $t = 1$.
3. Verify using the positive explicit branch.

Question 4 – Solution

Strategy. Differentiate the constraint along the path; its total rate must remain zero.

Step 1: Differentiate

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} = 0,$$

so

$$\boxed{\frac{dz}{dt} = -\frac{xx' + yy'}{z}}.$$

Step 2: Evaluate At $t = 1$, $x = 1$, $y = 1$, $z = \sqrt{14 - 1 - 1} = 2\sqrt{3}$, while $x' = 1$, $y' = 2$. Hence

$$\boxed{\left. \frac{dz}{dt} \right|_1 = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}}.$$

Step 3: Check The positive branch is $z(t) = \sqrt{14 - t^2 - t^4}$. Differentiation gives $z' = (-2t - 4t^3)/(2z)$, which produces the same value at $t = 1$.

Question 5

Let $u(x, y)$ be differentiable and set $x = r \cos \theta$, $y = r \sin \theta$.

Tasks

1. Derive formulas for u_r and u_θ .
2. Apply them to $u = x^2 - y^2$.
3. Verify against the polar form of u .

Question 5 – Solution

Strategy. Treat r and θ as independent inputs to the Cartesian coordinate map.

Step 1: General formulas

$$u_r = u_x \cos \theta + u_y \sin \theta,$$

$$u_\theta = -ru_x \sin \theta + ru_y \cos \theta.$$

Step 2: Apply For $u = x^2 - y^2$, $u_x = 2x$, $u_y = -2y$. Substituting $x = r \cos \theta$, $y = r \sin \theta$ gives

$$u_r = 2r(\cos^2 \theta - \sin^2 \theta) = 2r \cos 2\theta,$$

$$u_\theta = -4r^2 \sin \theta \cos \theta = -2r^2 \sin 2\theta.$$

Step 3: Verify Directly, $u = r^2 \cos 2\theta$; its two ordinary partial derivatives match both formulas.

Question 6

Let

$$F(x, y) = \begin{pmatrix} x^2 - y \\ xy \end{pmatrix}, \quad G(s, t) = \begin{pmatrix} e^s \cos t \\ e^s \sin t \end{pmatrix}.$$

Tasks

1. Find the Jacobian matrices J_F and J_G .
2. Use matrix chain rule to find $J_{F \circ G}(0, \pi/2)$.
3. Check one entry directly.

Question 6 – Solution

Strategy. Evaluate J_F at $G(s, t)$ before multiplying by J_G .

Step 1: Jacobians

$$J_F = \begin{pmatrix} 2x & -1 \\ y & x \end{pmatrix}, \quad J_G = \begin{pmatrix} e^s \cos t & -e^s \sin t \\ e^s \sin t & e^s \cos t \end{pmatrix}.$$

At $(0, \pi/2)$, $G = (0, 1)$, so

$$J_F(0, 1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Step 2: Multiply

$$J_{F \circ G}(0, \pi/2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The composite Jacobian is $-I_2$.

Step 3: Check The first component is $e^{2s} \cos^2 t - e^s \sin t$; its s -derivative at the point is $0 - 1 = -1$, matching the upper-left entry.

Question 7

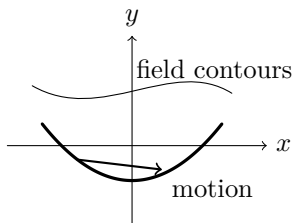
A temperature field is $T(x, y, t) = x^2 + 2xy + 3t$. A particle follows $x = t^2$, $y = 1 - t$.

Tasks

1. Find the rate of temperature change experienced by the particle.
2. Evaluate it at $t = 1$.
3. Separate spatial-motion and explicit-time contributions.

Question 7 – Solution

Strategy. The experienced rate includes motion through space and direct evolution of the field.



Step 1: Extended chain rule

$$\frac{dT}{dt} = T_x x' + T_y y' + T_t.$$

Here

$$T_x = 2x + 2y, \quad T_y = 2x, \quad T_t = 3, \quad x' = 2t, \quad y' = -1.$$

Thus

$$\boxed{\frac{dT}{dt} = (2x + 2y)(2t) - 2x + 3}.$$

Step 2: Evaluate At $t = 1$, $(x, y) = (1, 0)$:

$$\boxed{\frac{dT}{dt} = 2(2) - 2 + 3 = 5}.$$

Step 3: Contributions Motion in x contributes 4, motion in y contributes -2 , and explicit time dependence contributes 3. Their sum is 5 temperature units per time unit.

Question 8

Suppose $z = f(x, y)$ has $f_x(2, 0) = 3$ and $f_y(2, 0) = -4$. The inputs obey

$$x = u^2 + v, \quad y = u - v^2.$$

At $(u, v) = (1, 1)$:

Tasks

1. Find z_u and z_v .
2. Find all input changes (du, dv) producing zero first-order change in z .
3. Verify the cancellation using dx, dy .

Question 8 – Solution

Strategy. First propagate u, v changes to x, y , then combine them with the supplied partials.

Step 1: Chain derivatives At $(1, 1)$, $(x, y) = (2, 0)$ and

$$x_u = 2, \ x_v = 1, \ y_u = 1, \ y_v = -2.$$

Therefore

$$\boxed{z_u = 3(2) + (-4)(1) = 2}, \quad \boxed{z_v = 3(1) + (-4)(-2) = 11}.$$

Step 2: Zero-change inputs The total differential is

$$dz = 2 \, du + 11 \, dv.$$

Thus every first-order neutral change satisfies

$$\boxed{2 \, du + 11 \, dv = 0},$$

equivalently $(du, dv) = \lambda(11, -2)$.

Step 3: Verify For $(du, dv) = (11, -2)$, $dx = 2(11) - 2 = 20$ and $dy = 11 - 2(-2) = 15$. Then $dz = 3(20) - 4(15) = 0$.

Question 9

Variables x, y are defined implicitly by

$$x + y = s, \quad xy = t.$$

Near $(x, y) = (1, 2)$, regard x, y as functions of (s, t) .

Tasks

1. Find x_s, y_s, x_t, y_t by differentiating the system.
2. Evaluate them at $(1, 2)$.
3. Explain why the calculation fails when $x = y$.

Question 9 – Solution

Strategy. Differentiate both constraints and solve the resulting 2×2 linear systems.

Step 1: With respect to s

$$x_s + y_s = 1, \quad yx_s + xy_s = 0.$$

Solving gives

$$\boxed{x_s = \frac{x}{x-y}}, \quad \boxed{y_s = -\frac{y}{x-y}}.$$

Step 2: With respect to t

$$x_t + y_t = 0, \quad yx_t + xy_t = 1,$$

so

$$\boxed{x_t = \frac{1}{y-x}}, \quad \boxed{y_t = \frac{1}{x-y}}.$$

At $(1, 2)$:

$$\boxed{x_s = -1, \ y_s = 2, \ x_t = 1, \ y_t = -1}.$$

Step 3: Edge case The determinant of the coefficient matrix is $x - y$. When $x = y$, the system cannot locally distinguish the two roots; division by $x - y$ correctly signals failure of this local inverse description.

Question 10

Let $z = f(x(t), y(t))$, where f has continuous second partials.

Tasks

1. Derive a formula for d^2z/dt^2 .
2. Organize it into curvature-of- f and acceleration-of-path terms.
3. Apply it to $f = x^2 + xy + y^2$, $x = \cos t$, $y = \sin t$ at $t = 0$.

Question 10 – Solution

Strategy. Differentiate $z' = f_x x' + f_y y'$ and apply the chain and product rules to every factor.

Step 1: General formula

$$z'' = f_{xx}(x')^2 + 2f_{xy}x'y' + f_{yy}(y')^2 + f_x x'' + f_y y''.$$

The first three terms measure the function's second-order response along the velocity; the last two measure the effect of path acceleration on the first partials.

Step 2: Data For $f = x^2 + xy + y^2$,

$$f_x = 2x + y, \quad f_y = x + 2y, \quad f_{xx} = 2, \quad f_{xy} = 1, \quad f_{yy} = 2.$$

At $t = 0$, $(x, y) = (1, 0)$, $(x', y') = (0, 1)$, and $(x'', y'') = (-1, 0)$.

Step 3: Evaluate

$$z'' = 2(0)^2 + 2(1)(0)(1) + 2(1)^2 + (2)(-1) + (1)(0) = \boxed{0}.$$

Directly, $z = \cos^2 t + \sin t \cos t + \sin^2 t = 1 + \frac{1}{2} \sin 2t$, whose second derivative is $-\sin 2t$, also zero at $t = 0$.

Question 1

For

$$f(x, y) = x^2y + e^{x-y},$$

find the directional derivative at $P = (1, 0)$ in the direction $v = \langle 3, 4 \rangle$.

Tasks

1. Normalize the direction.
2. Compute the gradient at P .
3. Evaluate and interpret the directional derivative.

Question 1 – Solution

Strategy. A directional derivative uses a unit vector: $D_u f = \nabla f \cdot u$.

Step 1: Normalize

$$\|v\| = 5, \quad u = \boxed{\langle 3/5, 4/5 \rangle}.$$

Step 2: Gradient

$$f_x = 2xy + e^{x-y}, \quad f_y = x^2 - e^{x-y}.$$

At $(1, 0)$,

$$\nabla f(1, 0) = \langle e, 1 - e \rangle.$$

Step 3: Dot product

$$D_u f(1, 0) = e \frac{3}{5} + (1 - e) \frac{4}{5} = \boxed{\frac{4 - e}{5}}.$$

The positive value means f initially increases by approximately $(4 - e)/5$ output units per unit distance traveled in the specified direction.

Question 2

Let $g(x, y) = \ln(x^2 + y^2)$. Find its rate of change at $P = (1, -2)$ toward $Q = (4, 2)$.

Tasks

1. Construct the unit vector from P to Q .
2. Compute the directional derivative exactly.
3. Decide whether g initially increases or decreases.

Question 2 – Solution

Strategy. Subtract endpoint coordinates in the correct order before normalizing.

Step 1: Direction

$$\overrightarrow{PQ} = \langle 3, 4 \rangle, \quad u = \langle 3/5, 4/5 \rangle.$$

Step 2: Gradient

$$\nabla g = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle,$$

so

$$\nabla g(1, -2) = \langle 2/5, -4/5 \rangle.$$

Therefore

$$D_u g(P) = \frac{2}{5} \frac{3}{5} - \frac{4}{5} \frac{4}{5} = \boxed{-\frac{2}{5}}.$$

Step 3: Interpret The negative sign means g decreases initially toward Q . This is consistent with the initial motion having a component toward smaller distance from the origin.

Question 3

At a point P , a differentiable function has $\nabla f(P) = \langle -2, 5 \rangle$.

Tasks

1. Find the maximum and minimum directional derivatives and their directions.
2. Find all unit directions giving zero directional derivative.
3. Find the directional derivative at angle $\pi/6$ from the positive x -axis.

Question 3 – Solution

Strategy. Use $D_u f = \|\nabla f\| \cos \alpha$, where α is the angle from the gradient to the unit direction.

Step 1: Extremes Since $\|\nabla f\| = \sqrt{29}$,

$$D_{\max} = \sqrt{29} \text{ in direction } \frac{1}{\sqrt{29}} \langle -2, 5 \rangle,$$

$$D_{\min} = -\sqrt{29} \text{ in direction } \frac{1}{\sqrt{29}} \langle 2, -5 \rangle.$$

Step 2: Zero directions Unit vectors perpendicular to $\langle -2, 5 \rangle$ are

$$u = \pm \frac{1}{\sqrt{29}} \langle 5, 2 \rangle.$$

Step 3: Prescribed angle For $u = \langle \sqrt{3}/2, 1/2 \rangle$,

$$D_u f = -2 \frac{\sqrt{3}}{2} + 5 \frac{1}{2} = \frac{5}{2} - \sqrt{3}.$$

Question 4

At P , $\nabla f(P) = \langle 6, 8 \rangle$. Determine every unit direction u for which $D_u f(P) = 5$.

Tasks

1. Convert the condition to geometry.
2. Find both unit vectors exactly.
3. Verify normalization and the required rate.

Question 4 – Solution

Strategy. Decompose the desired unit vector into components parallel and perpendicular to the gradient.

Step 1: Unit basis Let

$$e = \frac{1}{10} \langle 6, 8 \rangle = \langle 3/5, 4/5 \rangle, \quad n = \langle -4/5, 3/5 \rangle.$$

These form an orthonormal basis. Write $u = ae + bn$ with $a^2 + b^2 = 1$.

Step 2: Rate condition

$$\nabla f \cdot u = 10a = 5,$$

so $a = 1/2$ and $b = \pm\sqrt{3}/2$. Thus

$$u = \frac{1}{2} \langle 3/5, 4/5 \rangle \pm \frac{\sqrt{3}}{2} \langle -4/5, 3/5 \rangle.$$

Equivalently,

$$u = \left\langle \frac{3 \mp 4\sqrt{3}}{10}, \frac{4 \pm 3\sqrt{3}}{10} \right\rangle.$$

Step 3: Verify Orthonormality gives $\|u\|^2 = 1/4 + 3/4 = 1$, and the perpendicular component contributes zero to the dot product, leaving $10(1/2) = 5$.

Question 5

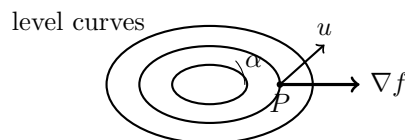
Let f be continuously differentiable near P . At P , its level curve is smooth and $\nabla f(P) \neq 0$. A unit direction u makes angle α with $\nabla f(P)$.

Tasks

1. Derive $D_u f(P)$ in terms of α .
2. Explain why tangent motion along the level curve gives zero rate.
3. Determine the sign of the rate for acute, right, and obtuse α .

Question 5 – Solution

Strategy. Combine the dot-product formula with the gradient's normal relationship to level curves.



Step 1: Angular formula

$$D_u f(P) = \nabla f(P) \cdot u = \|\nabla f(P)\| \cos \alpha.$$

Step 2: Tangent direction Along the level curve, f is constant, so its rate is zero. Geometrically the tangent is perpendicular to ∇f , giving $\alpha = \pi/2$ and $\cos \alpha = 0$.

Step 3: Signs The rate is positive for acute α , zero for a right angle, and negative for obtuse α . Its magnitude is greatest when u is parallel or antiparallel to the gradient.

Question 6

For the scalar field

$$F(x, y, z) = xe^{yz} + z^2,$$

find the directional derivative at $P = (2, 0, 1)$ toward $Q = (-1, 2, 3)$.

Tasks

1. Find the three-dimensional unit direction.
2. Compute $\nabla F(P)$.
3. Evaluate the rate and verify its units conceptually.

Question 6 – Solution

Strategy. The same gradient-dot-unit-vector formula applies in three dimensions.

Step 1: Direction

$$\overrightarrow{PQ} = \langle -3, 2, 2 \rangle, \quad \|\overrightarrow{PQ}\| = \sqrt{17}, \quad u = \frac{1}{\sqrt{17}} \langle -3, 2, 2 \rangle.$$

Step 2: Gradient

$$F_x = e^{yz}, \quad F_y = xze^{yz}, \quad F_z = xye^{yz} + 2z.$$

At $(2, 0, 1)$,

$$\nabla F(P) = \langle 1, 2, 2 \rangle.$$

Step 3: Rate

$$D_u F(P) = \frac{-3 + 4 + 4}{\sqrt{17}} = \boxed{\frac{5}{\sqrt{17}}}.$$

Because u is dimensionless and unit length, the result measures change in F per unit spatial distance.

Question 7

Define

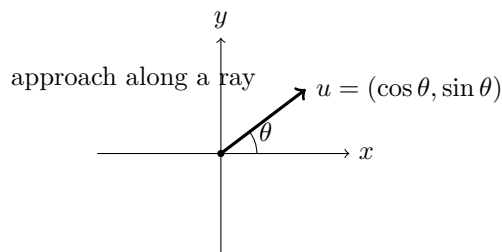
$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $D_u f(0, 0)$ from the definition for every unit $u = \langle \cos \theta, \sin \theta \rangle$.
2. Compare with $\nabla f(0, 0) \cdot u$.
3. Explain what this proves about directional derivatives and differentiability.

Question 7 – Solution

Strategy. Approach the origin along the ray hu and keep the signed parameter h .



Step 1: Definition For $h \neq 0$,

$$f(h \cos \theta, h \sin \theta) = \frac{h^3 \cos^3 \theta}{h^2} = h \cos^3 \theta.$$

Therefore

$$D_u f(0, 0) = \lim_{h \rightarrow 0} \frac{h \cos^3 \theta}{h} = \cos^3 \theta.$$

Step 2: Gradient comparison Axis definitions give $f_x(0, 0) = 1$ and $f_y(0, 0) = 0$, so

$$\nabla f(0, 0) \cdot u = \cos \theta,$$

which differs from $\cos^3 \theta$ for most angles.

Conclusion Every directional derivative exists, but its dependence on u is not linear. Thus f is not differentiable at the origin, and the gradient formula is not valid there.

Question 8

At $P = (1, 2)$, an unknown differentiable function satisfies

$$D_{u_1} f(P) = 4, \quad u_1 = \langle 1, 0 \rangle,$$

$$D_{u_2} f(P) = 1, \quad u_2 = \langle 3/5, 4/5 \rangle.$$

Tasks

1. Reconstruct $\nabla f(P)$.
2. Find the steepest-ascent direction and rate.
3. Predict the derivative in direction $\langle 0, -1 \rangle$ and verify consistency.

Question 8 – Solution

Strategy. Treat directional measurements as linear equations for the two gradient components.

Step 1: Reconstruct Write $\nabla f(P) = \langle a, b \rangle$. The first measurement gives $a = 4$. The second gives

$$\frac{3}{5}a + \frac{4}{5}b = 1.$$

Thus $12/5 + 4b/5 = 1$, so $b = -7/4$ and

$$\boxed{\nabla f(P) = \langle 4, -7/4 \rangle}.$$

Step 2: Maximum

$$\|\nabla f(P)\| = \sqrt{16 + \frac{49}{16}} = \frac{\sqrt{305}}{4}.$$

The steepest-ascent direction is

$$\boxed{\frac{1}{\sqrt{305}} \langle 16, -7 \rangle},$$

and its rate is $\boxed{\sqrt{305}/4}$.

Step 3: New direction For $\langle 0, -1 \rangle$,

$$D_{\langle 0, -1 \rangle} f = \langle 4, -7/4 \rangle \cdot \langle 0, -1 \rangle = \boxed{7/4}.$$

Substitution into both original dot products reproduces 4 and 1.

Question 9

Let $f(x, y, z) = x^2 + y^2 + z^2$. At $P = (1, 2, 2)$, restrict directions to be tangent to the plane $x + y + z = 5$.

Tasks

1. Find the unit tangent direction giving the greatest increase of f .
2. Find that maximum constrained rate.
3. Identify the exceptional case in which no preferred tangent direction exists.

Question 9 – Solution

Strategy. Project the gradient onto the plane's tangent subspace, then normalize the projection.

Step 1: Data $\nabla f(P) = \langle 2, 4, 4 \rangle$. A unit normal to the plane is

$$n = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle.$$

The tangential projection is

$$p = \nabla f - (\nabla f \cdot n)n = \langle 2, 4, 4 \rangle - \frac{10}{3} \langle 1, 1, 1 \rangle = \frac{1}{3} \langle -4, 2, 2 \rangle.$$

Step 2: Direction and rate Since $\|p\| = 2\sqrt{6}/3$,

$$\boxed{u_{\max} = \frac{1}{\sqrt{6}} \langle -2, 1, 1 \rangle}, \quad \boxed{D_{\max, \text{tangent}} f = \frac{2\sqrt{6}}{3}}.$$

Step 3: Verify $u_{\max} \cdot \langle 1, 1, 1 \rangle = 0$, so it is tangent, and $\nabla f \cdot u_{\max} = 2\sqrt{6}/3$. If the projected gradient were zero, every tangent directional derivative would be zero and no unique maximizing tangent direction would exist.

Question 10

At a point P , a differentiable function has unknown gradient. Its directional derivative equals 3 in every unit direction making angle $\pi/3$ with a fixed unit vector a .

Tasks

1. Determine what this implies about the component of $\nabla f(P)$ along a .
2. In two dimensions, use the two such directions to prove the perpendicular component is zero.
3. Determine $\nabla f(P)$ and the maximum directional derivative.

Question 10 – Solution

Strategy. Resolve the gradient into components parallel and perpendicular to a and compare symmetric directions.

Step 1: Decompose Let b be a unit vector perpendicular to a , and write

$$\nabla f = Aa + Bb.$$

The two unit directions at angles $\pm\pi/3$ from a are

$$u_{\pm} = \frac{1}{2}a \pm \frac{\sqrt{3}}{2}b.$$

Step 2: Use both measurements

$$3 = \nabla f \cdot u_{\pm} = \frac{A}{2} \pm \frac{\sqrt{3}B}{2}.$$

Adding gives $6 = A$, while subtracting gives $B = 0$.

Step 3: Result Therefore

$$\boxed{\nabla f(P) = 6a}.$$

The greatest directional derivative is the gradient magnitude,

$$\boxed{D_{\max}f(P) = 6},$$

attained uniquely in direction a . Checking either prescribed direction gives $6 \cos(\pi/3) = 3$.

Question 1

For the graph

$$z = f(x, y) = x^2 e^y,$$

consider the point above $(1, 0)$.

Tasks

1. Find the tangent plane and the linearization $L(x, y)$ at $(1, 0)$.
2. Use L to estimate $f(1.02, -0.03)$.
3. Compare the estimate with the calculator value $1.0404e^{-0.03}$.

Question 1 – Solution

Strategy. Use the value and both first partial derivatives at the base point to build the unique affine approximation.

Step 1: Local data

$$f(1, 0) = 1, \quad f_x = 2xe^y, \quad f_y = x^2e^y.$$

Thus $f_x(1, 0) = 2$ and $f_y(1, 0) = 1$.

Step 2: Plane and linearization The tangent-plane formula gives

$$z - 1 = 2(x - 1) + (y - 0).$$

Consequently,

$$\boxed{z = 1 + 2(x - 1) + y}, \quad \boxed{L(x, y) = 1 + 2(x - 1) + y}.$$

Step 3: Estimate Here $\Delta x = 0.02$ and $\Delta y = -0.03$, so

$$f(1.02, -0.03) \approx L(1.02, -0.03) = 1 + 2(0.02) - 0.03 = \boxed{1.0100}.$$

Verification Direct evaluation gives

$$1.0404e^{-0.03} \approx 1.0096515.$$

The linearization is high by about 0.0003485, a small error consistent with the small displacement from $(1, 0)$.

Question 2

The upper hemisphere

$$z = \sqrt{25 - x^2 - y^2}$$

is sampled near the point $(3, 0, 4)$.

Tasks

1. Derive the linearization at $(x, y) = (3, 0)$.
2. Estimate the height above $(3.04, 0.06)$.
3. Explain geometrically why the first-order estimate has no y -correction.

Question 2 – Solution

Strategy. Differentiate the height function, then evaluate its first-order change from the base point.

Step 1: Partial derivatives For $f(x, y) = (25 - x^2 - y^2)^{1/2}$,

$$f_x = -\frac{x}{\sqrt{25 - x^2 - y^2}}, \quad f_y = -\frac{y}{\sqrt{25 - x^2 - y^2}}.$$

At $(3, 0)$, these become $f_x = -3/4$ and $f_y = 0$.

Step 2: Linearization

$$L(x, y) = 4 - \frac{3}{4}(x - 3).$$

At the requested point,

$$f(3.04, 0.06) \approx 4 - \frac{3}{4}(0.04) = \boxed{3.9700}.$$

Step 3: Geometry The sphere is symmetric under $y \mapsto -y$. At $y = 0$, moving a small distance in either y -direction changes the height only to second order, so the tangent plane has no y -tilt.

Verification The exact height is

$$\sqrt{25 - 3.04^2 - 0.06^2} = \sqrt{15.7548} \approx 3.9692317.$$

The estimate is high by about 0.0007683.

Question 3

A differentiable function f has tangent plane

$$z = 7 + 3(x - 2) - 4(y + 1)$$

at the point whose input coordinates are $(2, -1)$.

Tasks

1. Recover $f(2, -1)$, $f_x(2, -1)$, and $f_y(2, -1)$.
2. Predict the first-order change along $x = 2 + t$, $y = -1 + 2t$.
3. Construct one nonlinear polynomial f having exactly this tangent plane.

Question 3 – Solution

Strategy. Read the constant and slope coefficients from the standard linearization form, then add terms that vanish to first order.

Step 1: Recover the data Comparing with

$$L = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

at $(a, b) = (2, -1)$ gives

$$\boxed{f(2, -1) = 7}, \quad \boxed{f_x(2, -1) = 3}, \quad \boxed{f_y(2, -1) = -4}.$$

Step 2: Change along the path The input changes are $\Delta x = t$ and $\Delta y = 2t$. Hence

$$\Delta f \approx 3t - 4(2t) = \boxed{-5t}.$$

Step 3: Construct an example One choice is

$$\boxed{f(x, y) = 7 + 3(x - 2) - 4(y + 1) + (x - 2)^2 + (y + 1)^2}.$$

The two quadratic terms and their first partial derivatives vanish at $(2, -1)$, so they do not alter the tangent plane.

Verification Substitution gives $f(2, -1) = 7$. Differentiation gives $f_x = 3 + 2(x - 2)$ and $f_y = -4 + 2(y + 1)$, which reproduce the recovered slopes at the base point.

Question 4

For the graph

$$z = f(x, y) = x^3 - 3x + y^2 - 4y,$$

locate every point at which the tangent plane is horizontal.

Tasks

1. Translate “horizontal tangent plane” into conditions on the partial derivatives.
2. Find all corresponding points on the graph.
3. Write each horizontal tangent plane and show that no others exist.

Question 4 – Solution

Strategy. A tangent plane to a graph is horizontal precisely when both of its slope coefficients vanish.

Step 1: Slope conditions

$$f_x = 3x^2 - 3, \quad f_y = 2y - 4.$$

Thus a horizontal plane requires

$$3x^2 - 3 = 0, \quad 2y - 4 = 0.$$

Therefore $x = \pm 1$ and $y = 2$.

Step 2: Heights

$$f(1, 2) = 1 - 3 + 4 - 8 = -6,$$

$$f(-1, 2) = -1 + 3 + 4 - 8 = -2.$$

The two points are

$$\boxed{(1, 2, -6) \quad \text{and} \quad (-1, 2, -2)}.$$

Step 3: Planes and completeness Because both slope coefficients are zero, the planes are

$$\boxed{z = -6} \quad \text{and} \quad \boxed{z = -2}.$$

The equations $x^2 = 1$ and $y = 2$ exhaust every simultaneous zero of f_x and f_y , so there are no other horizontal tangent planes.

Question 5

Define

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute both partial derivatives at the origin from their definitions.
2. Identify the candidate tangent plane suggested by those partial derivatives.
3. Test the linear-approximation remainder along $y = x^2$ and decide whether a tangent plane exists there in the differentiability sense.

Question 5 – Solution

Strategy. Existing partial derivatives only suggest a plane; differentiability requires the remainder to be negligible compared with the distance to the base point.

Step 1: Axis partial derivatives Along either coordinate axis, f is zero. Hence

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - 0}{h} = 0, \quad f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - 0}{k} = 0.$$

Step 2: Candidate plane The value and both proposed slopes are zero, so the formal candidate is

$$\boxed{z = 0}.$$

Step 3: Remainder test Along $(x, y) = (t, t^2)$ with $t \neq 0$,

$$f(t, t^2) = \frac{t^4}{t^4 + t^4} = \frac{1}{2}.$$

For the candidate linearization $L = 0$, the required quotient is

$$\frac{|f(t, t^2) - L(t, t^2)|}{\sqrt{t^2 + t^4}} = \frac{1/2}{|t|\sqrt{1 + t^2}} \rightarrow \infty.$$

Therefore the error is not $o(\sqrt{x^2 + y^2})$.

Conclusion The function is not even continuous at the origin along this curve, so it is not differentiable there. Thus the graph has $\boxed{\text{no tangent plane at the origin in the linear-approximation sense}}$.

Question 6

Let

$$f(x, y) = e^{x+2y}.$$

Use the linearization at $(0, 0)$ to estimate $f(0.02, 0.01)$ and rigorously bound the absolute error using the one-variable Taylor theorem for e^t .

Tasks

1. Find the linearization and the estimate.
2. Obtain a uniform error bound for $|x| \leq 0.02$, $|y| \leq 0.02$.
3. Compare the actual error at $(0.02, 0.01)$ with that bound.

Question 6 – Solution

Strategy. Write $t = x+2y$ so the two-variable approximation reduces to the first-order Taylor approximation of e^t at zero.

Step 1: Linearization At the origin, $f = 1$, $f_x = 1$, and $f_y = 2$. Therefore

$$\boxed{L(x, y) = 1 + x + 2y}.$$

At $(0.02, 0.01)$, $t = 0.04$, so

$$f(0.02, 0.01) \approx 1.04.$$

Step 2: Uniform bound Taylor's theorem gives

$$e^t = 1 + t + \frac{e^\xi}{2}t^2$$

for some ξ between 0 and t . In the given rectangle,

$$|t| = |x + 2y| \leq 0.06,$$

so $e^\xi \leq e^{0.06}$. Consequently,

$$\boxed{|f - L| \leq \frac{e^{0.06}}{2}(0.06)^2 < 0.001912}.$$

Step 3: Actual error

$$|e^{0.04} - 1.04| \approx \boxed{0.0008108},$$

which lies below the uniform bound. The bound is larger because it must cover the entire rectangle, not only the requested point.

Question 7

For the paraboloid

$$z = x^2 + 2y^2,$$

find the tangent plane parallel to the plane

$$2x - 4y + z = 7.$$

Tasks

1. Determine the point of tangency.
2. Write the tangent plane in standard form.
3. Find the perpendicular distance between the two parallel planes.

Question 7 – Solution

Strategy. Match the x - and y -slope coefficients of the tangent plane to those of the given plane.

Step 1: Match slopes The given plane is $z = 7 - 2x + 4y$, so its slopes are -2 and 4 . For $f = x^2 + 2y^2$,

$$f_x = 2x, \quad f_y = 4y.$$

Thus $2x = -2$ and $4y = 4$, giving $(x, y) = (-1, 1)$. The height is $z = 1 + 2 = 3$.

Step 2: Tangent plane At $(-1, 1, 3)$,

$$z - 3 = -2(x + 1) + 4(y - 1),$$

or

$$\boxed{2x - 4y + z = -3}.$$



Step 3: Separation For parallel planes $2x - 4y + z = 7$ and $2x - 4y + z = -3$, the distance is

$$\boxed{\frac{|7 - (-3)|}{\sqrt{2^2 + (-4)^2 + 1^2}} = \frac{10}{\sqrt{21}}}.$$

The identical left sides verify parallelism, and substituting $(-1, 1, 3)$ into the tangent plane gives -3 .

Question 8

Suppose a function f is differentiable at (a, b) . An affine function has the form

$$A(x, y) = c + p(x - a) + q(y - b).$$

Tasks

1. Prove that if A is a linear approximation to f at (a, b) , then $c = f(a, b)$.
2. Use displacements along the coordinate axes to prove $p = f_x(a, b)$ and $q = f_y(a, b)$.
3. Conclude that the tangent-plane linearization is unique.

Question 8 – Solution

Strategy. Apply the defining small-error condition first at zero displacement and then on each coordinate axis.

Step 1: Constant term For A to approximate f at the base point, it must agree there. Since $A(a, b) = c$,

$$\boxed{c = f(a, b)}.$$

Equivalently, the remainder must vanish when the displacement is zero.

Step 2: The x coefficient The approximation condition along $(h, 0)$ is

$$f(a + h, b) = f(a, b) + ph + o(|h|).$$

Subtract $f(a, b)$, divide by $h \neq 0$, and let $h \rightarrow 0$. Since $o(|h|)/h \rightarrow 0$ from both sides,

$$p = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h} = \boxed{f_x(a, b)}.$$

Step 3: The y coefficient The same argument along $(0, k)$ yields

$$q = \lim_{k \rightarrow 0} \frac{f(a, b + k) - f(a, b)}{k} = \boxed{f_y(a, b)}.$$

Conclusion Every affine first-order approximation must therefore be

$$\boxed{L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}.$$

All three coefficients are forced, proving uniqueness.

Question 9

A cylinder has volume

$$V(r, h) = \pi r^2 h.$$

Its nominal dimensions are $r = 10$ cm and $h = 20$ cm. A measured cylinder instead has $\Delta r = 0.03$ cm and $\Delta h = -0.08$ cm.

Tasks

1. Use linearization to estimate the change in volume.
2. Estimate the relative and percentage changes.
3. Compute the exact change and quantify the linearization error.

Question 9 – Solution

Strategy. Evaluate the tangent-plane increment $\Delta V \approx V_r \Delta r + V_h \Delta h$ at the nominal dimensions.

Step 1: Volume change

$$V_r = 2\pi r h, \quad V_h = \pi r^2.$$

At $(10, 20)$,

$$dV = (400\pi)(0.03) + (100\pi)(-0.08) = 12\pi - 8\pi.$$

Thus

$$\boxed{\Delta V \approx 4\pi \text{ cm}^3 \approx 12.566 \text{ cm}^3}.$$

Step 2: Relative change Dividing the differential by $V = \pi r^2 h$ gives

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h} = 2\frac{0.03}{10} - \frac{0.08}{20} = 0.002.$$

The estimated percentage increase is $\boxed{0.2\%}$.

Step 3: Exact comparison

$$\Delta V_{\text{exact}} = \pi(10.03)^2(19.92) - 2000\pi = \boxed{3.969928\pi \text{ cm}^3}.$$

The linear estimate exceeds the exact change by

$$(4 - 3.969928)\pi = \boxed{0.030072\pi \text{ cm}^3 \approx 0.09447 \text{ cm}^3}.$$

Question 10

Consider the family

$$f_{a,b}(x, y) = ax^2 + bxy + y^2.$$

At $(1, 1)$, its tangent plane passes through $(2, 0, 5)$, and its linearization predicts the value 3.06 at $(1.02, 0.99)$.

Tasks

1. Determine the parameters a and b .
2. Write the resulting tangent plane at $(1, 1)$.
3. Use it to estimate $f(0.98, 1.03)$ and verify both original constraints.

Question 10 – Solution

Strategy. Express the value and slopes at $(1, 1)$ in terms of a, b , then turn the two tangent-plane conditions into a linear system.

Step 1: General linearization At $(1, 1)$,

$$f = a + b + 1, \quad f_x = 2a + b, \quad f_y = b + 2.$$

Therefore

$$L(x, y) = a + b + 1 + (2a + b)(x - 1) + (b + 2)(y - 1).$$

Passing through $(2, 0, 5)$ requires

$$3a + b - 1 = 5, \quad \text{so} \quad 3a + b = 6.$$

The prediction at $(1.02, 0.99)$ requires

$$1.04a + 1.01b + 0.98 = 3.06.$$

Solving the two equations gives

$$\boxed{a = 2, \quad b = 0}.$$

Step 2: Tangent plane Now $f(1, 1) = 3$, $f_x(1, 1) = 4$, and $f_y(1, 1) = 2$. Hence

$$\boxed{z = 3 + 4(x - 1) + 2(y - 1)}.$$

Step 3: New estimate and verification At $(0.98, 1.03)$,

$$f(0.98, 1.03) \approx 3 + 4(-0.02) + 2(0.03) = \boxed{2.98}.$$

Also $L(2, 0) = 3 + 4 - 2 = 5$, and $L(1.02, 0.99) = 3 + 0.08 - 0.02 = 3.06$, verifying both supplied constraints.

Question 1

Consider the ellipsoid

$$F(x, y, z) = x^2 + 2y^2 + 3z^2 = 6$$

at $P = (1, 1, 1)$.

Tasks

1. Compute the gradient and verify that P is a regular point.
2. Find the tangent plane at P .
3. Find a parametric equation of the normal line and verify its direction.

Question 1 – Solution

Strategy. For a regular level surface, the gradient at the point is normal to the tangent plane.

Step 1: Gradient and regularity

$$\nabla F = \langle 2x, 4y, 6z \rangle, \quad \nabla F(1, 1, 1) = \langle 2, 4, 6 \rangle.$$

This vector is nonzero, so P is regular. Also $F(1, 1, 1) = 1 + 2 + 3 = 6$, confirming that P lies on the surface.

Step 2: Tangent plane Using $\nabla F(P)$ as a normal,

$$2(x - 1) + 4(y - 1) + 6(z - 1) = 0.$$

After division by 2,

$$\boxed{x + 2y + 3z = 6}.$$

Step 3: Normal line A simplified direction parallel to the gradient is $\langle 1, 2, 3 \rangle$. Hence

$$\boxed{(x, y, z) = (1, 1, 1) + t(1, 2, 3)},$$

or $x = 1 + t$, $y = 1 + 2t$, $z = 1 + 3t$.

Verification The line passes through P at $t = 0$, and $\langle 1, 2, 3 \rangle$ is perpendicular to every displacement $\langle u, v, w \rangle$ in the plane because the plane equation requires $u + 2v + 3w = 0$.

Question 2

Let the level surface

$$F(x, y, z) = xye^z = 2$$

contain $P = (1, 2, 0)$.

Tasks

1. Find the two unit normal vectors at P .
2. Write the tangent plane in standard form.
3. Give symmetric or parametric equations for the normal line.

Question 2 – Solution

Strategy. Differentiate the defining level function; its gradient supplies both the plane normal and the normal-line direction.

Step 1: Gradient

$$\nabla F = \langle ye^z, xe^z, xye^z \rangle.$$

At $P = (1, 2, 0)$,

$$\nabla F(P) = \langle 2, 1, 2 \rangle, \quad \|\nabla F(P)\| = 3.$$

Therefore the two unit normals are

$$\pm \frac{1}{3} \langle 2, 1, 2 \rangle.$$

Step 2: Tangent plane

$$2(x - 1) + (y - 2) + 2(z - 0) = 0,$$

so

$$2x + y + 2z = 4.$$

Step 3: Normal line Using the unsimplified integer direction,

$$(x, y, z) = (1, 2, 0) + t(2, 1, 2).$$

Equivalently,

$$\frac{x - 1}{2} = y - 2 = \frac{z}{2}.$$

Verification The point satisfies $1 \cdot 2 \cdot e^0 = 2$. The line direction equals the plane's coefficient vector, so it is perpendicular to the plane.

Question 3

The sphere

$$x^2 + y^2 + z^2 = 9$$

and the plane

$$x + y + z = 5$$

intersect in a curve through $P = (1, 2, 2)$.

Tasks

1. Find a direction tangent to the intersection curve at P .
2. Write the tangent line to the curve.
3. Verify that the direction is tangent to both surfaces.

Question 3 – Solution

Strategy. A curve lying on both surfaces has a tangent perpendicular to both surface normals, so use their cross product.

Step 1: Surface normals Let

$$F = x^2 + y^2 + z^2, \quad G = x + y + z.$$

At P ,

$$\nabla F(P) = \langle 2, 4, 4 \rangle, \quad \nabla G(P) = \langle 1, 1, 1 \rangle.$$

Their cross product is

$$\nabla F(P) \times \nabla G(P) = \langle 0, 2, -2 \rangle,$$

so a simplified tangent direction is $\langle 0, 1, -1 \rangle$.

Step 2: Tangent line

$$\boxed{(x, y, z) = (1, 2, 2) + t(0, 1, -1)}.$$

Step 3: Verification First, $1^2 + 2^2 + 2^2 = 9$ and $1 + 2 + 2 = 5$, so P lies on both surfaces. Next,

$$\langle 2, 4, 4 \rangle \cdot \langle 0, 1, -1 \rangle = 0, \quad \langle 1, 1, 1 \rangle \cdot \langle 0, 1, -1 \rangle = 0.$$

Thus the direction lies in both tangent planes, as required.

Question 4

On the ellipsoid

$$x^2 + 2y^2 + z^2 = 9,$$

find every point whose tangent plane is parallel to

$$2x - 4y + 2z = 7.$$

Tasks

1. Determine all points of tangency.
2. Write the tangent plane at each point.
3. Give a normal line at each point and verify completeness.

Question 4 – Solution

Strategy. Parallel tangent planes have parallel normal vectors; impose that condition together with the surface equation.

Step 1: Parallel-gradient condition For $F = x^2 + 2y^2 + z^2$,

$$\nabla F = \langle 2x, 4y, 2z \rangle.$$

The given plane has normal parallel to $\langle 1, -2, 1 \rangle$. Set

$$\langle 2x, 4y, 2z \rangle = \lambda \langle 1, -2, 1 \rangle.$$

Then $x = \lambda/2$, $y = -\lambda/2$, and $z = \lambda/2$. Substitution into the ellipsoid gives $\lambda^2 = 9$.

Thus the two points are

$$P_+ = \left(\frac{3}{2}, -\frac{3}{2}, \frac{3}{2} \right), \quad P_- = -P_+.$$

Step 2: Tangent planes Using the simplified common normal,

$$x - 2y + z = 6 \quad \text{at } P_+,$$

$$x - 2y + z = -6 \quad \text{at } P_-.$$

Step 3: Normal lines

$$\mathbf{r} = P_+ + t \langle 1, -2, 1 \rangle, \quad \mathbf{r} = P_- + t \langle 1, -2, 1 \rangle.$$

The equations $\lambda = \pm 3$ exhaust the surface constraint, proving there are no additional points.

Question 5

For the surface

$$xy + yz + zx = 3,$$

consider $P = (1, 1, 1)$.

Tasks

1. Find the normal line at P .
2. Find every point where that normal line intersects the sphere $x^2 + y^2 + z^2 = 12$.
3. Verify both the surface-normal direction and the sphere intersections.

Question 5 – Solution

Strategy. Parameterize the normal line using the gradient, then substitute that line into the sphere equation.

Step 1: Normal line Let $F = xy + yz + zx$. Then

$$\nabla F = \langle y + z, x + z, x + y \rangle, \quad \nabla F(P) = \langle 2, 2, 2 \rangle.$$

Using the simplified direction $\langle 1, 1, 1 \rangle$,

$$\boxed{(x, y, z) = (1 + t, 1 + t, 1 + t)}.$$

Step 2: Intersections with the sphere Substitution gives

$$3(1 + t)^2 = 12,$$

so $(1 + t)^2 = 4$ and

$$t = 1 \quad \text{or} \quad t = -3.$$

The intersection points are therefore

$$\boxed{(2, 2, 2) \quad \text{and} \quad (-2, -2, -2)}.$$

Step 3: Verification The base point satisfies $1 + 1 + 1 = 3$, and the line direction is parallel to $\nabla F(P)$, so the line is normal to the original surface. Finally,

$$2^2 + 2^2 + 2^2 = 12, \quad (-2)^2 + (-2)^2 + (-2)^2 = 12,$$

confirming both sphere intersections.

Question 6

Consider the double cone

$$F(x, y, z) = x^2 + y^2 - z^2 = 0.$$

Tasks

1. Find the tangent plane and normal line at the regular point $P = (1, 0, 1)$.
2. Explain why the gradient formula fails to select a plane at the origin.
3. Use generator curves through the origin to prove that no single tangent plane contains all tangent directions there.

Question 6 – Solution

Strategy. Contrast a regular point, where the gradient is nonzero, with the cone vertex, where many generator directions meet and the gradient vanishes.

Step 1: The regular point

$$\nabla F = \langle 2x, 2y, -2z \rangle, \quad \nabla F(1, 0, 1) = \langle 2, 0, -2 \rangle.$$

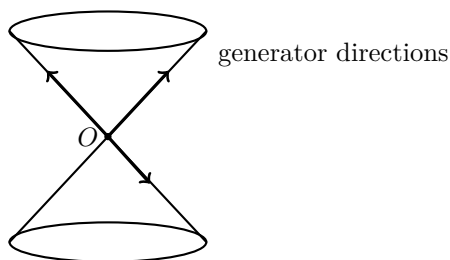
Thus

$$2(x - 1) - 2(z - 1) = 0, \quad \boxed{x - z = 0},$$

and a normal line is

$$\boxed{(x, y, z) = (1, 0, 1) + t(1, 0, -1)}.$$

Step 2: The vertex At $O = (0, 0, 0)$, $\nabla F(O) = \langle 0, 0, 0 \rangle$, so the usual plane equation reduces to $0 = 0$ and supplies no normal.



Step 3: Generator argument For every θ , the curve

$$\mathbf{r}_\theta(s) = \langle s \cos \theta, s \sin \theta, s \rangle$$

lies on the cone and has tangent $\langle \cos \theta, \sin \theta, 1 \rangle$ at O . The directions for $\theta = 0, \pi$, and $\pi/2$ span all of $\mathbb{R}^3$: the first two yield the x - and z -directions by differences and sums, and the third then yields the y -direction. No two-dimensional plane can contain them all. Hence $\boxed{\text{there is no unique tangent plane at the vertex}}.$

Question 7

An unknown member of the family

$$F(x, y, z) = x^2 + ay^2 + bz^2 = c$$

passes through $P = (1, 1, 1)$ and has tangent plane

$$x + 2y + 3z = 6$$

there.

Tasks

1. Determine a , b , and c .
2. Confirm that P is regular for the resulting surface.
3. Write its normal line at P and verify the prescribed tangent plane.

Question 7 – Solution

Strategy. Match the unknown gradient to the plane's normal; the first gradient component fixes the proportionality factor.

Step 1: Match normals

$$\nabla F(P) = \langle 2, 2a, 2b \rangle.$$

This must equal $\lambda \langle 1, 2, 3 \rangle$. The first component gives $\lambda = 2$. Therefore

$$2a = 4, \quad 2b = 6,$$

so

$$\boxed{a = 2, \quad b = 3}.$$

Since P lies on the surface,

$$c = 1^2 + 2(1)^2 + 3(1)^2 = \boxed{6}.$$

Step 2: Regularity For the resulting surface,

$$\nabla F(P) = \langle 2, 4, 6 \rangle \neq \langle 0, 0, 0 \rangle,$$

so P is regular.

Step 3: Line and plane verification The normal line is

$$\boxed{(x, y, z) = (1, 1, 1) + t(1, 2, 3)}.$$

The plane obtained from the gradient is

$$2(x - 1) + 4(y - 1) + 6(z - 1) = 0.$$

Dividing by 2 produces $x + 2y + 3z = 6$, exactly the prescribed plane.

Question 8

For the level surface

$$xyz = 8,$$

let $P = (2, 2, 2)$.

Tasks

1. Find the tangent plane and normal line at P .
2. Show that the normal line passes through the origin.
3. Prove that P is the perpendicular foot from the origin to the tangent plane and find that distance in two ways.

Question 8 – Solution

Strategy. Use the gradient for the plane and line, then compare the point-to-plane formula with the length of the normal segment.

Step 1: Plane and line For $F = xyz$,

$$\nabla F = \langle yz, xz, xy \rangle, \quad \nabla F(P) = \langle 4, 4, 4 \rangle.$$

Thus the tangent plane is

$$4(x - 2) + 4(y - 2) + 4(z - 2) = 0,$$

or

$$\boxed{x + y + z = 6}.$$

A normal line is

$$\boxed{(x, y, z) = (2, 2, 2) + t(1, 1, 1)}.$$

Step 2: Origin on the line At $t = -2$, all three coordinates are zero. Thus the segment from the origin to P follows the plane's normal direction, making P the perpendicular foot.

Step 3: Distance checks The normal-segment length is

$$\| \langle 2, 2, 2 \rangle \| = \boxed{2\sqrt{3}}.$$

The point-to-plane formula independently gives

$$\frac{|0 + 0 + 0 - 6|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{6}{\sqrt{3}} = \boxed{2\sqrt{3}}.$$

The agreement verifies both the geometry and the scale of the normal vector.

Question 9

Let S be a smooth level surface $F(x, y, z) = c$, and let P be a point of S with $\nabla F(P) \neq 0$.

Tasks

1. For any differentiable curve $\mathbf{r}(t)$ on S through P , prove that its tangent at P is perpendicular to $\nabla F(P)$.
2. Deduce the tangent-plane equation at $P = (x_0, y_0, z_0)$.
3. Deduce the normal-line equation and explain the role of the regularity condition.

Question 9 – Solution

Strategy. Differentiate the constant identity $F(\mathbf{r}(t)) = c$ and interpret the chain rule geometrically.

Step 1: Tangent vectors Suppose $\mathbf{r}(t_0) = P$ and the curve lies on S . Then

$$F(\mathbf{r}(t)) = c.$$

Differentiating at t_0 by the multivariable chain rule gives

$$0 = \left. \frac{d}{dt} F(\mathbf{r}(t)) \right|_{t=t_0} = \nabla F(P) \cdot \mathbf{r}'(t_0).$$

Thus every surface-tangent velocity is perpendicular to $\nabla F(P)$.

Step 2: Tangent plane Because $\nabla F(P)$ is nonzero, it supplies a valid normal. Therefore

$$\boxed{F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0}.$$

Step 3: Normal line The line through P in the gradient direction is

$$\boxed{\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \nabla F(P)}.$$

The condition $\nabla F(P) \neq 0$ ensures a genuine normal direction and a two-dimensional tangent plane. At a singular point, the zero vector cannot determine either object; additional geometric analysis is required.

Question 10

On the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6,$$

find every point at which the normal line passes through the origin.

Tasks

1. Derive the condition relating a point to its gradient.
2. Prove that at most one coordinate of such a point can be nonzero.
3. List all points, tangent planes, and normal lines, and verify completeness.

Question 10 – Solution

Strategy. If the origin lies on the normal line at P , then the vector from P to the origin must be parallel to $\nabla F(P)$.

Step 1: Parallelism condition Write $P = (p, q, r)$. Since

$$\nabla F(P) = \langle 2p, 4q, 6r \rangle,$$

the origin lies on the normal line exactly when some t satisfies

$$\langle -p, -q, -r \rangle = t \langle 2p, 4q, 6r \rangle.$$

If $p \neq 0$, then $t = -1/2$; if $q \neq 0$, then $t = -1/4$; if $r \neq 0$, then $t = -1/6$. One value of t cannot meet two of these conditions, so at most one coordinate is nonzero.

Step 2: Points and planes Using the ellipsoid equation on each coordinate axis gives exactly six points:

$$(\pm\sqrt{6}, 0, 0), \quad (0, \pm\sqrt{3}, 0), \quad (0, 0, \pm\sqrt{2}).$$

Their tangent planes are, respectively,

$$\boxed{x = \pm\sqrt{6}}, \quad \boxed{y = \pm\sqrt{3}}, \quad \boxed{z = \pm\sqrt{2}}.$$

Step 3: Normal lines and verification The corresponding normal lines are

$$\boxed{\mathbf{r} = (\pm\sqrt{6}, 0, 0) + t \langle 1, 0, 0 \rangle},$$

$$\boxed{\mathbf{r} = (0, \pm\sqrt{3}, 0) + t \langle 0, 1, 0 \rangle}, \quad \boxed{\mathbf{r} = (0, 0, \pm\sqrt{2}) + t \langle 0, 0, 1 \rangle}.$$

These are the coordinate axes written through their stated points. Every one passes through the origin and is perpendicular to its listed coordinate plane. The parallelism argument rules out mixed-coordinate points, while the surface equation rules out the origin itself, so the list is complete.

Question 1

For

$$f(x, y) = x^2 + 2y^2 - 4x + 8y + 1,$$

analyze all relative extrema.

Tasks

1. Find every critical point.
2. Classify each point with the second derivative test.
3. Complete the square to verify the classification and find the extremal value.

Question 1 – Solution

Strategy. Solve the first-derivative equations, classify with the Hessian determinant, and then use an exact algebraic form as an independent check.

Step 1: Critical point

$$f_x = 2x - 4, \quad f_y = 4y + 8.$$

Both vanish only when

$$(x, y) = (2, -2).$$

Step 2: Second derivative test

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = 0.$$

The Hessian determinant is

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 8 > 0.$$

Since $f_{xx} > 0$, the point is a strict relative minimum. Its value is

$$f(2, -2) = -11.$$

Step 3: Algebraic verification Completing squares gives

$$f(x, y) = (x - 2)^2 + 2(y + 2)^2 - 11.$$

Both squared terms are nonnegative and vanish simultaneously only at $(2, -2)$. This directly confirms the strict local minimum and rules out any other critical point or relative extremum.

Question 2

For

$$f(x, y) = x^3 + y^3 - 3xy,$$

find and classify every critical point.

Tasks

1. Solve the nonlinear critical-point equations without losing solutions.
2. Apply the second derivative test at each point.
3. Verify the saddle classification by displaying nearby values of opposite signs relative to the critical value.

Question 2 – Solution

Strategy. Use the coupled equations to reduce the possible coordinates, then apply the Hessian determinant separately at each solution.

Step 1: Critical points

$$f_x = 3x^2 - 3y, \quad f_y = 3y^2 - 3x.$$

Thus $y = x^2$ and $x = y^2$. Substitution gives

$$x = x^4, \quad x(x^3 - 1) = 0.$$

The real possibilities are $x = 0$ and $x = 1$, giving

$$\boxed{(0, 0) \quad \text{and} \quad (1, 1)}.$$

Step 2: Hessian test

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3,$$

so $D = 36xy - 9$. At $(0, 0)$, $D = -9 < 0$, hence it is a saddle point. At $(1, 1)$, $D = 27 > 0$ and $f_{xx} = 6 > 0$, so it is a strict relative minimum with

$$f(1, 1) = \boxed{-1}.$$

Step 3: Saddle verification At the origin the critical value is 0. Along $y = 0$,

$$f(x, 0) = x^3,$$

which is positive for small $x > 0$ and negative for small $x < 0$. Thus every neighborhood contains values above and below 0, independently verifying the saddle classification.

Question 3

Analyze the origin for

$$f(x, y) = x^2 + 4xy + y^2.$$

Tasks

1. Verify that the origin is the only critical point.
2. Classify it using the second derivative test.
3. Give two explicit paths that make the saddle geometry visible.

Question 3 – Solution

Strategy. The quadratic form itself records the local geometry; the Hessian test and carefully chosen diagonal paths should agree.

Step 1: Critical point

$$f_x = 2x + 4y, \quad f_y = 4x + 2y.$$

Solving $x + 2y = 0$ and $2x + y = 0$ gives $x = y = 0$. Therefore the origin is the unique critical point.

Step 2: Hessian test

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 4.$$

Hence

$$D = (2)(2) - 4^2 = -12 < 0.$$

Therefore

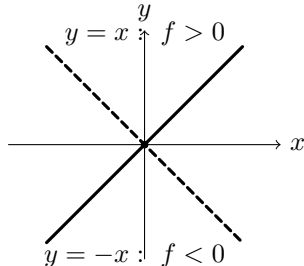
$(0, 0)$ is a saddle point.

Step 3: Path verification Along $y = x$,

$$f(x, x) = x^2 + 4x^2 + x^2 = 6x^2 > 0$$

for $x \neq 0$. Along $y = -x$,

$$f(x, -x) = x^2 - 4x^2 + x^2 = -2x^2 < 0.$$



Both paths approach the origin, whose value is zero, so arbitrarily nearby values occur on both sides of the critical value.

Question 4

Consider the “monkey saddle”

$$f(x, y) = x^3 - 3xy^2.$$

Tasks

1. Find all critical points.
2. Show precisely why the second derivative test is inconclusive there.
3. Classify the point using both simple paths and the polar form of the function.

Question 4 – Solution

Strategy. When the Hessian degenerates, return to the definition and compare the function along paths through the critical point.

Step 1: Critical point

$$f_x = 3x^2 - 3y^2, \quad f_y = -6xy.$$

The second equation requires $x = 0$ or $y = 0$. In either case the first equation then forces the other coordinate to be zero. Thus the only critical point is $(0, 0)$.

Step 2: Failed Hessian test

$$f_{xx} = 6x, \quad f_{yy} = -6x, \quad f_{xy} = -6y.$$

All three second derivatives vanish at the origin, so

$$D(0, 0) = 0.$$

The test gives no classification.

Step 3: Direct classification

Along $y = 0$,

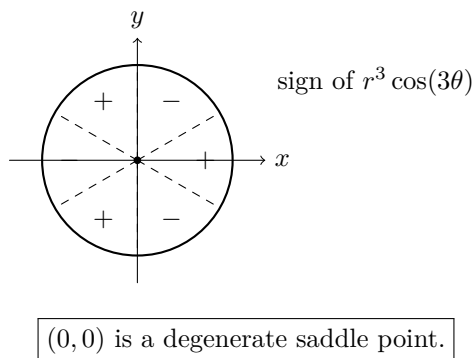
$$f(x, 0) = x^3,$$

which takes both signs arbitrarily close to zero. Hence the origin is a saddle.

For a structural check, let $x = r \cos \theta$ and $y = r \sin \theta$. Then

$$f = r^3 (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) = r^3 \cos(3\theta).$$

The factor $\cos(3\theta)$ alternates sign in six angular sectors. Therefore



Question 5

Let

$$f(x, y) = (x^2 + y^2)e^{-(x^2 + y^2)}.$$

Tasks

1. Find the complete critical set.
2. Classify the origin and every point on the critical circle as relative extrema.
3. Explain why the circular extrema are not strict and why the ordinary Hessian test degenerates there.

Question 5 – Solution

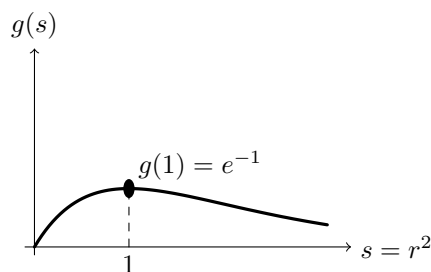
Strategy. Set $s = x^2 + y^2$ and analyze the one-variable radial profile $g(s) = se^{-s}$.

Step 1: Critical set Since

$$f_x = 2xe^{-s}(1-s), \quad f_y = 2ye^{-s}(1-s),$$

both derivatives vanish when $(x, y) = (0, 0)$ or when $s = 1$. Thus

$$\text{critical set} = \{(0, 0)\} \cup \{x^2 + y^2 = 1\}.$$



Step 2: Radial classification For $s \geq 0$,

$$g'(s) = e^{-s}(1-s).$$

It is positive for $0 \leq s < 1$ and negative for $s > 1$. Thus $g(0) = 0$ increases immediately away from the origin, so

$$(0, 0) \text{ is a strict relative minimum.}$$

The radial profile peaks at $s = 1$, so every point of the unit circle is a relative maximum with value

$$e^{-1}.$$

Step 3: Non-strictness and degeneracy Moving tangentially along the circle leaves s and f unchanged, so none of these maxima is strict. That zero-curvature tangential direction makes the Hessian determinant zero on the critical circle, which is why the standard second derivative test is inconclusive there.

Question 6

Consider the parameterized quadratic

$$f_{a,b}(x, y) = x^2 + axy + by^2 - 4x + 2y.$$

The point $P = (1, -1)$ is known to be critical.

Tasks

1. Determine a and b from the critical-point condition.
2. Classify P for the resulting function.
3. Verify the classification with two lines through P .

Question 6 – Solution

Strategy. Use the two first-derivative equations to reconstruct the parameters, then test the resulting quadratic form around P .

Step 1: Determine the parameters

$$f_x = 2x + ay - 4, \quad f_y = ax + 2by + 2.$$

At $(1, -1)$, the equations $f_x = f_y = 0$ become

$$-2 - a = 0, \quad a - 2b + 2 = 0.$$

Therefore

$$\boxed{a = -2, \quad b = 0}.$$

Step 2: Hessian classification For the resulting function,

$$f_{xx} = 2, \quad f_{yy} = 0, \quad f_{xy} = -2.$$

Thus

$$D = (2)(0) - (-2)^2 = -4 < 0,$$

so P is a saddle point.

Step 3: Direct verification Write $x = 1 + h$, $y = -1 + k$. Since the function is quadratic and P is critical,

$$f(1 + h, -1 + k) - f(1, -1) = h^2 - 2hk.$$

Along $k = 0$, the change is $h^2 > 0$. Along $k = h$, it is $-h^2 < 0$. Hence every neighborhood of P contains values above and below $f(P)$, confirming

$$\boxed{P = (1, -1) \text{ is a saddle point.}}$$

Question 7

A twice differentiable function has the local expansion near a critical point P :

$$f(P + (h, k)) = 5 + 3h^2 + 4hk - k^2 + o(h^2 + k^2).$$

Tasks

1. Reconstruct the Hessian matrix at P .
2. Apply the second derivative test.
3. Use the remainder term carefully to prove the classification along two coordinate directions.

Question 7 – Solution

Strategy. Match the quadratic terms with $\frac{1}{2}(h, k)H(h, k)^T$, then show that the leading signs dominate the smaller remainder.

Step 1: Hessian The second-order Taylor term is

$$\frac{1}{2} (f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2).$$

Matching coefficients gives

$$f_{xx} = 6, \quad f_{xy} = 4, \quad f_{yy} = -2,$$

so

$$H(P) = \begin{pmatrix} 6 & 4 \\ 4 & -2 \end{pmatrix}.$$

Step 2: Second derivative test

$$D = (6)(-2) - 4^2 = -28 < 0.$$

Therefore P is a saddle point.

Step 3: Remainder verification Along $k = 0$,

$$f(P + (h, 0)) - 5 = 3h^2 + o(h^2),$$

which is positive for all sufficiently small nonzero h . Along $h = 0$,

$$f(P + (0, k)) - 5 = -k^2 + o(k^2),$$

which is negative for all sufficiently small nonzero k . The little- o terms are eventually smaller in magnitude than, for example, half of the displayed quadratic terms. Thus nearby values occur on both sides of 5, rigorously confirming the saddle.

Question 8

Let f be differentiable at an interior point $P = (a, b)$ of its domain.

Tasks

1. Prove that if f has a relative maximum or minimum at P , then $\nabla f(P) = \mathbf{0}$.
2. Explain why the converse is false.
3. Apply the result to show that $g(x, y) = x + e^y$ has no relative extrema on $\mathbb{R}^2$.

Question 8 – Solution

Strategy. Restrict the function to coordinate lines through the interior point and apply the one-variable necessary condition.

Step 1: Necessary condition Suppose f has a relative extremum at (a, b) . The one-variable restriction

$$\phi(x) = f(x, b)$$

has a relative extremum at $x = a$. Differentiability gives

$$f_x(a, b) = \phi'(a) = 0.$$

Likewise, $\psi(y) = f(a, y)$ has a relative extremum at b , so

$$f_y(a, b) = \psi'(b) = 0.$$

Therefore

$$\boxed{\nabla f(P) = \mathbf{0}}.$$

Step 2: The converse A zero gradient makes P a critical point, not necessarily an extremum. For instance, $f(x, y) = x^2 - y^2$ has gradient zero at the origin but takes both positive and negative values nearby, so the origin is a saddle.

Step 3: Application For $g(x, y) = x + e^y$,

$$g_x = 1, \quad g_y = e^y.$$

In particular, the gradient is never zero. Since every point of $\mathbb{R}^2$ is interior and g is differentiable everywhere, the necessary condition rules out a relative maximum or minimum at every point. Hence

$$\boxed{g \text{ has no relative extrema on } \mathbb{R}^2}.$$

Question 9

Consider

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Tasks

1. Prove directly that the origin is a strict relative minimum.
2. Determine whether $f_x(0, 0)$ and $f_y(0, 0)$ exist.
3. Explain how this example refines the usual critical-point search procedure.

Question 9 – Solution

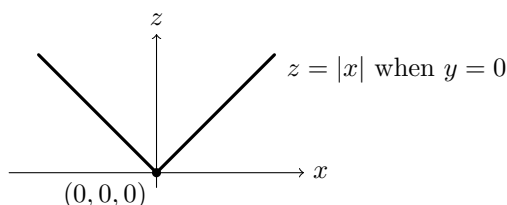
Strategy. Use the definition of a relative minimum first, then test each partial derivative with a signed two-sided limit.

Step 1: Direct extremum test At the origin, $f(0, 0) = 0$. For every $(x, y) \neq (0, 0)$,

$$f(x, y) = \sqrt{x^2 + y^2} > 0 = f(0, 0).$$

Therefore

$(0, 0)$ is a strict relative minimum.



Step 2: Partial derivatives Along the x -axis,

$$\frac{f(h, 0) - f(0, 0)}{h} = \frac{|h|}{h},$$

whose right-hand limit is 1 and left-hand limit is -1 . Thus $f_x(0, 0)$ does not exist. The identical calculation on the y -axis shows that $f_y(0, 0)$ does not exist.

Step 3: Methodological conclusion Fermat's theorem says that a *differentiable* interior extremum has zero gradient. It does not say every extremum is differentiable. Consequently, a complete candidate search must include interior points where first partial derivatives vanish *and* points where they fail to exist. This example supplies the latter case.

Question 10

A local profit model, in suitable units, is

$$P(x, y) = 80x + 60y - 2x^2 - xy - y^2.$$

Assume the critical point found below lies in the interior of the operational region.

Tasks

1. Find the critical production levels.
2. Classify the critical point using the Hessian.
3. Complete the square in translated coordinates to verify the local classification and compute the modeled profit there.

Question 10 – Solution

Strategy. Solve the marginal-profit equations, then verify the Hessian conclusion by expressing every nearby change as a negative quadratic form.

Step 1: Critical production levels

$$P_x = 80 - 4x - y, \quad P_y = 60 - x - 2y.$$

The equations $4x + y = 80$ and $x + 2y = 60$ yield

$$\boxed{x = \frac{100}{7}, \quad y = \frac{160}{7}}.$$

Step 2: Hessian test

$$P_{xx} = -4, \quad P_{yy} = -2, \quad P_{xy} = -1.$$

Thus

$$D = (-4)(-2) - (-1)^2 = 7 > 0.$$

Because $P_{xx} < 0$, the critical point is a strict relative maximum. Direct substitution gives

$$\boxed{P\left(\frac{100}{7}, \frac{160}{7}\right) = \frac{8800}{7}}.$$

Step 3: Translated verification Let $h = x - 100/7$ and $k = y - 160/7$. Since P is quadratic and its linear terms vanish at the critical point,

$$P(x, y) - \frac{8800}{7} = -2h^2 - hk - k^2.$$

Complete the square:

$$-2h^2 - hk - k^2 = -2\left(h + \frac{k}{4}\right)^2 - \frac{7}{8}k^2 < 0$$

for every $(h, k) \neq (0, 0)$. This independently verifies the strict relative maximum.

Question 1

Find the absolute extrema of

$$f(x, y) = x^2 + y^2 - 2x - 4y$$

on the rectangle $R = [0, 3] \times [0, 4]$.

Tasks

1. Find all interior critical points.
2. Analyze all four boundary segments, including their endpoints.
3. Compare the complete candidate list and report every absolute extremum.

Question 1 – Solution

Strategy. On a closed, bounded rectangle, compare values at interior critical points, one-variable boundary critical points, and corners.

Step 1: Interior

$$f_x = 2x - 2, \quad f_y = 2y - 4.$$

The sole interior critical point is $(1, 2)$, where

$$f(1, 2) = 1 + 4 - 2 - 8 = \boxed{-5}.$$

Step 2: Boundaries On $x = 0$ and $x = 3$,

$$f(0, y) = y^2 - 4y, \quad f(3, y) = y^2 - 4y + 3.$$

Both have a boundary critical value at $y = 2$, giving -4 and -1 . On $y = 0$ and $y = 4$,

$$f(x, 0) = f(x, 4) = x^2 - 2x,$$

whose critical point $x = 1$ gives -1 . The four corner values are

$$f(0, 0) = 0, \quad f(3, 0) = 3, \quad f(0, 4) = 0, \quad f(3, 4) = 3.$$

Step 3: Comparison The smallest candidate value is -5 and the largest is 3 . Hence

$$\boxed{f_{\min} = -5 \text{ at } (1, 2)},$$

$$\boxed{f_{\max} = 3 \text{ at } (3, 0) \text{ and } (3, 4)}.$$

Completing the square, $f = (x - 1)^2 + (y - 2)^2 - 5$, independently confirms the comparison.

Question 2

Find the absolute extrema of

$$f(x, y) = x^2 + y^2 - 2x$$

on the closed unit disk $D : x^2 + y^2 \leq 1$.

Tasks

1. Check for interior critical points.
2. Parameterize the circular boundary and find all boundary candidates.
3. Compare the candidates and verify the result geometrically by completing the square.

Question 2 – Solution

Strategy. Treat the interior and the circular boundary separately; on the boundary the radial term is constant.

Step 1: Interior

$$f_x = 2x - 2, \quad f_y = 2y.$$

The only solution is $(1, 0)$, which lies on the boundary rather than in the interior. Thus there are no interior critical points.

Step 2: Boundary Set $x = \cos \theta$, $y = \sin \theta$. Then

$$f(\cos \theta, \sin \theta) = 1 - 2 \cos \theta.$$

The smallest value occurs when $\cos \theta = 1$ and the largest when $\cos \theta = -1$. Therefore

$$\boxed{f_{\min} = -1 \text{ at } (1, 0)}, \quad \boxed{f_{\max} = 3 \text{ at } (-1, 0)}.$$

Step 3: Geometric verification

$$f(x, y) = (x - 1)^2 + y^2 - 1.$$

Thus $f + 1$ is squared distance from (x, y) to $(1, 0)$. Within the disk, that distance is smallest at $(1, 0)$ and largest at the diametrically opposite point $(-1, 0)$. This confirms both locations and values.

Question 3

On the closed triangle

$$T = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 4\},$$

find the absolute extrema of

$$f(x, y) = xy(4 - x - y).$$

Tasks

1. Find every interior critical point.
2. Analyze all three edges and vertices.
3. Compare the candidates and interpret why the minimum occurs along an entire boundary.

Question 3 – Solution

Strategy. The compact triangle guarantees extrema; compare the interior stationary value with the three edge restrictions.

Step 1: Interior

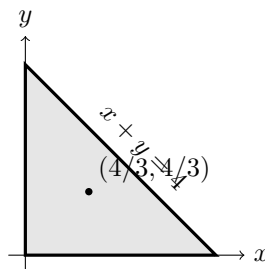
$$f_x = y(4 - 2x - y), \quad f_y = x(4 - x - 2y).$$

In the interior $x, y > 0$, so

$$2x + y = 4, \quad x + 2y = 4.$$

Thus $x = y = 4/3$, and

$$f\left(\frac{4}{3}, \frac{4}{3}\right) = \frac{4}{3} \frac{4}{3} \frac{4}{3} = \boxed{\frac{64}{27}}.$$



Step 2: Boundary On $x = 0$ or $y = 0$, one factor vanishes. On $x + y = 4$, the third factor vanishes. Hence

$$f = 0 \quad \text{on all three edges, including the vertices.}$$

Step 3: Comparison Inside T , all three factors x , y , and $4 - x - y$ are positive, so $f > 0$. Therefore

$$\boxed{f_{\min} = 0 \text{ at every boundary point of } T},$$

$$\boxed{f_{\max} = \frac{64}{27} \text{ at } (4/3, 4/3)}.$$

The candidate list is complete because the polynomial is continuous and T is closed and bounded.

Question 4

Find the absolute extrema of

$$f(x, y) = x + 2y$$

on and inside the ellipse

$$x^2 + 4y^2 \leq 4.$$

Do not use Lagrange multipliers.

Tasks

1. Transform the ellipse to a disk with a change of variables.
2. Use an inequality to derive sharp upper and lower bounds.
3. Find all equality points and verify they lie on the ellipse.

Question 4 – Solution

Strategy. Let $u = x$ and $v = 2y$; then the region is a radius-2 disk and the objective is $u + v$.

Step 1: Transform Set

$$u = x, \quad v = 2y.$$

The constraint becomes $u^2 + v^2 \leq 4$, while

$$f = u + v = \langle u, v \rangle \cdot \langle 1, 1 \rangle.$$

Step 2: Sharp bounds By Cauchy–Schwarz,

$$|u + v| \leq \sqrt{u^2 + v^2} \sqrt{2} \leq 2\sqrt{2}.$$

Thus $-2\sqrt{2} \leq f \leq 2\sqrt{2}$.

Equality on the upper side requires $\langle u, v \rangle$ to point with $\langle 1, 1 \rangle$ and have length 2, giving

$$(u, v) = (\sqrt{2}, \sqrt{2}).$$

Lower equality gives its negative.

Step 3: Return to (x, y) Because $x = u$ and $y = v/2$,

$$f_{\max} = 2\sqrt{2} \text{ at } \left(\sqrt{2}, \frac{\sqrt{2}}{2} \right),$$

$$f_{\min} = -2\sqrt{2} \text{ at } \left(-\sqrt{2}, -\frac{\sqrt{2}}{2} \right).$$

For either point, $x^2 + 4y^2 = 2 + 2 = 4$, verifying boundary membership and sharpness.

Question 5

Find the absolute extrema of

$$f(x, y) = x^2 + y^2 - 3x$$

on the closed annulus

$$1 \leq x^2 + y^2 \leq 4.$$

Tasks

1. Find all interior critical points that belong to the annulus.
2. Analyze both circular boundary components.
3. Compare every candidate and identify the absolute extrema.

Question 5 – Solution

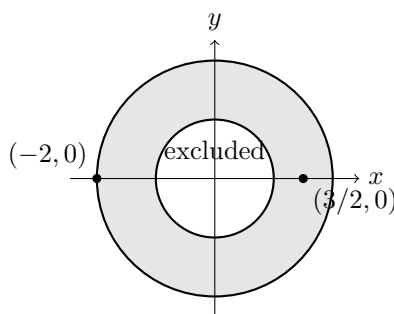
Strategy. Check the annulus interior, then write each boundary circle as $x = r \cos \theta$, $y = r \sin \theta$ with fixed r .

Step 1: Interior

$$f_x = 2x - 3, \quad f_y = 2y.$$

The critical point is $(3/2, 0)$, whose radius $3/2$ lies strictly between 1 and 2. Its value is

$$f(3/2, 0) = \frac{9}{4} - \frac{9}{2} = \boxed{-\frac{9}{4}}.$$



Step 2: Circular boundaries On a circle of radius r ,

$$f = r^2 - 3r \cos \theta.$$

For $r = 1$, its range is $[-2, 4]$. For $r = 2$, its range is $[-2, 10]$. The value 10 occurs on the outer circle when $\cos \theta = -1$, at $(-2, 0)$.

Step 3: Comparison Comparing $-9/4$, the boundary minimum -2 , and the boundary maximum 10 gives

$$\boxed{f_{\min} = -\frac{9}{4} \text{ at } (3/2, 0)}, \quad \boxed{f_{\max} = 10 \text{ at } (-2, 0)}.$$

Both boundary components were checked, so the hole in the region causes no missing candidates.

Question 6

Let

$$D = \{(x, y) : x^2 + y^2 < 1\}$$

be the open unit disk, and let $f(x, y) = x + y$.

Tasks

1. Determine whether f has an absolute maximum or minimum on D .
2. Find its supremum and infimum and give sequences approaching them.
3. Explain exactly which hypothesis of the Extreme Value Theorem is absent.

Question 6 – Solution

Strategy. Bound the linear function with Cauchy–Schwarz, then determine whether equality points belong to the open disk.

Step 1: Bounds For $(x, y) \in D$,

$$|x + y| \leq \sqrt{x^2 + y^2}\sqrt{2} < \sqrt{2}.$$

Therefore every function value lies strictly between $-\sqrt{2}$ and $\sqrt{2}$.

Step 2: Approaching the bounds For $0 < r < 1$, the point

$$\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right)$$

lies in D and gives $f = r\sqrt{2} \rightarrow \sqrt{2}$ as $r \rightarrow 1^-$. Its negative gives values tending to $-\sqrt{2}$. Thus

$$\boxed{\sup_D f = \sqrt{2}}, \quad \boxed{\inf_D f = -\sqrt{2}}.$$

Neither value is attained, so f has no absolute maximum or minimum on D .

Step 3: The theorem The function is continuous and the disk is bounded, but D is not closed: its boundary circle is omitted. Hence the domain is not compact, and the Extreme Value Theorem does not guarantee attained extrema. The missing equality points are precisely

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Question 7

On the unbounded domain $\mathbb{R}^2$, consider

$$f(x, y) = x^2 + y^2 + \frac{1}{1 + x^2 + y^2}.$$

Tasks

1. Determine all absolute minima and their value.
2. Decide whether an absolute maximum exists.
3. Explain why this example shows that compactness is sufficient, but not necessary, for an absolute minimum to exist.

Question 7 – Solution

Strategy. Reduce the function to the radial variable $s = x^2 + y^2 \geq 0$ and study a one-variable function on an unbounded interval.

Step 1: Radial reduction Let

$$g(s) = s + \frac{1}{1+s}, \quad s \geq 0.$$

Then

$$g'(s) = 1 - \frac{1}{(1+s)^2}.$$

At $s = 0$, $g'(0) = 0$, and for every $s > 0$, $(1+s)^2 > 1$, so $g'(s) > 0$. Therefore g increases strictly away from $s = 0$.

It follows that

$$\boxed{f_{\min} = 1 \text{ at } (0,0)}.$$

Step 2: No maximum As $x^2 + y^2 = s \rightarrow \infty$,

$$g(s) = s + \frac{1}{1+s} \rightarrow \infty.$$

Thus f is unbounded above and has $\boxed{\text{no absolute maximum}}$.

Step 3: Interpretation The domain $\mathbb{R}^2$ is closed but not bounded, so the Extreme Value Theorem does not apply. Nevertheless, direct analysis proves that an absolute minimum is attained. Compactness guarantees both extrema for every continuous function, but a particular continuous function may still attain one or both extrema on a noncompact domain.

Question 8

For a real parameter a , define

$$f_a(x, y) = (x - a)^2 + y^2$$

on the square $S = [-1, 1] \times [-1, 1]$.

Tasks

1. Determine the absolute minimum value and all minimizing points for every a .
2. Determine the absolute maximum value and all maximizing points for every a .
3. Treat the tie case $a = 0$ explicitly and interpret the answer as squared distance.

Question 8 – Solution

Strategy. The function is squared distance from (x, y) to $(a, 0)$; minimize and maximize the two nonnegative coordinate contributions independently.

Step 1: Minimum The y^2 term is minimized uniquely at $y = 0$. The closest $x \in [-1, 1]$ to a is the clipped value

$$x_* = \begin{cases} -1, & a < -1, \\ a, & -1 \leq a \leq 1, \\ 1, & a > 1. \end{cases}$$

Hence the unique minimizing point is $(x_*, 0)$, and

$$f_{\min} = (\max\{|a| - 1, 0\})^2.$$

Step 2: Maximum The largest possible y^2 is 1, attained at $y = \pm 1$. The endpoint of $[-1, 1]$ farthest from a is at distance $|a| + 1$. Therefore

$$f_{\max} = (|a| + 1)^2 + 1.$$

If $a > 0$, the maximizers are $(-1, \pm 1)$; if $a < 0$, they are $(1, \pm 1)$.

Step 3: Tie case When $a = 0$, both x -endpoints are equally far away, so all four corners $(\pm 1, \pm 1)$ maximize f_0 with value 2. The point $(0, 0)$ uniquely minimizes it with value 0. This agrees exactly with the squared-distance interpretation.

Question 9

Let R be the closed region between

$$y = x^2 \quad \text{and} \quad y = 2 - x.$$

Find the absolute extrema of

$$f(x, y) = x^2 + y$$

on R .

Tasks

1. Find the boundary intersection points and describe the compact region.
2. Check the interior and each curved boundary piece.
3. Compare all candidates and verify the result from the geometry of the region.

Question 9 – Solution

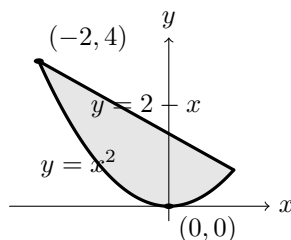
Strategy. Determine the finite interval where the line lies above the parabola, then reduce each boundary to one variable.

Step 1: Region The boundaries meet when

$$x^2 = 2 - x,$$

so $(x + 2)(x - 1) = 0$. The intersection points are $(-2, 4)$ and $(1, 1)$, and

$$-2 \leq x \leq 1, \quad x^2 \leq y \leq 2 - x.$$



Step 2: Candidate search Since $f_y = 1$, there are no interior critical points. On the lower boundary $y = x^2$,

$$f = 2x^2,$$

whose minimum is 0 at $x = 0$ and whose largest endpoint value is 8 at $x = -2$. On the upper boundary $y = 2 - x$,

$$f = x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + \frac{7}{4}.$$

Its boundary minimum is $7/4$ at $x = 1/2$; its endpoint values are 8 and 2.

Step 3: Comparison Therefore

$$\boxed{f_{\min} = 0 \text{ at } (0, 0)}, \quad \boxed{f_{\max} = 8 \text{ at } (-2, 4)}.$$

Because $f = x^2 + y$ increases with y , it is also geometrically consistent that vertical comparisons reduce to the lower curve for the minimum and upper curve for the maximum.

Question 10

Find the absolute extrema of the nondifferentiable function

$$f(x, y) = |x| + y^2$$

on the closed unit disk $x^2 + y^2 \leq 1$.

Tasks

1. Analyze the interior, including the crease $x = 0$ where f_x does not exist.
2. Reduce the circular boundary problem to a single variable $u = |x|$.
3. Find and verify all absolute minimizing and maximizing points.

Question 10 – Solution

Strategy. A complete candidate search must include nondifferentiable points. On the circle, replace y^2 by $1 - x^2$.

Step 1: Interior and crease Where $x \neq 0$,

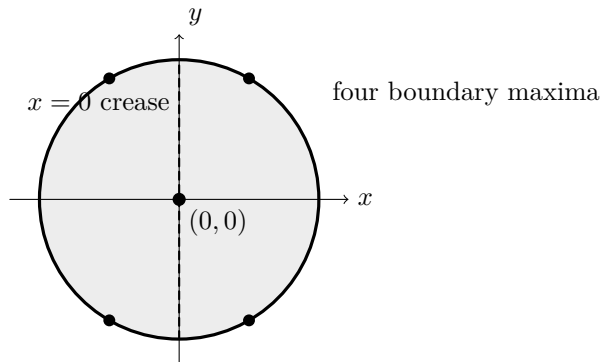
$$f_x = \operatorname{sgn}(x), \quad f_y = 2y,$$

so there are no differentiable interior critical points. On the crease $x = 0$,

$$f(0, y) = y^2.$$

Its smallest value is 0 at the origin. Since $|x| \geq 0$ and $y^2 \geq 0$ everywhere,

$$\boxed{f_{\min} = 0 \text{ at } (0, 0)}.$$



Step 2: Circular boundary Let $u = |x| \in [0, 1]$. On $x^2 + y^2 = 1$,

$$f = u + 1 - u^2 = \frac{5}{4} - \left(u - \frac{1}{2}\right)^2.$$

Thus the largest boundary value is $5/4$, attained when $|x| = 1/2$. Then

$$y^2 = 1 - \frac{1}{4} = \frac{3}{4},$$

so $y = \pm\sqrt{3}/2$.

Step 3: Final comparison For a fixed x , increasing y^2 increases f , so an absolute maximum must lie on the circle; the boundary calculation is therefore decisive. Hence

$$\boxed{f_{\max} = \frac{5}{4} \text{ at } \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)},$$

where the two signs are independent, giving four points. Direct substitution returns $1/2 + 3/4 = 5/4$ at each.

Question 1

Find the constrained extrema of

$$f(x, y) = x + y$$

subject to

$$x^2 + y^2 = 1.$$

Tasks

1. Write and solve the Lagrange multiplier equations.
2. Determine the maximum and minimum values and their locations.
3. Interpret the multiplier condition as tangency of level curves.

Question 1 – Solution

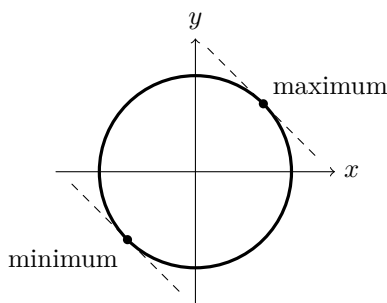
Strategy. Solve $\nabla f = \lambda \nabla g$ together with the circular constraint, then compare the finite candidate set.

Step 1: Lagrange system Let $g = x^2 + y^2$. Then

$$\langle 1, 1 \rangle = \lambda \langle 2x, 2y \rangle, \quad x^2 + y^2 = 1.$$

The first two equations imply $x = y$. Hence $2x^2 = 1$, so

$$(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad \text{or} \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$



Step 2: Values Direct evaluation gives

$$f_{\max} = \sqrt{2} \text{ at } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),$$

$$f_{\min} = -\sqrt{2} \text{ at } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$

Step 3: Geometry and verification The level curves $x + y = c$ are parallel lines with normal ∇f . At an extreme one such line is tangent to the circle, so its normal is parallel to the radial normal ∇g . The circle is compact and the two candidates exhaust the multiplier system, completing the verification.

Question 2

Find the maximum and minimum of

$$f(x, y) = xy$$

subject to the ellipse

$$x^2 + 4y^2 = 4.$$

Tasks

1. Solve the multiplier equations, treating possible zero coordinates carefully.
2. List all constrained extrema and values.
3. Verify the result with a substitution that converts the constraint to a circle.

Question 2 – Solution

Strategy. Solve the component equations without dividing until zero-coordinate cases have been checked.

Step 1: Multiplier equations With $g = x^2 + 4y^2$,

$$\langle y, x \rangle = \lambda \langle 2x, 8y \rangle, \quad x^2 + 4y^2 = 4.$$

If $x = 0$, the first multiplier equation forces $y = 0$, violating the constraint; similarly $y = 0$ is impossible. Thus $xy \neq 0$, and

$$y = 2\lambda x, \quad x = 8\lambda y.$$

Consequently $16\lambda^2 = 1$ and $y = \pm x/2$. The constraint then gives $2x^2 = 4$, so $x = \pm\sqrt{2}$.

Step 2: Candidates and values Equal signs for x and y give $xy = 1$; opposite signs give $xy = -1$. Therefore

$$f_{\max} = 1 \text{ at } (\sqrt{2}, \sqrt{2}/2), (-\sqrt{2}, -\sqrt{2}/2),$$

$$f_{\min} = -1 \text{ at } (\sqrt{2}, -\sqrt{2}/2), (-\sqrt{2}, \sqrt{2}/2).$$

Step 3: Verification Let $u = x$ and $v = 2y$. Then $u^2 + v^2 = 4$ and $f = uv/2$. Since

$$|uv| \leq \frac{u^2 + v^2}{2} = 2,$$

we have $|f| \leq 1$, with equality precisely when $|u| = |v|$. This reproduces all four points.

Question 3

Find the points on the parabola

$$y = x^2$$

closest to $A = (0, 2)$.

Tasks

1. Formulate the squared-distance objective and the constraint.
2. Solve the Lagrange system, including all branches.
3. Compare candidates and justify that the reported minimum is absolute despite the noncompact constraint.

Question 3 – Solution

Strategy. Minimize squared distance, which has the same minimizers as distance but simpler derivatives.

Step 1: Set up Let

$$f(x, y) = x^2 + (y - 2)^2, \quad g(x, y) = y - x^2.$$

The system $\nabla f = \lambda \nabla g$ is

$$2x = -2\lambda x, \quad 2(y - 2) = \lambda, \quad y = x^2.$$

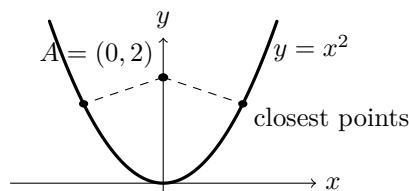
Step 2: Solve branches If $x = 0$, then $y = 0$, producing the candidate $(0, 0)$ with squared distance 4.

If $x \neq 0$, the first equation gives $\lambda = -1$. The second gives $2(y - 2) = -1$, so $y = 3/2$. Hence

$$x = \pm \sqrt{\frac{3}{2}}.$$

At either point the squared distance is

$$\frac{3}{2} + \left(-\frac{1}{2}\right)^2 = \boxed{\frac{7}{4}}.$$



Step 3: Conclusion Comparing candidates gives the two closest points

$$\boxed{\left(\pm \sqrt{\frac{3}{2}}, \frac{3}{2}\right)},$$

each at distance $\boxed{\sqrt{7}/2}$ from A . Along the parabola, $f(x, x^2) = x^2 + (x^2 - 2)^2 \rightarrow \infty$ as $|x| \rightarrow \infty$. Therefore no minimizing sequence can escape to infinity, and the smaller candidate value is the absolute minimum.

Question 4

A rectangular box has positive side lengths x, y, z and fixed sum of its three side lengths

$$x + y + z = 12.$$

Find the dimensions giving the largest volume $V = xyz$.

Tasks

1. Solve the three-variable Lagrange multiplier system.
2. Find the maximum volume.
3. Verify global maximality by checking the boundary of the closed simplex and using an inequality.

Question 4 – Solution

Strategy. The multiplier equations force equal positive side lengths; then compare with the zero-volume boundary and verify by AM–GM.

Step 1: Lagrange system For $g = x + y + z$,

$$\nabla V = \langle yz, xz, xy \rangle = \lambda \langle 1, 1, 1 \rangle.$$

Because $x, y, z > 0$, equality of the three components gives

$$yz = xz = xy,$$

and division by positive factors yields $x = y = z$. The constraint then gives

$$\boxed{x = y = z = 4}.$$

Step 2: Volume

$$\boxed{V_{\max} = 4 \cdot 4 \cdot 4 = 64}.$$

Step 3: Global verification Allowing $x, y, z \geq 0$ closes the feasible simplex. Its boundary has at least one zero side, so $V = 0$ there. In the positive interior, AM–GM gives

$$\frac{x + y + z}{3} \geq (xyz)^{1/3}.$$

Since the left side is 4,

$$xyz \leq 64,$$

with equality only when $x = y = z$. Thus the Lagrange candidate is the unique positive absolute maximizer.

Question 5

The sphere

$$x^2 + y^2 + z^2 = 9$$

meets the plane

$$x + y + z = 3.$$

Find the highest and lowest points of their intersection curve.

Tasks

1. Use two Lagrange multipliers to extremize the height $f(x, y, z) = z$.
2. Solve the complete system and identify both points.
3. Verify the result geometrically or by direct substitution.

Question 5 – Solution

Strategy. At a constrained extreme with two regular constraints, ∇f lies in the span of both constraint gradients.

Step 1: Two-multiplier system Let

$$g = x^2 + y^2 + z^2, \quad h = x + y + z.$$

Then

$$\langle 0, 0, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle + \mu \langle 1, 1, 1 \rangle.$$

The first two component equations imply $2\lambda(x - y) = 0$. If $\lambda = 0$, then the first equation gives $\mu = 0$, contradicting the third. Hence $x = y = a$.

Step 2: Enforce both constraints The plane gives $z = 3 - 2a$. Substitution into the sphere gives

$$2a^2 + (3 - 2a)^2 = 9,$$

or

$$6a(a - 2) = 0.$$

Thus $a = 0$, $z = 3$, or $a = 2$, $z = -1$. Therefore

$$\boxed{\text{highest point } (0, 0, 3)}, \quad \boxed{\text{lowest point } (2, 2, -1)}.$$

Step 3: Verification Both points satisfy the sphere and plane equations. Their heights are 3 and -1 . The intersection is a closed circle, hence compact; the two candidates exhaust the multiplier equations, so these are the absolute height extrema.

Question 6

Consider maximizing and minimizing

$$f(x, y) = x$$

subject to the constraint

$$g(x, y) = x^2 + y^2 = 0.$$

Tasks

1. Determine the feasible set and its constrained extrema directly.
2. Show that no multiplier λ satisfies $\nabla f = \lambda \nabla g$ at the feasible point.
3. Explain which constraint-qualification hypothesis fails and why this does not contradict the method.

Question 6 – Solution

Strategy. Inspect the feasible set before applying the multiplier equation; the constraint has a singular gradient at its only point.

Step 1: Direct analysis The equation $x^2 + y^2 = 0$ over the reals forces

$$x = 0, \quad y = 0.$$

Thus the feasible set is the singleton $\{(0, 0)\}$. Its only function value is 0, so

$$\boxed{f_{\min} = f_{\max} = 0 \text{ at } (0, 0)}.$$

Step 2: Multiplier failure

$$\nabla f(0, 0) = \langle 1, 0 \rangle, \quad \nabla g(0, 0) = \langle 0, 0 \rangle.$$

The equation

$$\langle 1, 0 \rangle = \lambda \langle 0, 0 \rangle$$

has no solution for any real λ .

Step 3: Interpretation The standard Lagrange multiplier theorem assumes $\nabla g \neq \mathbf{0}$ at the constrained extremum. That regularity condition fails here, so the theorem makes no promise. The example shows that singular feasible points must be checked separately; it does not invalidate the multiplier method at regular constraint points.

Question 7

Find the constrained extrema of

$$f(x, y) = x^2 + y^2$$

subject to

$$x^2 + xy + y^2 = 3.$$

Tasks

1. Solve the Lagrange equations without assuming x or y is nonzero.
2. Determine all maximizing and minimizing points and values.
3. Verify the result by rewriting the constraint in rotated directions $y = x$ and $y = -x$.

Question 7 – Solution

Strategy. Eliminate the multiplier by combining the two component equations; the resulting factorization identifies the principal directions.

Step 1: Multiplier equations

$$\langle 2x, 2y \rangle = \lambda \langle 2x + y, x + 2y \rangle, \quad x^2 + xy + y^2 = 3.$$

Multiply the first component equation by y , the second by x , and subtract. The left sides cancel, leaving

$$\lambda(y^2 - x^2) = 0.$$

The constraint rules out $(0, 0)$, and $\lambda = 0$ would force $x = y = 0$. Hence

$$y = \pm x.$$

Step 2: Candidate values If $y = x$, the constraint is $3x^2 = 3$, so $(x, y) = (1, 1)$ or $(-1, -1)$ and $f = 2$.

If $y = -x$, the constraint is $x^2 = 3$, giving $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ with $f = 6$. Therefore

$$f_{\min} = 2 \text{ at } (1, 1), (-1, -1),$$

$$f_{\max} = 6 \text{ at } (\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}).$$

Step 3: Verification The constraint ellipse is compact. Along its $y = x$ principal direction its radius squared is 2, while along $y = -x$ its radius squared is 6. These are exactly the values of $x^2 + y^2$, confirming the classifications and completeness.

Question 8

For the family

$$f_a(x, y) = x^2 + ay^2,$$

the point $P = (1, 2)$ is a constrained extremum on the line $x + y = 3$.

Tasks

1. Determine the parameter a using the Lagrange condition.
2. Classify the constrained extremum.
3. Verify the classification by substituting the constraint into the resulting function.

Question 8 – Solution

Strategy. Parallel gradients at the specified regular point determine the parameter; a one-variable restriction then establishes the classification.

Step 1: Recover a For $g = x + y$,

$$\nabla f_a(1, 2) = \langle 2, 4a \rangle, \quad \nabla g = \langle 1, 1 \rangle.$$

The equation $\langle 2, 4a \rangle = \lambda \langle 1, 1 \rangle$ gives $\lambda = 2$ and $4a = 2$. Hence

$$\boxed{a = \frac{1}{2}}.$$

Step 2: Restrict to the line Set $y = 3 - x$. Then

$$f_{1/2}(x, 3 - x) = x^2 + \frac{1}{2}(3 - x)^2.$$

Completing the square,

$$x^2 + \frac{1}{2}(3 - x)^2 = \frac{3}{2}(x - 1)^2 + 3.$$

Therefore the constrained value is minimized uniquely at $x = 1$, $y = 2$.

Step 3: Result and verification

$$\boxed{P = (1, 2) \text{ is a strict constrained minimum}}, \quad \boxed{f_{1/2}(1, 2) = 3}.$$

The restricted quadratic tends to infinity as $|x| \rightarrow \infty$, so the minimum is also absolute along the entire constraint line; no constrained maximum exists.

Question 9

A firm must choose positive inputs x, y satisfying

$$xy = 100.$$

Its cost is

$$C(x, y) = 3x + 2y.$$

Tasks

1. Use Lagrange multipliers to find the cost-minimizing inputs.
2. Compute the minimum cost and interpret the multiplier balance.
3. Verify global minimality by substitution or an inequality.

Question 9 – Solution

Strategy. The multiplier equations balance marginal costs against the constraint normal; then a one-variable check rules out other behavior.

Step 1: Multiplier equations With $g = xy$,

$$\langle 3, 2 \rangle = \lambda \langle y, x \rangle, \quad xy = 100.$$

Thus $3 = \lambda y$ and $2 = \lambda x$. Eliminating λ gives

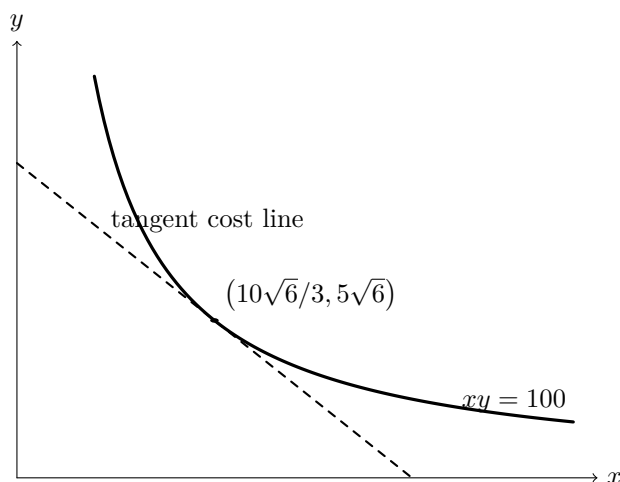
$$3x = 2y, \quad y = \frac{3}{2}x.$$

The positive constraint then yields

$$x^2 = \frac{200}{3},$$

so

$$x = \frac{10\sqrt{6}}{3}, \quad y = 5\sqrt{6}.$$



Step 2: Cost and interpretation

$$C_{\min} = 3 \left(\frac{10\sqrt{6}}{3} \right) + 2(5\sqrt{6}) = 20\sqrt{6}.$$

The relation $3/y = 2/x = \lambda$ says the cost-gradient components are proportional to the production-constraint gradient components.

Step 3: Global verification Using $y = 100/x$ for $x > 0$,

$$C(x) = 3x + \frac{200}{x} \longrightarrow \infty$$

as $x \rightarrow 0^+$ or $x \rightarrow \infty$. Its only critical point is the one found above, and $C''(x) = 400/x^3 > 0$. Hence it is the unique absolute constrained minimum.

Question 10

Find the absolute extrema of

$$f(x, y, z) = xyz$$

on the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6.$$

Tasks

1. Solve the Lagrange equations for candidates with nonzero product.
2. Treat zero-coordinate candidates and compare all possible values.
3. List every maximizing and minimizing point and verify the sign pattern.

Question 10 – Solution

Strategy. Multiplying each component equation by its corresponding coordinate reveals fixed ratios among x^2, y^2, z^2 .

Step 1: Nonzero-product candidates The multiplier system is

$$yz = 2\lambda x, \quad xz = 4\lambda y, \quad xy = 6\lambda z.$$

When $xyz \neq 0$, multiply these equations by x, y, z , respectively:

$$xyz = 2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2.$$

Thus

$$2x^2 = 4y^2 = 6z^2 = K.$$

Substitution into the constraint gives

$$\frac{K}{2} + 2\frac{K}{4} + 3\frac{K}{6} = \frac{3K}{2} = 6,$$

so $K = 4$. Therefore

$$x^2 = 2, \quad y^2 = 1, \quad z^2 = \frac{2}{3}.$$

Step 2: Values and zero cases At these eight sign choices,

$$|xyz| = \sqrt{2} \cdot 1 \cdot \sqrt{\frac{2}{3}} = \frac{2}{\sqrt{3}}.$$

Any feasible candidate with a zero coordinate has $f = 0$, which lies strictly between the positive and negative nonzero values.

Step 3: All extrema The ellipsoid is compact, so comparison is decisive:

$$f_{\max} = \frac{2}{\sqrt{3}}$$

at the four points $(\pm\sqrt{2}, \pm 1, \pm\sqrt{2/3})$ having an even number of minus signs, and

$$f_{\min} = -\frac{2}{\sqrt{3}}$$

at the four such points having an odd number of minus signs. The parity rule follows because the product is positive for an even number of negative factors and negative for an odd number.

Question 1

Let

$$R = [0, 2] \times [0, 1], \quad f(x, y) = 6 - x^2 - y.$$

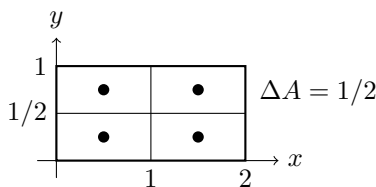
Partition R into two equal columns and two equal rows, and use the midpoint of each subrectangle as its sample point.

Tasks

1. List the four sample points and the common subrectangle area.
2. Compute the midpoint Riemann sum.
3. Interpret the sum geometrically and check it against bounds from the minimum and maximum of f on R .

Question 1 – Solution

Strategy. Translate the partition into four base areas, evaluate the height at each midpoint, and add the resulting prism volumes.



Step 1: Partition data Here $\Delta x = 1$, $\Delta y = 1/2$, and

$$\Delta A = \Delta x \Delta y = \frac{1}{2}.$$

The midpoints are

$$\left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{3}{2}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{3}{2}, \frac{3}{4}\right).$$

Step 2: Heights and sum In the same order, the four function values are

$$\frac{11}{2}, \quad \frac{7}{2}, \quad 5, \quad 3.$$

Therefore

$$S = \frac{1}{2} \left(\frac{11}{2} + \frac{7}{2} + 5 + 3 \right) = \boxed{\frac{17}{2}}.$$

Step 3: Interpretation and bounds The sum is the total volume of four midpoint prisms. On R , f decreases as either x or y increases, so

$$1 = f(2, 1) \leq f(x, y) \leq f(0, 0) = 6.$$

Since $\text{area}(R) = 2$, every Riemann-sum estimate lies between 2 and 12; indeed $17/2$ does. This also verifies that every sampled height was positive and on the correct scale.

Question 2

On $R = [0, 1] \times [0, 2]$, let $f(x, y) = x + 2y$. Divide the x -interval into m equal parts and the y -interval into n equal parts. Use the upper-right corner of each subrectangle as its sample point.

Tasks

1. Write the resulting double Riemann sum $S_{m,n}$.
2. Simplify the finite sum exactly using summation formulas.
3. Take the limit as $m, n \rightarrow \infty$ to evaluate the double integral from its definition.

Question 2 – Solution

Strategy. Express every sample point and area in terms of its indices, then separate the finite sums algebraically before taking the two-parameter limit.

Step 1: Grid and samples We have

$$\Delta x = \frac{1}{m}, \quad \Delta y = \frac{2}{n}, \quad \Delta A = \frac{2}{mn},$$

and the upper-right sample in cell (i, j) is $(i/m, 2j/n)$. Thus

$$S_{m,n} = \sum_{i=1}^m \sum_{j=1}^n \left(\frac{i}{m} + \frac{4j}{n} \right) \frac{2}{mn}.$$

Step 2: Exact finite sum Using $\sum_{k=1}^N k = N(N+1)/2$,

$$\begin{aligned} S_{m,n} &= \frac{2}{m^2 n} \sum_{i=1}^m \sum_{j=1}^n i + \frac{8}{mn^2} \sum_{i=1}^m \sum_{j=1}^n j \\ &= \frac{2}{m^2 n} \left(n \frac{m(m+1)}{2} \right) + \frac{8}{mn^2} \left(m \frac{n(n+1)}{2} \right) \\ &= \frac{m+1}{m} + 4 \frac{n+1}{n}. \end{aligned}$$

Step 3: Definition limit Both correction terms vanish independently, so

$$\boxed{\iint_R (x + 2y) dA = \lim_{m,n \rightarrow \infty} S_{m,n} = 1 + 4 = 5}.$$

The answer lies between $0 \cdot 2$ and $5 \cdot 2 = 10$, the bounds obtained from the extreme values of f on the area-2 rectangle.

Question 3

Let $f(x, y) = xy$ on the unit square $R = [0, 1] \times [0, 1]$. Partition R into an $n \times n$ uniform grid.

Tasks

1. Compute the lower sum L_n and upper sum U_n .
2. Use them to determine $\iint_R xy \, dA$ without iterated integration.
3. Find the least integer n for which the gap $U_n - L_n$ is less than 0.01.

Question 3 – Solution

Strategy. Because xy increases in each coordinate on the unit square, lower-left and upper-right corners produce the lower and upper sums.

Step 1: Lower sum Every cell has area $1/n^2$. Using indices $i, j = 0, \dots, n-1$ at lower-left corners,

$$\begin{aligned} L_n &= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{i}{n} \frac{j}{n} = \frac{1}{n^4} \left(\sum_{i=0}^{n-1} i \right)^2 \\ &= \frac{1}{n^4} \left(\frac{n(n-1)}{2} \right)^2 = \frac{(n-1)^2}{4n^2}. \end{aligned}$$

Step 2: Upper sum and squeeze Upper-right corners give

$$U_n = \frac{1}{n^4} \left(\sum_{i=1}^n i \right)^2 = \frac{(n+1)^2}{4n^2}.$$

Both sums approach $1/4$. Since $L_n \leq \iint_R xy \, dA \leq U_n$,

$$\boxed{\iint_R xy \, dA = \frac{1}{4}}.$$

Step 3: Resolution The uncertainty is

$$U_n - L_n = \frac{(n+1)^2 - (n-1)^2}{4n^2} = \frac{1}{n}.$$

Thus $1/n < 0.01$ requires $n > 100$, and the least integer is n=101. The strict inequality rules out $n = 100$.

Question 4

On $R = [-2, 2] \times [0, 3]$, define

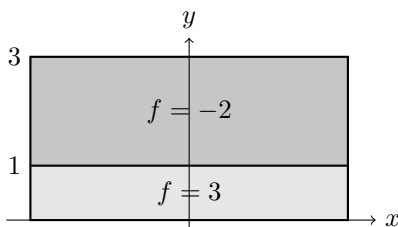
$$f(x, y) = \begin{cases} 3, & 0 \leq y \leq 1, \\ -2, & 1 < y \leq 3. \end{cases}$$

Tasks

1. Evaluate $\iint_R f \, dA$ by signed volume.
2. Evaluate $\iint_R |f| \, dA$ and explain the difference.
3. Find the average value of f on R .

Question 4 – Solution

Strategy. Split the base into the two constant-height bands and distinguish algebraic volume from total unsigned volume.



Step 1: Signed volume The lower band has area $4(1) = 4$ and height 3, so it contributes 12. The upper band has area $4(2) = 8$ and height -2 , so it contributes -16 . Hence

$$\iint_R f \, dA = 12 - 16 = -4.$$

Step 2: Unsigned volume Absolute value changes the second height from -2 to 2. Therefore

$$\iint_R |f| \, dA = 12 + 16 = 28.$$

The first integral records volume below the xy -plane negatively; the second records all geometric volume positively.

Step 3: Average value Since $\text{area}(R) = 4 \cdot 3 = 12$,

$$f_{\text{avg}} = \frac{1}{12} \iint_R f \, dA = -\frac{1}{3}.$$

As a check, $-1/3$ lies between the only two function values, -2 and 3 , and the wider negative band explains its negative sign.

Question 5

Let $R = [-1, 2] \times [0, 2]$. A continuous function f satisfies

$$2 \leq f(x, y) \leq 5, \quad \iint_R f \, dA = 19.$$

A second integrable function g satisfies $|g(x, y) - f(x, y)| \leq 0.1$ throughout R .

Tasks

1. Check that the stated integral of f is consistent with its pointwise bounds.
2. Evaluate $\iint_R (3f - 4) \, dA$.
3. Give the sharp bounds guaranteed by the data for $\iint_R g \, dA$.

Question 5 – Solution

Strategy. Convert pointwise inequalities to integral inequalities using the area of the rectangle, and use linearity for the transformed function.

Step 1: Comparison bounds The rectangle has area

$$A = (2 - (-1))(2 - 0) = 6.$$

Integrating $2 \leq f \leq 5$ over R gives

$$12 \leq \iint_R f \, dA \leq 30.$$

The given value 19 lies in this interval, so the data are consistent.

Step 2: Linearity Because the integral of the constant 4 is $4A = 24$,

$$\iint_R (3f - 4) \, dA = 3 \iint_R f \, dA - \iint_R 4 \, dA = 57 - 24 = \boxed{33}.$$

Step 3: Stability bound From $|g - f| \leq 0.1$,

$$-0.1 \leq g - f \leq 0.1.$$

Integration over an area-6 region yields

$$-0.6 \leq \iint_R (g - f) \, dA \leq 0.6.$$

Consequently,

$$\boxed{18.4 \leq \iint_R g \, dA \leq 19.6}.$$

These endpoints are attainable by $g = f - 0.1$ and $g = f + 0.1$, so no tighter bounds follow from the supplied information.

Question 6

On the centrally symmetric rectangle

$$R = [-2, 2] \times [-1, 1],$$

consider $f(x, y) = 5 + x^3 + xy^2 + 2y$.

Tasks

1. Show that the nonconstant part cancels under central reflection.
2. Evaluate $\iint_R f \, dA$ and find the average value of f .
3. Explain precisely which symmetry of the region makes the argument valid.

Question 6 – Solution

Strategy. Pair each point (x, y) with its central reflection $(-x, -y)$ and average the two function values.

Step 1: Pairing Let $h(x, y) = x^3 + xy^2 + 2y$. Then

$$h(-x, -y) = -x^3 - xy^2 - 2y = -h(x, y).$$

Thus the contributions of h at every reflected pair cancel. Equivalently,

$$f(x, y) + f(-x, -y) = 10.$$

Step 2: Integral and average Central reflection maps R onto itself without changing area. Therefore the paired average of f is 5 everywhere, and

$$\iint_R f \, dA = 5 \operatorname{area}(R) = 5(4)(2) = \boxed{40}.$$

Dividing by the area 8 gives

$$\boxed{f_{\text{avg}} = 5}.$$

Step 3: Symmetry check The required property is

$$(x, y) \in R \iff (-x, -y) \in R.$$

Symmetry in only one coordinate would not automatically cancel the entire h : xy^2 is odd in x , while $2y$ is odd in y . Central symmetry reverses both coordinates at once and cancels all three terms.

Question 7

The plane $z = 6 + x - 2y$ lies above the rectangle

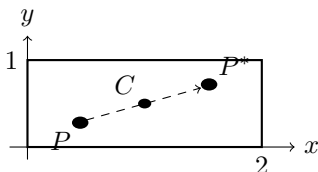
$$R = [0, 2] \times [0, 1].$$

Tasks

1. Pair each point of R with its reflection through the center $(1, 1/2)$ and compare their heights.
2. Use the pairing to find the volume under the plane without iterated integration.
3. Verify the result using the four corner heights.

Question 7 – Solution

Strategy. An affine function has complementary values at points reflected through a rectangle's center, so every reflected pair has the same average height.



Step 1: Reflected heights The reflection of $P = (x, y)$ through $C = (1, 1/2)$ is

$$P^* = (2 - x, 1 - y).$$

For $f(x, y) = 6 + x - 2y$,

$$\begin{aligned} f(P) + f(P^*) &= (6 + x - 2y) + (6 + (2 - x) - 2(1 - y)) \\ &= 12. \end{aligned}$$

Thus every pair has average height $6 = f(C)$.

Step 2: Volume The base area is 2, and the plane's minimum corner height is $4 > 0$, so the double integral is geometric volume:

$$V = \iint_R (6 + x - 2y) dA = 6(2) = 12.$$

Step 3: Corner check The corner heights are

$$f(0, 0) = 6, \quad f(2, 0) = 8, \quad f(0, 1) = 4, \quad f(2, 1) = 6.$$

Their average is $(6 + 8 + 4 + 6)/4 = 6$, agreeing with the center and pairing calculation. Multiplication by base area again gives 12.

Question 8

An affine function $f(x, y) = a + bx + cy$ is defined on the unit square. Its average value is 4 on the whole square, 5 on the right half $[1/2, 1] \times [0, 1]$, and 3 on the top half $[0, 1] \times [1/2, 1]$.

Tasks

1. Use center values of affine functions to obtain three equations for a, b, c .
2. Determine f uniquely.
3. Predict the integral of f over the lower-left quarter $[0, 1/2] \times [0, 1/2]$ and verify consistency.

Question 8 – Solution

Strategy. The average of an affine function over a rectangle equals its value at that rectangle's center; use the three given centers to reconstruct the coefficients.

Step 1: Average equations The relevant centers are $(1/2, 1/2)$, $(3/4, 1/2)$, and $(1/2, 3/4)$. Hence

$$\begin{aligned}a + \frac{b}{2} + \frac{c}{2} &= 4, \\a + \frac{3b}{4} + \frac{c}{2} &= 5, \\a + \frac{b}{2} + \frac{3c}{4} &= 3.\end{aligned}$$

The center-value rule follows from the same reflection pairing used for any affine graph over a rectangle.

Step 2: Reconstruct Subtracting the first equation from the second gives $b/4 = 1$, so $b = 4$. Subtracting the first from the third gives $c/4 = -1$, so $c = -4$. The first equation then gives $a = 4$. Therefore

$$\boxed{f(x, y) = 4 + 4x - 4y}.$$

Step 3: New prediction The lower-left quarter has center $(1/4, 1/4)$ and area $1/4$. Its average is

$$f\left(\frac{1}{4}, \frac{1}{4}\right) = 4 + 1 - 1 = 4,$$

so

$$\boxed{\iint_{[0, 1/2]^2} f \, dA = 4 \left(\frac{1}{4}\right) = 1}.$$

The equal x - and y -center coordinates cancel the opposite slopes, which checks the predicted average.

Question 9

Define the bounded function on the unit square by

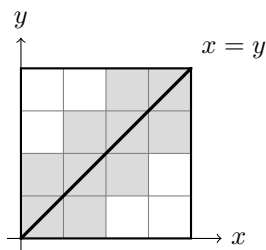
$$f(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

Tasks

1. Explain why every lower sum is zero.
2. Use a uniform $n \times n$ grid to bound an upper sum by $3/n$.
3. Prove that f is Riemann integrable and find its double integral, despite its discontinuities.

Question 9 – Solution

Strategy. The diagonal has no area: trap every possible nonzero upper-sum contribution in a shrinking collection of grid squares.



Step 1: Lower sums Every subrectangle with positive area contains a point off the diagonal, where $f = 0$. Its infimum is therefore 0, so every lower sum equals 0.

Step 2: Upper-sum cover In an $n \times n$ grid, each square has area $1/n^2$. The n diagonal squares meet $x = y$. At each of the $n - 1$ interior diagonal grid vertices, two additional off-diagonal squares have closures meeting the diagonal. Thus at most

$$n + 2(n - 1) = 3n - 2$$

closed grid squares have supremum 1; all others have supremum 0. Consequently,

$$0 \leq U_n \leq \frac{3n - 2}{n^2} < \frac{3}{n}.$$

Step 3: Integrability Given $\varepsilon > 0$, choose $n > 3/\varepsilon$. Then

$$0 \leq U_n - L_n < \frac{3}{n} < \varepsilon.$$

The Darboux criterion proves integrability, and the integral is squeezed between zero lower sums and upper sums tending to zero:

$$\boxed{\iint_{[0,1]^2} f \, dA = 0}.$$

The function is discontinuous at every point of the diagonal, but that one-dimensional set contributes no area.

Question 10

A function f on the unit square satisfies the Lipschitz estimate

$$|f(P) - f(Q)| \leq 2\|P - Q\|$$

for all points P, Q . Let S_n be the midpoint Riemann sum on an $n \times n$ uniform grid.

Tasks

1. Prove the error bound $|\iint_R f \, dA - S_n| \leq \sqrt{2}/n$.
2. Find the least n certified by this bound to give error less than 0.01.
3. If $S_{142} = 2.731$, give a guaranteed interval containing the true integral.

Question 10 – Solution

Strategy. Control the variation within each cell by its maximum distance from the midpoint, then add the cellwise error bounds.

Step 1: One-cell error The Lipschitz estimate makes f continuous, hence integrable on the closed square. A grid square has side $1/n$, area $1/n^2$, and maximum distance from its midpoint to a corner

$$d_n = \frac{\sqrt{2}}{2n}.$$

If M_{ij} is its midpoint, the Lipschitz condition gives

$$|f(P) - f(M_{ij})| \leq 2d_n = \frac{\sqrt{2}}{n}$$

throughout that cell. Its integral differs from midpoint height times area by at most $\sqrt{2}/n^3$.

Step 2: Global error There are n^2 cells, so the triangle inequality yields

$$\left| \iint_R f \, dA - S_n \right| \leq n^2 \frac{\sqrt{2}}{n^3} = \frac{\sqrt{2}}{n}.$$

To guarantee strict error below 0.01, require $n > 100\sqrt{2} \approx 141.421$. The least integer is therefore n=142.

Step 3: Numerical enclosure For $n = 142$,

$$\frac{\sqrt{2}}{142} \approx 0.009959.$$

Thus

$$2.731 - \frac{\sqrt{2}}{142} \leq \iint_R f \, dA \leq 2.731 + \frac{\sqrt{2}}{142},$$

or approximately

$$2.72104 \leq \iint_R f \, dA \leq 2.74096.$$

The half-diagonal, rather than the full cell diagonal, is essential to the stated constant.

Question 1

Evaluate the double integral of

$$f(x, y) = 3x^2y - 2x + 4$$

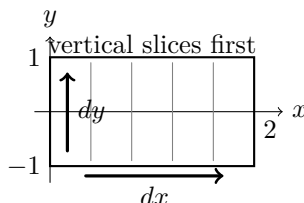
over $R = [0, 2] \times [-1, 1]$ by iterated integration.

Tasks

1. Evaluate by integrating with respect to y first.
2. Evaluate again by integrating with respect to x first.
3. Explain why the two answers must agree.

Question 1 – Solution

Strategy. Write both constant-bound orders explicitly and simplify the inner integral before performing the outer one.



Step 1: Integrate in y first

$$\begin{aligned}\int_0^2 \int_{-1}^1 (3x^2y - 2x + 4) dy dx &= \int_0^2 \left[\frac{3}{2}x^2y^2 - 2xy + 4y \right]_{-1}^1 dx \\ &= \int_0^2 (-4x + 8) dx \\ &= [-2x^2 + 8x]_0^2 = 8.\end{aligned}$$

The $3x^2y$ term vanishes because it is odd in y over a symmetric interval.

Step 2: Integrate in x first

$$\begin{aligned}\int_{-1}^1 \int_0^2 (3x^2y - 2x + 4) dx dy &= \int_{-1}^1 [x^3y - x^2 + 4x]_0^2 dy \\ &= \int_{-1}^1 (8y + 4) dy = 8.\end{aligned}$$

Thus

$$\boxed{\iint_R f dA = 8}.$$

Verification The polynomial is continuous on the closed rectangle, so Fubini's theorem permits either order. In the second order the odd term $8y$ cancels, leaving constant height 4 across a y -interval of length 2, again giving 8.

Question 2

Evaluate

$$\iint_R e^{2x} \cos y \, dA, \quad R = [0, \ln 3] \times \left[0, \frac{\pi}{3}\right].$$

Tasks

1. Recognize and use the product structure of the integrand.
2. Evaluate the integral in each order.
3. Verify the exact result by multiplying the two one-variable integrals.

Question 2 – Solution

Strategy. The integrand is a product of an x -only factor and a y -only factor, so constant rectangular bounds make the two integrations separate.

Step 1: Integrate in y first

$$\begin{aligned}\int_0^{\ln 3} \int_0^{\pi/3} e^{2x} \cos y \, dy \, dx &= \int_0^{\ln 3} e^{2x} [\sin y]_0^{\pi/3} \, dx \\ &= \frac{\sqrt{3}}{2} \left[\frac{1}{2} e^{2x} \right]_0^{\ln 3} \\ &= \frac{\sqrt{3}}{4} (9 - 1) = 2\sqrt{3}.\end{aligned}$$

Step 2: Reverse the order

$$\begin{aligned}\int_0^{\pi/3} \int_0^{\ln 3} e^{2x} \cos y \, dx \, dy &= \int_0^{\pi/3} \cos y \left[\frac{1}{2} e^{2x} \right]_0^{\ln 3} \, dy \\ &= 4[\sin y]_0^{\pi/3} = 2\sqrt{3}.\end{aligned}$$

Therefore

$$\boxed{\iint_R e^{2x} \cos y \, dA = 2\sqrt{3}}.$$

Verification Direct factorization gives

$$\left(\int_0^{\ln 3} e^{2x} \, dx \right) \left(\int_0^{\pi/3} \cos y \, dy \right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3},$$

confirming both orders and the endpoint $e^{2\ln 3} = 9$.

Question 3

Consider

$$I = \int_0^1 \int_0^2 x e^{xy} dy dx.$$

Tasks

1. Evaluate I using the displayed order.
2. Write the reversed iterated integral and its inner antiderivative for $y \neq 0$.
3. Explain why the displayed order is substantially more efficient and check the behavior at $x = 0$.

Question 3 – Solution

Strategy. Treat x as constant in the inner integral; the factor x is exactly the derivative needed for e^{xy} with respect to y .

Step 1: Efficient order For $x \neq 0$,

$$\int_0^2 x e^{xy} dy = [e^{xy}]_0^2 = e^{2x} - 1.$$

At $x = 0$ the original integrand is identically zero and the same expression gives $e^0 - 1 = 0$, so the formula extends continuously. Hence

$$\begin{aligned} I &= \int_0^1 (e^{2x} - 1) dx = \left[\frac{1}{2} e^{2x} - x \right]_0^1 \\ &= \boxed{\frac{e^2 - 3}{2}}. \end{aligned}$$

Step 2: Reversed order The rectangle permits

$$I = \int_0^2 \int_0^1 x e^{xy} dx dy.$$

For $y \neq 0$, integration by parts gives

$$\int x e^{xy} dx = \frac{e^{xy}(xy - 1)}{y^2} + C,$$

so the inner result is

$$\frac{e^y(y - 1) + 1}{y^2}.$$

Step 3: Efficiency and verification The reversed expression has a removable singular appearance at $y = 0$ and requires another nonobvious antiderivative. The original order collapses immediately by substitution. Continuity of $x e^{xy}$ on the rectangle guarantees by Fubini's theorem that either completed order has the same value.

Question 4

A student is given

$$I = \int_{-1}^1 \int_0^2 (x - 2y) \, dx \, dy$$

and claims the inner integral is $[x^2/2 - 2xy]_{-1}^1$.

Tasks

1. Diagnose the student's error using the differential and bounds.
2. Evaluate the integral correctly as written.
3. Reverse the order on the same rectangle and verify the result.

Question 4 – Solution

Strategy. Read an iterated integral from the inside out: the inner differential determines both the variable of integration and which pair of bounds belongs to it.

Step 1: Diagnose The inner differential is dx , so x runs from 0 to 2 while y is held constant. The student incorrectly used the outer y -bounds, -1 and 1 , as x -bounds.

Step 2: Evaluate as written

$$\begin{aligned} I &= \int_{-1}^1 \left[\frac{x^2}{2} - 2yx \right]_0^2 dy = \int_{-1}^1 (2 - 4y) dy \\ &= [2y - 2y^2]_{-1}^1 = \boxed{4}. \end{aligned}$$

The odd contribution $-4y$ cancels over $[-1, 1]$.

Step 3: Reverse correctly Constant rectangular bounds reverse to

$$I = \int_0^2 \int_{-1}^1 (x - 2y) dy dx.$$

Then

$$I = \int_0^2 [xy - y^2]_{-1}^1 dx = \int_0^2 2x dx = [x^2]_0^2 = 4.$$

The agreement verifies both the order reversal and the association of each bound pair with its variable.

Question 5

Evaluate

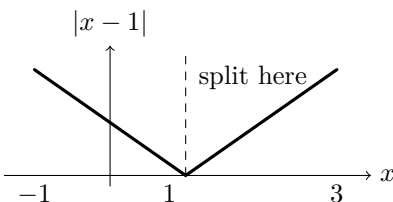
$$\iint_R |x - 1|(1 + y) \, dA, \quad R = [-1, 3] \times [0, 2].$$

Tasks

1. Choose an efficient order of integration.
2. Split at the correct location to remove the absolute value.
3. Evaluate the integral and its average value over R .

Question 5 – Solution

Strategy. Integrate the positive y -factor first, then split the remaining x -integral where $x - 1$ changes sign.



Step 1: Inner integral Using dy first,

$$\begin{aligned}\int_{-1}^3 \int_0^2 |x - 1|(1 + y) dy dx &= \int_{-1}^3 |x - 1| \left[y + \frac{y^2}{2} \right]_0^2 dx \\ &= 4 \int_{-1}^3 |x - 1| dx.\end{aligned}$$

Step 2: Absolute-value split

$$\begin{aligned}\int_{-1}^3 |x - 1| dx &= \int_{-1}^1 (1 - x) dx + \int_1^3 (x - 1) dx \\ &= 2 + 2 = 4.\end{aligned}$$

Therefore

$$\boxed{\iint_R |x - 1|(1 + y) dA = 16}.$$

Step 3: Average and check The rectangle has area $4 \cdot 2 = 8$, so

$$\boxed{f_{\text{avg}} = \frac{16}{8} = 2}.$$

The two x -pieces are mirror images about $x = 1$, and the nonnegative integrand confirms the positive sign.

Question 6

For a real parameter a , define

$$I(a) = \int_0^2 \int_{-1}^1 (axy^2 + x^2 - 2y) \, dy \, dx.$$

Tasks

1. Evaluate $I(a)$ exactly.
2. Find the unique value of a for which the integrand's average value on the rectangle is 1.
3. Verify the resulting integral directly from the inner accumulation function.

Question 6 – Solution

Strategy. Use parity in the symmetric y -interval, retain the parameter symbolically, and convert the prescribed average into a total integral.

Step 1: Inner accumulation Since $-2y$ is odd,

$$\begin{aligned}\int_{-1}^1 (axy^2 + x^2 - 2y) dy &= ax \left[\frac{y^3}{3} \right]_{-1}^1 + x^2[y]_{-1}^1 \\ &= \frac{2ax}{3} + 2x^2.\end{aligned}$$

Thus

$$I(a) = \int_0^2 \left(\frac{2ax}{3} + 2x^2 \right) dx = \frac{4a}{3} + \frac{16}{3}.$$

Step 2: Prescribed average The rectangle's area is $2 \cdot 2 = 4$. Average value 1 therefore requires $I(a) = 4$:

$$\frac{4a + 16}{3} = 4 \implies \boxed{a = -1}.$$

Step 3: Direct verification With $a = -1$, the inner accumulation is

$$-\frac{2x}{3} + 2x^2.$$

Consequently,

$$\int_0^2 \left(-\frac{2x}{3} + 2x^2 \right) dx = \left[-\frac{x^2}{3} + \frac{2x^3}{3} \right]_0^2 = -\frac{4}{3} + \frac{16}{3} = 4.$$

Dividing by area 4 returns the required average 1.

Question 7

A temperature field on a rectangular plate is

$$T(x, y) = 18 + 2x + y^2, \quad 0 \leq x \leq 3, \quad -1 \leq y \leq 1.$$

Tasks

1. Compute the average temperature using an iterated integral.
2. Compare it with the minimum and maximum temperatures on the plate.
3. Exhibit two distinct points where the temperature equals its average.

Question 7 – Solution

Strategy. Integrate the three terms separately, divide by plate area, and then solve the resulting level equation at convenient choices of y .

Step 1: Total and average The plate area is $A = 3 \cdot 2 = 6$. Its total temperature integral is

$$\begin{aligned}\int_0^3 \int_{-1}^1 (18 + 2x + y^2) dy dx &= 18(6) + \left(\int_0^3 2x dx \right) (2) + 3 \int_{-1}^1 y^2 dy \\ &= 108 + 18 + 2 = 128.\end{aligned}$$

Hence

$$T_{\text{avg}} = \frac{128}{6} = \frac{64}{3}.$$

Step 2: Range check The minimum is $T(0, 0) = 18$. The maximum 25 occurs at $(3, 1)$ and $(3, -1)$. Since

$$18 < \frac{64}{3} < 25,$$

the average lies strictly inside the temperature range, as required.

Step 3: Points at the average Solve

$$18 + 2x + y^2 = \frac{64}{3}.$$

Taking $y = 0$ gives $x = 5/3$, while taking $y = 1$ gives $x = 7/6$. Thus two examples are

$$\left(\frac{5}{3}, 0 \right) \quad \text{and} \quad \left(\frac{7}{6}, 1 \right).$$

Both lie on the plate, and substitution gives $64/3$ at each.

Question 8

Let p be continuous on $[a, b]$ and q continuous on $[c, d]$.

Tasks

1. Prove using iterated integrals that

$$\iint_{[a,b] \times [c,d]} p(x)q(y) \, dA = \left(\int_a^b p(x) \, dx \right) \left(\int_c^d q(y) \, dy \right).$$

2. Explain why the same proof works in either order.
3. Apply the result to $(1 + x^2) \cos y$ on $[-1, 1] \times [-\pi/2, \pi/2]$.

Question 8 – Solution

Strategy. In the inner integral, the factor depending only on the outer variable is constant and can be pulled outside.

Step 1: Factorization proof Integrating in y first gives

$$\begin{aligned}\int_a^b \int_c^d p(x)q(y) dy dx &= \int_a^b p(x) \left(\int_c^d q(y) dy \right) dx \\ &= \left(\int_a^b p(x) dx \right) \left(\int_c^d q(y) dy \right).\end{aligned}$$

The bracketed q -integral is a number, so it is constant in the outer x -integration.

Step 2: Other order Reversing the order first produces the constant $\int_a^b p(x) dx$ and then the q -integral. Continuity on the closed rectangle guarantees Fubini's theorem applies, so both forms represent the same double integral.

Step 3: Application Here

$$\int_{-1}^1 (1+x^2) dx = \left[x + \frac{x^3}{3} \right]_{-1}^1 = \frac{8}{3}$$

and

$$\int_{-\pi/2}^{\pi/2} \cos y dy = [\sin y]_{-\pi/2}^{\pi/2} = 2.$$

Therefore

$$\boxed{\iint_R (1+x^2) \cos y dA = \frac{16}{3}}.$$

Both factors are nonnegative on their intervals, verifying the sign.

Question 9

Evaluate

$$\iint_R (x^3 \cos y + x^2 \sin y + x^2) dA, \quad R = [-2, 2] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Tasks

1. Integrate with respect to y first and identify the term eliminated by y -symmetry.
2. Use x -symmetry in the outer integral.
3. Evaluate the surviving contribution and verify its sign.

Question 9 – Solution

Strategy. Use parity at each stage: the symmetric inner interval cancels the sine term, and the symmetric outer interval cancels the remaining odd power of x .

Step 1: Inner y -integration

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} (x^3 \cos y + x^2 \sin y + x^2) dy &= x^3 [\sin y]_{-\pi/2}^{\pi/2} - x^2 [\cos y]_{-\pi/2}^{\pi/2} + \pi x^2 \\ &= 2x^3 + \pi x^2.\end{aligned}$$

The $x^2 \sin y$ term vanishes because sine is odd in y .

Step 2: Outer x -integration Now $2x^3$ is odd on $[-2, 2]$, so it contributes zero. Therefore

$$\begin{aligned}\iint_R f dA &= \pi \int_{-2}^2 x^2 dx = \pi \left[\frac{x^3}{3} \right]_{-2}^2 \\ &= \boxed{\frac{16\pi}{3}}.\end{aligned}$$

Step 3: Verification The only surviving term is $x^2 \geq 0$, integrated over a y -interval of length π . Thus a positive result is necessary. Its one-variable integral is $16/3$, so multiplication by π confirms the scale and constant.

Question 10

Over the square $R = [-1, 1] \times [-1, 1]$, a solid lies between

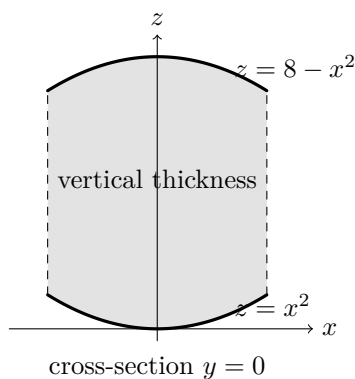
$$z_{\text{top}} = 8 - x^2 - y^2 \quad \text{and} \quad z_{\text{bottom}} = x^2 + y^2.$$

Tasks

1. Verify that the named top surface stays above the bottom surface on all of R .
2. Set up and evaluate an iterated integral for the volume.
3. Check the answer using symmetry and separate one-variable integrals.

Question 10 – Solution

Strategy. Integrate vertical thickness over the rectangular base, after confirming that the thickness never changes sign.



Step 1: Thickness The vertical thickness is

$$h(x, y) = 8 - 2x^2 - 2y^2.$$

On R , $x^2 + y^2 \leq 2$, so $h \geq 8 - 4 = 4 > 0$. Thus no top-bottom reversal or absolute value is needed.

Step 2: Iterated integral

$$\begin{aligned} V &= \int_{-1}^1 \int_{-1}^1 (8 - 2x^2 - 2y^2) dy dx \\ &= \int_{-1}^1 \left(16 - 4x^2 - \frac{4}{3} \right) dx \\ &= \left[\frac{44}{3}x - \frac{4}{3}x^3 \right]_{-1}^1 = \boxed{\frac{80}{3}}. \end{aligned}$$

Step 3: Independent check The constant term contributes $8 \text{ area}(R) = 32$. Also,

$$\iint_R x^2 dA = \left(\int_{-1}^1 x^2 dx \right) (2) = \frac{4}{3},$$

and identically $\iint_R y^2 dA = 4/3$. Hence

$$V = 32 - 2 \left(\frac{4}{3} \right) - 2 \left(\frac{4}{3} \right) = 32 - \frac{16}{3} = \frac{80}{3},$$

confirming the iterated calculation.

Question 1

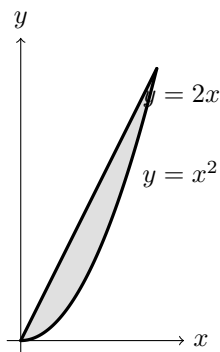
Let D be bounded by $y = x^2$ and $y = 2x$.

Tasks

1. Find the intersection points and describe D as a Type I region.
2. Set up and evaluate $\iint_D (x + y) dA$.
3. Verify that the bounds keep the lower curve below the upper curve.

Question 1 – Solution

Strategy. Solve the boundary intersection first, then use vertical slices from the parabola to the line.



Step 1: Region The curves meet where $x^2 = 2x$, so $x = 0, 2$. For $0 \leq x \leq 2$, $x^2 \leq 2x$. Thus

$$D = \{(x, y) : 0 \leq x \leq 2, x^2 \leq y \leq 2x\}.$$

Step 2: Integral

$$\begin{aligned} \iint_D (x + y) dA &= \int_0^2 \int_{x^2}^{2x} (x + y) dy dx \\ &= \int_0^2 \left[xy + \frac{y^2}{2} \right]_{x^2}^{2x} dx \\ &= \int_0^2 \left(4x^2 - x^3 - \frac{x^4}{2} \right) dx \\ &= \frac{32}{3} - 4 - \frac{16}{5} = \boxed{\frac{52}{15}}. \end{aligned}$$

Verification The slice height is $2x - x^2 = x(2 - x) \geq 0$ on $[0, 2]$. Also $x + y \geq 0$ throughout D , consistent with the positive answer.

Question 2

Let D be bounded by $x = y^2$ and $x = 2y$.

Tasks

1. Describe D as a Type II region.
2. Evaluate $\iint_D x \, dA$ using horizontal slices.
3. Check the sign and the intersection endpoints.

Question 2 – Solution

Strategy. Treat x as the inner variable and determine which curve is leftmost between the intersection values of y .

Step 1: Region From $y^2 = 2y$, the curves meet at $y = 0, 2$. On this interval, $y^2 \leq 2y$, so

$$D = \{(x, y) : 0 \leq y \leq 2, y^2 \leq x \leq 2y\}.$$

Step 2: Integral

$$\begin{aligned}\iint_D x \, dA &= \int_0^2 \int_{y^2}^{2y} x \, dx \, dy = \frac{1}{2} \int_0^2 (4y^2 - y^4) \, dy \\ &= \left[\frac{2y^3}{3} - \frac{y^5}{5} \right]_0^2 = \frac{16}{3} - \frac{16}{5} = \boxed{\frac{32}{15}}.\end{aligned}$$

Verification Both boundaries have $x \geq 0$ on $0 \leq y \leq 2$, so the integral of x must be nonnegative. The width $2y - y^2 = y(2 - y)$ vanishes at both intersection endpoints as the geometry requires.

Question 3

The region D lies between $y = x$ and $y = \sqrt{x}$ in the first quadrant.

Tasks

1. Describe D using vertical slices and using horizontal slices.
2. Write $\iint_D 1 \, dA$ in both orders.
3. Evaluate the area and verify the two descriptions agree.

Question 3 – Solution

Strategy. Solve each boundary for the inner variable and use the common endpoints $(0, 0)$ and $(1, 1)$.

Step 1: Two descriptions Since $x \leq \sqrt{x}$ on $[0, 1]$,

$$D = \{0 \leq x \leq 1, x \leq y \leq \sqrt{x}\}.$$

Solving for x , the parabola is $x = y^2$ and the line is $x = y$. For $0 \leq y \leq 1$, $y^2 \leq y$, so

$$D = \{0 \leq y \leq 1, y^2 \leq x \leq y\}.$$

Step 2: Both orders

$$\iint_D 1 \, dA = \int_0^1 \int_x^{\sqrt{x}} 1 \, dy \, dx = \int_0^1 \int_{y^2}^y 1 \, dx \, dy.$$

Step 3: Area The horizontal description gives

$$\text{area}(D) = \int_0^1 (y - y^2) \, dy = \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \boxed{\frac{1}{6}}.$$

Directly, $\int_0^1 (\sqrt{x} - x) \, dx = 2/3 - 1/2 = 1/6$, verifying the reversal.

Question 4

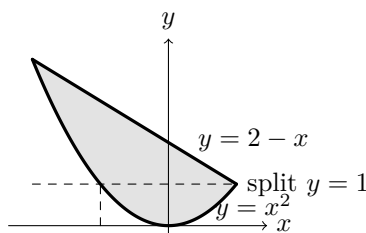
Let D be the region between $y = x^2$ and $y = 2 - x$.

Tasks

1. Describe D with vertical slices.
2. Reverse the order, explaining why horizontal slices require two integrals.
3. Compute the area in the reversed order and check it in the original order.

Question 4 – Solution

Strategy. Locate the intersection points and the height where the right boundary changes from the parabola to the line.



Step 1: Vertical description Solving $x^2 = 2 - x$ gives $x = -2, 1$. Hence

$$D = \{(x, y) : -2 \leq x \leq 1, x^2 \leq y \leq 2 - x\}.$$

Step 2: Reverse and split Horizontal slices begin at $x = -\sqrt{y}$. For $0 \leq y \leq 1$, they end at $x = \sqrt{y}$; for $1 \leq y \leq 4$, they end on the line $x = 2 - y$. Thus

$$\iint_D 1 \, dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} 1 \, dx \, dy + \int_1^4 \int_{-\sqrt{y}}^{2-y} 1 \, dx \, dy.$$

Step 3: Area

$$\begin{aligned} A &= \int_0^1 2\sqrt{y} \, dy + \int_1^4 (2 - y + \sqrt{y}) \, dy \\ &= \frac{4}{3} + \left[2y - \frac{y^2}{2} + \frac{2}{3}y^{3/2} \right]_1^4 = \frac{4}{3} + \frac{19}{6} = \boxed{\frac{9}{2}}. \end{aligned}$$

The original order gives $\int_{-2}^1 (2 - x - x^2) dx = 9/2$, confirming the split.

Question 5

Let

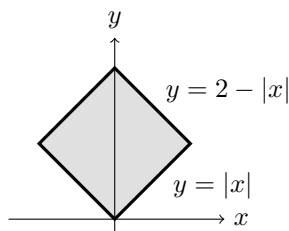
$$D = \{(x, y) : -1 \leq x \leq 1, |x| \leq y \leq 2 - |x|\}.$$

Tasks

1. Sketch or describe the region and identify its symmetry.
2. Evaluate $\iint_D y \, dA$ by splitting at $x = 0$ or using $|x|$ directly.
3. Find the area of D and hence the average value of y .

Question 5 – Solution

Strategy. Each vertical slice has endpoints symmetric about the horizontal line $y = 1$, making its average height immediate.



Step 1: Geometry The vertices are $(0, 0), (1, 1), (0, 2), (-1, 1)$. The diamond is symmetric about both $x = 0$ and $y = 1$.

Step 2: Integral of y

$$\begin{aligned}\iint_D y \, dA &= \int_{-1}^1 \int_{|x|}^{2-|x|} y \, dy \, dx \\ &= \frac{1}{2} \int_{-1}^1 ((2 - |x|)^2 - |x|^2) \, dx \\ &= \int_{-1}^1 (2 - 2|x|) \, dx = \boxed{2}.\end{aligned}$$

Step 3: Area and average The slice width is $2 - 2|x|$, so

$$A = \int_{-1}^1 (2 - 2|x|) \, dx = 2.$$

Therefore $y_{\text{avg}} = 2/2 = \boxed{1}$, exactly the symmetry line. This independently checks the integral: total equals average times area.

Question 6

Consider

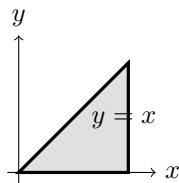
$$I = \int_0^1 \int_y^1 e^{x^2} dx dy.$$

Tasks

1. Describe the triangular region represented by the bounds.
2. Explain why the displayed order is not convenient for elementary integration.
3. Reverse the order and evaluate I exactly.

Question 6 – Solution

Strategy. Draw the triangle $0 \leq y \leq x \leq 1$; reversing the order creates an inner interval whose length supplies the factor needed for substitution.



Step 1: Region The displayed bounds say $0 \leq y \leq 1$ and $y \leq x \leq 1$. Equivalently,

$$D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}.$$

Step 2: Why reverse In the given order, the inner antiderivative $\int e^{x^2} dx$ is not elementary. With dy first, e^{x^2} is constant in the inner variable.

Step 3: Reversed evaluation

$$\begin{aligned} I &= \int_0^1 \int_0^x e^{x^2} dy dx = \int_0^1 x e^{x^2} dx \\ &= \frac{1}{2} \int_0^1 e^u du = \left[\frac{e-1}{2} \right], \quad u = x^2. \end{aligned}$$

The integrand and region are positive, so the positive result is consistent. Differentiating $e^{x^2}/2$ returns $x e^{x^2}$.

Question 7

A solid lies above the triangle

$$D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$$

and below the plane $z = 1 - x - y$.

Tasks

1. Write Cartesian bounds for D .
2. Evaluate the volume using a double integral.
3. Verify the answer using the geometry of a tetrahedron.

Question 7 – Solution

Strategy. Use vertical slices of the triangular base, then compare the integral with the intercept-form tetrahedron.

Step 1: Bounds For $0 \leq x \leq 1$, the variable y runs from 0 to $1 - x$. Therefore

$$V = \int_0^1 \int_0^{1-x} (1 - x - y) dy dx.$$

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx \\ &= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[x - x^2 + \frac{x^3}{3} \right]_0^1 \\ &= \boxed{\frac{1}{6}}. \end{aligned}$$

Step 3: Geometric check The coordinate planes and $x + y + z = 1$ enclose a tetrahedron with mutually perpendicular intercept lengths 1, 1, 1. Its volume is

$$\frac{1}{3}(\text{base area})(\text{height}) = \frac{1}{3} \left(\frac{1}{2} \right) (1) = \frac{1}{6}.$$

This matches the integral and verifies the top surface remains nonnegative over D .

Question 8

For $k > 0$, let D_k be the region enclosed by $y = x^2$ and $y = kx$.

Tasks

1. Find the intersections and write bounds for D_k .
2. Derive a formula for the area as a function of k .
3. Determine the unique $k > 0$ for which the area is $9/2$.

Question 8 – Solution

Strategy. Keep the line slope symbolic; the nonzero intersection becomes the outer endpoint of a vertical-slice area integral.

Step 1: Intersections and bounds From $x^2 = kx$,

$$x(x - k) = 0,$$

so the intersections occur at $x = 0, k$. Since $k > 0$, $x^2 \leq kx$ on $[0, k]$, and

$$D_k = \{(x, y) : 0 \leq x \leq k, x^2 \leq y \leq kx\}.$$

Step 2: Area formula

$$\begin{aligned} A(k) &= \int_0^k \int_{x^2}^{kx} 1 \, dy \, dx = \int_0^k (kx - x^2) \, dx \\ &= \left[\frac{kx^2}{2} - \frac{x^3}{3} \right]_0^k = \boxed{\frac{k^3}{6}}. \end{aligned}$$

Step 3: Inverse condition Set $k^3/6 = 9/2$. Then $k^3 = 27$, and the positive restriction gives

$$\boxed{k = 3}.$$

Substitution yields $A(3) = 27/6 = 9/2$. Since $A(k) = k^3/6$ is strictly increasing for $k > 0$, this positive solution is unique.

Question 9

Let

$$D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq \min(x, 2 - x)\}.$$

Tasks

1. Explain why the vertical-slice description must split at $x = 1$.
2. Give a single Type II description of D .
3. Evaluate $\iint_D (x + 2y) \, dA$ using the more efficient description.

Question 9 – Solution

Strategy. Recognize the triangle with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$; horizontal slices use one left and one right boundary.

Step 1: Vertical split The smaller upper boundary is x on $[0, 1]$ and $2 - x$ on $[1, 2]$. Thus vertical integration would require

$$\int_0^1 \int_0^x (\cdots) dy dx + \int_1^2 \int_0^{2-x} (\cdots) dy dx.$$

Step 2: Single horizontal description For $0 \leq y \leq 1$, the line $y = x$ gives $x = y$ and $y = 2 - x$ gives $x = 2 - y$. Hence

$$D = \{(x, y) : 0 \leq y \leq 1, y \leq x \leq 2 - y\}.$$

Step 3: Evaluate

$$\begin{aligned} \iint_D (x + 2y) dA &= \int_0^1 \int_y^{2-y} (x + 2y) dx dy \\ &= \int_0^1 \left[\frac{x^2}{2} + 2yx \right]_y^{2-y} dy \\ &= \int_0^1 (2 + 2y - 4y^2) dy = \left[2y + y^2 - \frac{4}{3}y^3 \right]_0^1 \\ &= \boxed{\frac{5}{3}}. \end{aligned}$$

The integrand is nonnegative on D , agreeing with the sign.

Question 10

Let

$$D = \{(x, y) : 0 \leq y \leq 1, -y \leq x \leq y\}.$$

Tasks

1. Describe the region in the reverse order, including any required split.
2. Evaluate $\iint_D (x^2 + y) \, dA$ in the given order.
3. Use symmetry and the reversed description to check the structure of the answer.

Question 10 – Solution

Strategy. Use horizontal symmetry about the y -axis for evaluation, then express the lower boundary as $y = |x|$ when reversing.

Step 1: Reverse description The triangle has vertices $(0, 0), (-1, 1), (1, 1)$. In one compact form,

$$-1 \leq x \leq 1, \quad |x| \leq y \leq 1.$$

Without absolute values it splits into $-1 \leq x \leq 0, -x \leq y \leq 1$, and $0 \leq x \leq 1, x \leq y \leq 1$.

Step 2: Given order

$$\begin{aligned} \iint_D (x^2 + y) dA &= \int_0^1 \int_{-y}^y (x^2 + y) dx dy \\ &= \int_0^1 \left(\frac{2y^3}{3} + 2y^2 \right) dy \\ &= \frac{1}{6} + \frac{2}{3} = \boxed{\frac{5}{6}}. \end{aligned}$$

Step 3: Structural check In the reversed order, the two x -halves contribute equally because $x^2 + y$ is even in x . Thus

$$\iint_D (x^2 + y) dA = 2 \int_0^1 \int_x^1 (x^2 + y) dy dx.$$

Evaluating gives $2(5/12) = 5/6$. This verifies both the factor of two and the lower boundary $y = |x|$.

Question 1

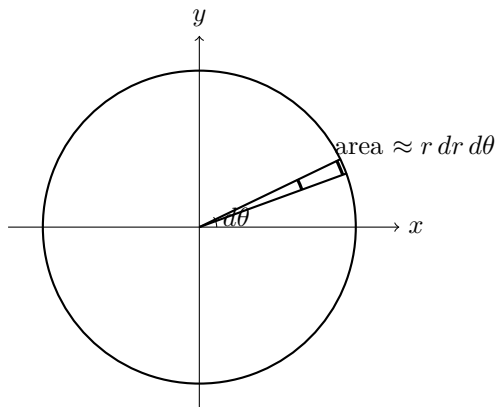
Let D be the disk $x^2 + y^2 \leq 4$.

Tasks

1. Convert $x^2 + y^2$ and dA to polar form.
2. Set up and evaluate $\iint_D (x^2 + y^2) dA$.
3. Explain the Jacobian factor geometrically.

Question 1 – Solution

Strategy. Radial symmetry turns the integrand into r^2 and the area element into $r \, dr \, d\theta$.



Step 1: Conversion The disk is $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, and $x^2 + y^2 = r^2$. Hence

$$\iint_D (x^2 + y^2) dA = \int_0^{2\pi} \int_0^2 r^2 (r \, dr \, d\theta).$$

Step 2: Evaluate

$$\int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\theta = \int_0^{2\pi} 4 \, d\theta = \boxed{8\pi}.$$

Verification A thin polar cell has radial side dr and arc-length side approximately $r \, d\theta$, so its area is $r \, dr \, d\theta$. Omitting r would give the wrong units and value.

Question 2

Let D be the annular sector

$$1 \leq x^2 + y^2 \leq 9, \quad \frac{\pi}{6} \leq \arg(x + iy) \leq \frac{\pi}{2}.$$

Tasks

1. Write polar bounds.
2. Evaluate $\iint_D 1/\sqrt{x^2 + y^2} \, dA$.
3. Check finiteness and units.

Question 2 – Solution

Strategy. The factor $1/r$ cancels the polar Jacobian, leaving a rectangular integral in (r, θ) .

Step 1: Bounds The radial inequality gives $1 \leq r \leq 3$, while $\pi/6 \leq \theta \leq \pi/2$.

Step 2: Evaluate

$$\begin{aligned}\iint_D \frac{1}{\sqrt{x^2 + y^2}} dA &= \int_{\pi/6}^{\pi/2} \int_1^3 \frac{1}{r} (r \, dr \, d\theta) \\ &= \int_{\pi/6}^{\pi/2} 2 \, d\theta = \boxed{\frac{2\pi}{3}}.\end{aligned}$$

Verification The region stays at least one unit from the origin, so the integrand is bounded and no singularity occurs. After cancellation the radial length is 2 and the angular width is $\pi/3$.

Question 3

Find the area enclosed by the cardioid $r = 1 + \cos \theta$.

Tasks

1. Choose a complete nonduplicating angular interval.
2. Derive the polar area integral from a double integral.
3. Evaluate it exactly.

Question 3 – Solution

Strategy. Integrate 1 with radial bounds from the pole to the cardioid over one full turn.

Step 1: Bounds Since $1 + \cos \theta \geq 0$, one traversal is $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1 + \cos \theta$.

Step 2: Area integral

$$A = \int_0^{2\pi} \int_0^{1+\cos \theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 \, d\theta.$$

Step 3: Evaluate Over a full period, $\int \cos \theta = 0$ and $\int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$. Thus

$$A = \frac{1}{2}(2\pi + 0 + \pi) = \boxed{\frac{3\pi}{2}}.$$

The radial bound vanishes only at $\theta = \pi$, so the interval neither omits nor duplicates positive-area portions.

Question 4

Find the area inside $r = 2 \cos \theta$ but outside the unit circle $r = 1$.

Tasks

1. Determine the angles where the radial bounds are ordered.
2. Set up and evaluate the area integral.
3. Interpret the boundary intersections.

Question 4 – Solution

Strategy. Solve $2 \cos \theta = 1$ to locate the two intersection rays, then integrate from the unit circle to the offset circle.

Step 1: Angles The curves meet when $\cos \theta = 1/2$, so $\theta = \pm\pi/3$. Between these angles, $2 \cos \theta \geq 1$.

Step 2: Area

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \int_1^{2 \cos \theta} r \, dr \, d\theta \\ &= \frac{1}{2} \int_{-\pi/3}^{\pi/3} (4 \cos^2 \theta - 1) d\theta \\ &= \frac{1}{2} \left(4 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) - \frac{2\pi}{3} \right) = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}. \end{aligned}$$

Verification At both endpoint rays the inner and outer radii equal 1, so slice width vanishes. At $\theta = 0$ the slice runs from 1 to 2, confirming the order.

Question 5

For $a > 0$, evaluate

$$\iint_{x^2+y^2 \leq a^2} e^{-(x^2+y^2)} dA.$$

Tasks

1. Convert to polar coordinates.
2. Evaluate as a function of a .
3. Find the limit as $a \rightarrow \infty$ and verify monotonicity.

Question 5 – Solution

Strategy. The exponent and disk are radial; use $u = r^2$ after including the Jacobian.

Step 1: Set up

$$I(a) = \int_0^{2\pi} \int_0^a e^{-r^2} r \, dr \, d\theta.$$

Step 2: Evaluate With $u = r^2$, $du = 2r \, dr$,

$$I(a) = 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^a = \boxed{\pi(1 - e^{-a^2})}.$$

Step 3: Checks Since $I'(a) = 2\pi a e^{-a^2} > 0$, enlarging the disk increases the integral. Also

$$\boxed{\lim_{a \rightarrow \infty} I(a) = \pi}.$$

The result approaches a finite limit because the exponential decay dominates the growing circumference.

Question 6

Let D be the first-quadrant triangle $x \geq 0$, $y \geq 0$, $x + y \leq 2$.

Tasks

1. Convert the line boundary to a radial bound.
2. Use polar coordinates to compute the area.
3. Verify with elementary triangle geometry.

Question 6 – Solution

Strategy. A ray at angle θ meets $x + y = 2$ when $r(\cos \theta + \sin \theta) = 2$.

Step 1: Polar bounds First-quadrant rays give $0 \leq \theta \leq \pi/2$, and

$$0 \leq r \leq \frac{2}{\cos \theta + \sin \theta}.$$

Step 2: Area

$$A = \frac{1}{2} \int_0^{\pi/2} \frac{4}{(\cos \theta + \sin \theta)^2} d\theta.$$

Let $u = \tan \theta$. Since $(\cos \theta + \sin \theta)^2 = (1 + u)^2 / (1 + u^2)$ and $d\theta = du / (1 + u^2)$,

$$A = 2 \int_0^\infty \frac{du}{(1 + u)^2} = \boxed{2}.$$

Verification The Cartesian triangle has legs of length 2, so its area is $\frac{1}{2}(2)(2) = 2$. The nonconstant radial bound is essential; a constant bound would describe a sector, not the triangle.

Question 7

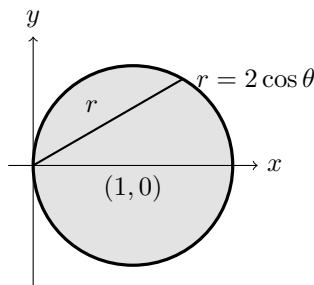
Let D be the disk $(x - 1)^2 + y^2 \leq 1$. Evaluate $\iint_D x \, dA$ in polar coordinates.

Tasks

1. Convert the offset circle to polar bounds.
2. Evaluate the integral.
3. Verify the result using area and symmetry.

Question 7 – Solution

Strategy. Rewrite the circle as $r = 2 \cos \theta$ and remember that $x = r \cos \theta$ contributes another factor of r with dA .



Step 1: Bounds The boundary equation becomes $r^2 - 2r \cos \theta = 0$. Thus

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2 \cos \theta.$$

Step 2: Evaluate

$$\begin{aligned} \iint_D x \, dA &= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^2 \cos \theta \, dr \, d\theta \\ &= \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^4 \theta \, d\theta = \frac{8}{3} \cdot \frac{3\pi}{8} = \boxed{\pi}. \end{aligned}$$

Verification The disk has area π and is symmetric about the vertical line $x = 1$, so its average x -coordinate is 1. Hence its first moment is $1 \cdot \pi = \pi$.

Question 8

Find the area of one petal of the rose $r = \cos(2\theta)$ centered on the positive x -axis.

Tasks

1. Determine the angular interval for exactly that petal.
2. Set up and evaluate its area.
3. Explain why negative r values are excluded.

Question 8 – Solution

Strategy. The desired petal begins and ends where $\cos(2\theta) = 0$ around $\theta = 0$.

Step 1: Bounds The adjacent zeros are $\theta = \pm\pi/4$, and $\cos(2\theta) \geq 0$ between them. Thus $0 \leq r \leq \cos(2\theta)$.

Step 2: Area

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r \, dr \, d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2(2\theta) d\theta \\ &= \frac{1}{4} \left[\theta + \frac{\sin 4\theta}{4} \right]_{-\pi/4}^{\pi/4} = \boxed{\frac{\pi}{8}}. \end{aligned}$$

Verification The sine endpoint terms vanish. Extending beyond these zeros introduces negative radial values, which trace different petals rather than enlarging the positive- x petal.

Question 9

An unknown annulus $a \leq r \leq b$, with $0 < a < b$, has area 3π and satisfies

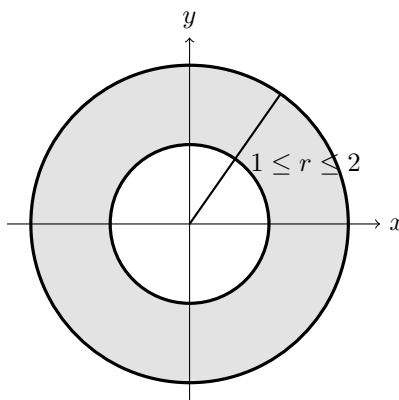
$$\iint_D (x^2 + y^2) dA = \frac{15\pi}{2}.$$

Tasks

1. Translate both conditions into equations for a^2, b^2 .
2. Recover the radii.
3. Verify both measurements.

Question 9 – Solution

Strategy. Set $A = a^2$ and $B = b^2$; area gives their difference, while the radial moment gives a difference of squares.



Step 1: Equations Area gives

$$\pi(b^2 - a^2) = 3\pi, \quad \text{so} \quad B - A = 3.$$

Also

$$\int_0^{2\pi} \int_a^b r^3 dr d\theta = \frac{\pi}{2}(b^4 - a^4) = \frac{15\pi}{2},$$

so $(B - A)(B + A) = 15$.

Step 2: Solve Since $B - A = 3$, we get $B + A = 5$. Therefore $B = 4$, $A = 1$, and positivity gives

$$\boxed{a = 1, \quad b = 2}.$$

Verification The area is $\pi(4 - 1) = 3\pi$, and the moment is $\frac{\pi}{2}(16 - 1) = 15\pi/2$. Both conditions hold exactly.

Question 10

Let D be the upper half-disk $x^2 + y^2 \leq 9$, $y \geq 0$. Evaluate

$$\iint_D (x^2 + y^2 + xy) dA.$$

Tasks

1. Convert the region and integrand to polar coordinates.
2. Use angular symmetry to eliminate the mixed term.
3. Evaluate the remaining integral and verify its sign.

Question 10 – Solution

Strategy. In polar form the radial term is r^2 , while $xy = r^2 \cos \theta \sin \theta$ cancels over $0 \leq \theta \leq \pi$.

Step 1: Set up

$$0 \leq \theta \leq \pi, \quad 0 \leq r \leq 3,$$

so

$$I = \int_0^\pi \int_0^3 r^3 (1 + \cos \theta \sin \theta) \, dr \, d\theta.$$

Step 2: Symmetry Since

$$\int_0^\pi \cos \theta \sin \theta \, d\theta = 0,$$

the xy contribution vanishes, matching reflection symmetry across the y -axis.

Step 3: Evaluate

$$I = \int_0^\pi \left[\frac{r^4}{4} \right]_0^3 d\theta = \int_0^\pi \frac{81}{4} d\theta = \boxed{\frac{81\pi}{4}}.$$

The surviving integrand $x^2 + y^2$ is nonnegative, so the positive result is necessary. Its scale is radius squared times half-disk area, as expected.

Question 1

Evaluate $\iiint_B (x + 2y + 3z) \, dV$ on $B = [0, 1] \times [0, 2] \times [-1, 1]$.

Tasks

1. Write a Cartesian iterated integral.
2. Evaluate it.
3. Verify by average values.

Question 1 – Solution

Strategy. Integrate termwise over the box.

Step 1: Set up $I = \int_0^1 \int_0^2 \int_{-1}^1 (x + 2y + 3z) \, dz \, dy \, dx.$

Step 2: Evaluate The z term is odd; the other terms give $2x + 4y$. Thus $I = \int_0^1 \int_0^2 (2x + 4y) \, dy \, dx = \int_0^1 (4x + 8) \, dx = \boxed{10}.$

Verification The box volume is 4 and its coordinate averages are $(1/2, 1, 0)$, so the integrand average is $5/2$ and $4(5/2) = 10$.

Question 2

Let $E = \{0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 2 - x - y\}$.

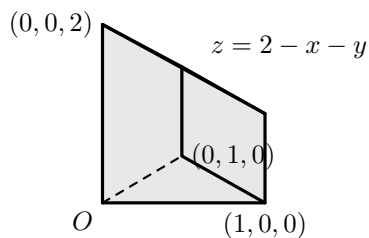
Tasks

1. Interpret the bounds.
2. Evaluate $\iiint_E 1 \, dV$.
3. Check positivity of every bound.

Question 2 – Solution

Strategy. Integrate vertical height over the triangular projection.

Step 1: Geometry The projection is the triangle $x \geq 0$, $y \geq 0$, $x + y \leq 1$. Above it, the solid runs from $z = 0$ to the sloping plane $z = 2 - x - y$.



Step 2: Evaluate $V = \int_0^1 \int_0^{1-x} \int_0^{2-x-y} 1 \, dz \, dy \, dx = \int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2} \right) dx = \boxed{\frac{2}{3}}.$

Verification On the projection, $2 - x - y \geq 1$, so every vertical bound is positive.

Question 3

Find the volume of the first-octant tetrahedron $x/2 + y/3 + z/6 \leq 1$.

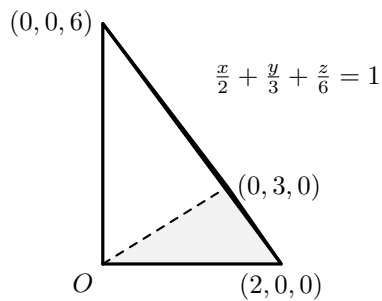
Tasks

1. Write bounds in the order $dz\,dy\,dx$.
2. Evaluate.
3. Verify geometrically.

Question 3 – Solution

Strategy. Solve the plane successively for z and for the projected y -boundary.

Step 1: Geometry The coordinate-plane faces meet the sloping face at the three axis intercepts shown.



Step 2: Bounds and integral $0 \leq x \leq 2$, $0 \leq y \leq 3 - 3x/2$, and $0 \leq z \leq 6 - 3x - 2y$. Hence $V = \int_0^2 \int_0^{3-3x/2} (6 - 3x - 2y) dy dx = \int_0^2 \frac{9}{4} (2 - x)^2 dx = \boxed{6}$.

Verification The intercept formula gives $V = (1/6)(2)(3)(6) = 6$.

Question 4

A box $0 \leq x \leq 2$, $0 \leq y \leq 1$, $0 \leq z \leq 3$ has density $\rho = 1 + x + z$.

Tasks

1. Find its mass.
2. Find the average density.
3. Check against the density range.

Question 4 – Solution

Strategy. Integrate density and divide by volume.

Step 1: Mass $M = \int_0^2 \int_0^1 \int_0^3 (1 + x + z) dz dy dx = \int_0^2 (15/2 + 3x) dx = \boxed{21}$.

Step 2: Average The volume is 6, so $\boxed{\rho_{\text{avg}} = 21/6 = 7/2}$.

Verification Here $1 \leq \rho \leq 6$; $7/2$ lies in this interval. Also the coordinate-average rule gives $1 + 1 + 3/2 = 7/2$.

Question 5

Find the average of $f = x^2 + y^2 + z^2$ on $[-1, 1]^3$.

Tasks

1. Use symmetry and separability.
2. Compute the average.
3. Compare with the range.

Question 5 – Solution

Strategy. The three squared-coordinate contributions are identical.

Step 1: Integral $\iiint_B x^2 dV = (\int_{-1}^1 x^2 dx)(2)(2) = 8/3$. Thus $\iiint_B f dV = 3(8/3) = 8$.

Step 2: Average Since $\text{vol}(B) = 8$, $\boxed{f_{\text{avg}} = 1}$.

Verification $0 \leq f \leq 3$, and each coordinate square has average $1/3$, so their sum has average 1 .

Question 6

Evaluate $\iiint_B (x^3 y z^2 + 4) \, dV$ on $B = [-2, 2] \times [-1, 1] \times [0, 3]$.

Tasks

1. Identify a cancellation symmetry.
2. Evaluate the integral.
3. State why the region matters.

Question 6 – Solution

Strategy. Pair x with $-x$; the nonconstant term is odd in x .

Step 1: Cancellation x^3yz^2 integrates to zero over the symmetric x -interval.

Step 2: Constant term The box volume is $(4)(2)(3) = 24$, so $\boxed{\iiint_B (x^3yz^2 + 4)dV = 4(24) = 96}.$

Verification Reflection $(x, y, z) \mapsto (-x, y, z)$ preserves B and reverses only the nonconstant term. Without that regional symmetry the cancellation would not follow.

Question 7

For $E = \{x, y, z \geq 0 : x + y + z \leq 1\}$, rewrite $\iiint_E z \, dV$ in the orders $dz \, dy \, dx$ and $dx \, dz \, dy$, then evaluate.

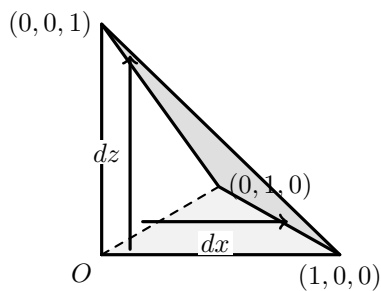
Tasks

1. Give both sets of bounds.
2. Evaluate one order.
3. Verify by symmetry.

Question 7 – Solution

Strategy. Each order peels coordinates from the simplex inequality.

Step 1: Geometry The simplex is bounded by the coordinate planes and $x + y + z = 1$. The two marked fibers show which coordinate is peeled off first.



Step 2: Orders $I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \int_0^1 \int_0^{1-y} \int_0^{1-y-z} z \, dx \, dz \, dy$. The first form gives $I = \frac{1}{6} \int_0^1 (1-x)^3 dx = \boxed{\frac{1}{24}}$.

Verification The simplex volume is $1/6$ and its centroid has $\bar{z} = 1/4$, so the moment is $(1/6)(1/4) = 1/24$.

Question 8

Let $I(a) = \int_0^1 \int_0^2 \int_0^3 (a + x - y + z) \, dz \, dy \, dx$. Find a so the average integrand is 5.

Tasks

1. Evaluate $I(a)$.
2. Solve the average condition.
3. Verify using coordinate averages.

Question 8 – Solution

Strategy. Convert average 5 into total 5 times box volume.

Step 1: Integral The volume is 6 and the coordinate averages are $(1/2, 1, 3/2)$, hence $I(a) = 6(a + 1)$.

Step 2: Solve $I(a)/6 = 5$ gives $a + 1 = 5$, so $\boxed{a = 4}$.

Verification At $a = 4$, the average is $4 + 1/2 - 1 + 3/2 = 5$, and the total integral is 30.

Question 9

For continuous f, g, h on their respective intervals, prove the product rule for $f(x)g(y)h(z)$ on $[a, b] \times [c, d] \times [e, k]$, then apply it to $(1+x)y^2e^z$ on $[0, 1] \times [-1, 1] \times [0, \ln 2]$.

Tasks

1. Prove factorization.
2. Evaluate the example.
3. Check the sign.

Question 9 – Solution

Strategy. Pull factors independent of the current integration variable outside each inner integral.

Step 1: Proof Successive integration gives $\iiint fgh \, dV = (\int_a^b f)(\int_c^d g)(\int_e^k h)$. Continuity guarantees the orders are valid.

Step 2: Example $\int_0^1 (1+x)dx = 3/2$, $\int_{-1}^1 y^2 dy = 2/3$, and $\int_0^{\ln 2} e^z dz = 1$. Their product is $\boxed{1}$.

Verification All factors are nonnegative on the box, matching the positive result.

Question 10

A solid E has volume 12, and $2 \leq f \leq 7$ throughout E . Another integrable function g satisfies $|g - f| \leq 0.05$. Given $\iiint_E f \, dV = 50$, bound $\iiint_E g \, dV$.

Tasks

1. Check consistency of the data.
2. Derive sharp bounds for the new integral.
3. Explain sharpness.

Question 10 – Solution

Strategy. Integrate pointwise inequalities and scale uniform error by volume.

Step 1: Consistency $24 \leq \iiint_E f \, dV \leq 84$, so 50 is possible.

Step 2: Perturbation From $-0.05 \leq g - f \leq 0.05$ and $\text{vol}(E) = 12$,

$$-0.6 \leq \iiint_E (g - f) \, dV \leq 0.6.$$

Therefore $\boxed{49.4 \leq \iiint_E g \, dV \leq 50.6}.$

Verification The endpoints occur for $g = f - 0.05$ and $g = f + 0.05$, so the supplied data guarantee no tighter interval.

Question 1

Let

$$E = \{(x, y, z) : x^2 + y^2 \leq 4, x \geq 0, y \geq 0, -1 \leq z \leq 2\}.$$

Evaluate $\iiint_E (x^2 + y^2) dV$ using cylindrical coordinates.

Tasks

1. Convert the region, integrand, and volume element.
2. Evaluate the cylindrical integral.
3. Verify the result using an average value.

Question 1 – Solution

Strategy. Treat the solid as a quarter-cylinder; radial symmetry makes the integrand r^2 .

Step 1: Convert Cylindrical coordinates give

$$x = r \cos \theta, \quad y = r \sin \theta, \quad x^2 + y^2 = r^2, \quad dV = r \, dz \, dr \, d\theta.$$

The first-quadrant quarter-cylinder has bounds

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2, \quad -1 \leq z \leq 2.$$

Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^2 \int_{-1}^2 r^2 (r \, dz \, dr \, d\theta) \\ &= \left(\frac{\pi}{2}\right) (3) \left[\frac{r^4}{4}\right]_0^2 = \boxed{6\pi}. \end{aligned}$$

Verification The volume is $(\pi 2^2/4)(3) = 3\pi$. On a radius-2 disk, the average of r^2 is $2^2/2 = 2$; restricting to one quadrant does not change that radial average. Thus $I = (3\pi)(2) = 6\pi$.

Question 2

Find the volume inside the paraboloid

$$z = 9 - x^2 - y^2$$

and above the xy -plane.

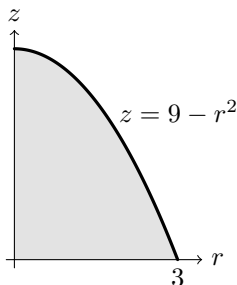
Tasks

1. Determine the cylindrical bounds from the surface intersection.
2. Evaluate the volume.
3. Check the answer against a containing cylinder.

Question 2 – Solution

Strategy. In cylindrical coordinates the paraboloid becomes $z = 9 - r^2$; its intersection with $z = 0$ determines the radial endpoint.

Step 1: Geometry Setting $z = 0$ gives $r = 3$. A meridian half-plane shows the height above each radius.



Thus $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 3$, and $0 \leq z \leq 9 - r^2$.

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r \, dz \, dr \, d\theta \\ &= 2\pi \int_0^3 (9r - r^3) \, dr = 2\pi \left[\frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_0^3 = \boxed{\frac{81\pi}{2}}. \end{aligned}$$

Verification The solid lies inside the cylinder of radius 3 and height 9, whose volume is 81π . The computed volume is positive and exactly half that containing-cylinder volume.

Question 3

Let E lie above the cone $z = \sqrt{x^2 + y^2}$ and below the paraboloid $z = 2 - x^2 - y^2$.

Tasks

1. Find the curve where the two surfaces meet.
2. Set up and evaluate $\iiint_E 1 \, dV$ in cylindrical coordinates.
3. Verify that the vertical bounds are correctly ordered.

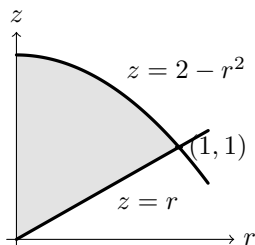
Question 3 – Solution

Strategy. Convert both surfaces to functions of r , solve for their positive intersection radius, and integrate top minus bottom.

Step 1: Intersection The surfaces are $z = r$ and $z = 2 - r^2$. Their intersection satisfies

$$r = 2 - r^2 \iff (r - 1)(r + 2) = 0.$$

Since $r \geq 0$, the boundary circle is $r = 1, z = 1$.



Step 2: Integral

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r \, dz \, dr \, d\theta \\ &= 2\pi \int_0^1 (2 - r^2 - r)r \, dr = 2\pi \left(1 - \frac{1}{4} - \frac{1}{3} \right) = \boxed{\frac{5\pi}{6}}. \end{aligned}$$

Verification For $0 \leq r \leq 1$, $2 - r^2 - r = (1 - r)(r + 2) \geq 0$, so the upper and lower z -bounds never reverse.

Question 4

Let

$$E = \{(x, y, z) : x^2 + y^2 \leq 4x, 0 \leq z \leq \sqrt{x^2 + y^2}\}.$$

Find the volume of E using cylindrical coordinates.

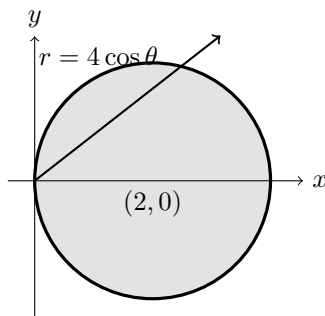
Tasks

1. Convert the offset cylindrical boundary and determine a nonduplicating angular interval.
2. Evaluate the volume.
3. Check the radial bounds at the endpoint and central rays.

Question 4 – Solution

Strategy. Convert the footprint to $r \leq 4 \cos \theta$ and use the conical height $z = r$.

Step 1: Footprint From $r^2 \leq 4r \cos \theta$, nonnegative radii satisfy $0 \leq r \leq 4 \cos \theta$. This requires $-\pi/2 \leq \theta \leq \pi/2$ and describes the disk centered at $(2, 0)$ with radius 2.



Step 2: Evaluate

$$\begin{aligned} V &= \int_{-\pi/2}^{\pi/2} \int_0^{4 \cos \theta} \int_0^r r \, dz \, dr \, d\theta \\ &= \frac{64}{3} \int_{-\pi/2}^{\pi/2} \cos^3 \theta \, d\theta = \frac{64}{3} \left(\frac{4}{3} \right) = \boxed{\frac{256}{9}}. \end{aligned}$$

Verification At $\theta = \pm\pi/2$ the radial interval collapses to $r = 0$; at $\theta = 0$ it reaches $r = 4$. These are exactly the leftmost and rightmost points of the footprint, so no rays are omitted or duplicated.

Question 5

A solid annular cylinder is described by

$$1 \leq x^2 + y^2 \leq 4, \quad 0 \leq z \leq 3,$$

and has density $\rho(x, y, z) = \sqrt{x^2 + y^2} + z$.

Tasks

1. Find its mass using cylindrical coordinates.
2. Find its average density.
3. Compare the average with the density range.

Question 5 – Solution

Strategy. The annular cylinder becomes a rectangular box in (r, θ, z) , while the density becomes $r + z$.

Step 1: Mass The bounds are $0 \leq \theta \leq 2\pi$, $1 \leq r \leq 2$, and $0 \leq z \leq 3$. Therefore

$$\begin{aligned} M &= \int_0^{2\pi} \int_1^2 \int_0^3 (r + z)r \, dz \, dr \, d\theta \\ &= 2\pi \int_1^2 \left(3r^2 + \frac{9}{2}r \right) dr = 2\pi \left[r^3 + \frac{9}{4}r^2 \right]_1^2 = \boxed{\frac{55\pi}{2}}. \end{aligned}$$

Step 2: Average density The volume is $\pi(2^2 - 1^2)(3) = 9\pi$. Hence

$$\boxed{\rho_{\text{avg}} = \frac{M}{V} = \frac{55}{18}}.$$

Verification On the solid, $1 \leq r + z \leq 5$, and $55/18 \approx 3.06$ lies strictly in that range. The units of M/V are density units, as required.

Question 6

The cone

$$0 \leq z \leq 3 - \sqrt{x^2 + y^2}$$

has constant density $\rho = 2$.

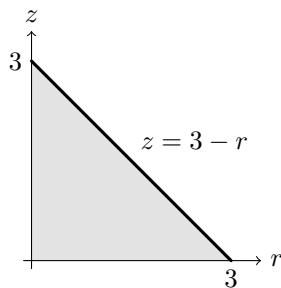
Tasks

1. Find its mass.
2. Find its moment of inertia I_z about the z -axis.
3. Find its radius of gyration $k_z = \sqrt{I_z/M}$.

Question 6 – Solution

Strategy. Use $x^2 + y^2 = r^2$ for the squared distance to the axis and $dV = r \, dz \, dr \, d\theta$.

Step 1: Region The meridian triangle gives $0 \leq r \leq 3$ and $0 \leq z \leq 3 - r$.



Step 2: Mass

$$M = \int_0^{2\pi} \int_0^3 \int_0^{3-r} 2r \, dz \, dr \, d\theta = 4\pi \int_0^3 r(3-r) \, dr = \boxed{18\pi}.$$

Step 3: Moment and radius of gyration

$$I_z = \iiint_E (x^2 + y^2) \rho \, dV = 4\pi \int_0^3 r^3(3-r) \, dr = \boxed{\frac{243\pi}{5}},$$

$$k_z = \sqrt{\frac{I_z}{M}} = \boxed{\sqrt{\frac{27}{10}}}.$$

Verification $k_z \approx 1.64$ lies between the minimum and maximum distances 0 and 3 from the axis. Also $M = 2 \cdot \frac{1}{3}\pi(3^2)(3) = 18\pi$ agrees with the cone-volume formula.

Question 7

Let E be the ball $x^2 + y^2 + z^2 \leq 4$. Without using spherical coordinates, evaluate

$$\iiint_E (x^2 + y^2) \, dV.$$

Tasks

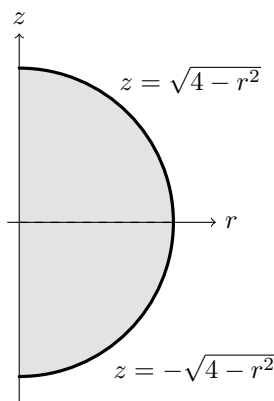
1. Express the ball with cylindrical bounds.
2. Evaluate the integral exactly.
3. Check positivity and magnitude.

Question 7 – Solution

Strategy. At fixed cylindrical radius r , the sphere supplies symmetric upper and lower z -bounds.

Step 1: Bounds Since $r^2 + z^2 \leq 4$,

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}.$$



Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2 (r \, dz \, dr \, d\theta) \\ &= 4\pi \int_0^2 r^3 \sqrt{4-r^2} \, dr. \end{aligned}$$

With $u = 4 - r^2$, $r^3 dr = -(4 - u)du/2$, so

$$\int_0^2 r^3 \sqrt{4-r^2} \, dr = \frac{1}{2} \int_0^4 (4-u)u^{1/2} du = \frac{64}{15}.$$

Thus $\boxed{I = 256\pi/15}$.

Verification The integrand is nonnegative and at most 4. Since the ball volume is $32\pi/3$, the required bound $0 < I < 128\pi/3$ holds.

Question 8

Let E be the solid between the paraboloid $z = x^2 + y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.

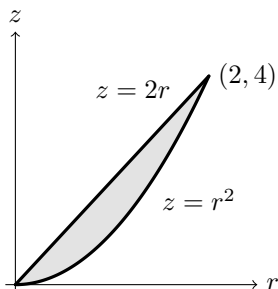
Tasks

1. Determine the cylindrical bounds.
2. Compute $\iiint_E z \, dV$.
3. Find the average value of z on E .

Question 8 – Solution

Strategy. Find the nonzero intersection radius, then integrate z between the radial graphs.

Step 1: Geometry The surfaces meet where $r^2 = 2r$, so $r = 0$ or $r = 2$. On $0 \leq r \leq 2$, the cone lies above the paraboloid.



Thus $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$, and $r^2 \leq z \leq 2r$.

Step 2: Integral of z

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^2 \int_{r^2}^{2r} zr \, dz \, dr \, d\theta \\ &= 2\pi \int_0^2 \left(2r^3 - \frac{1}{2}r^5 \right) dr = \boxed{\frac{16\pi}{3}}. \end{aligned}$$

Step 3: Average The volume is

$$V = 2\pi \int_0^2 (2r - r^2)r \, dr = \frac{8\pi}{3}.$$

Therefore $\boxed{z_{\text{avg}} = I/V = 2}$. This lies between the solid's minimum and maximum heights, 0 and 4.

Question 9

Explain why cylindrical coordinates use

$$dV = r \, dz \, dr \, d\theta,$$

then apply the formula to evaluate $\iiint_E (x^2 + y^2) \, dV$, where

$$1 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{3}, \quad 0 \leq z \leq 4.$$

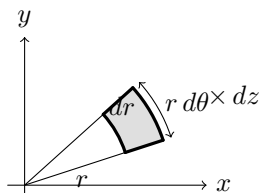
Tasks

1. Derive the scale factor r geometrically.
2. Evaluate the integral.
3. Verify the dimensional scaling.

Question 9 – Solution

Strategy. Approximate a small cylindrical cell by a rectangular box whose angular side is an arc of length $r d\theta$.

Step 1: Jacobian factor A small cell has radial length dr , vertical length dz , and circumferential length approximately $r d\theta$.



Consequently,

$$dV \approx (dr)(r d\theta)(dz) = r dz dr d\theta,$$

and the approximation becomes exact in the integral limit.

Step 2: Apply Since $x^2 + y^2 = r^2$,

$$\begin{aligned} I &= \int_0^{\pi/3} \int_1^2 \int_0^4 r^2 (r dz dr d\theta) \\ &= \left(\frac{\pi}{3}\right) (4) \left[\frac{r^4}{4}\right]_1^2 = \boxed{5\pi}. \end{aligned}$$

Verification The factors scale as angular width times height times radius to the fourth power. That produces length^5 , the correct units for integrating a squared distance over volume.

Question 10

For $a > 0$, let E_a be the solid

$$0 \leq z \leq a^2, \quad z \geq x^2 + y^2.$$

Its average z -coordinate is 6.

Tasks

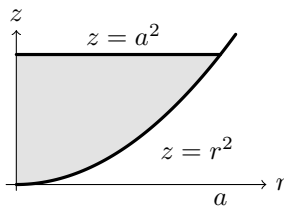
1. Express E_a in cylindrical coordinates.
2. Derive its volume and average z -coordinate as functions of a .
3. Recover a and the corresponding volume.

Question 10 – Solution

Strategy. The horizontal plane meets the paraboloid at $r = a$; compute both volume and first z -moment before imposing the average.

Step 1: Bounds The solid lies above $z = r^2$ and below $z = a^2$:

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq a, \quad r^2 \leq z \leq a^2.$$



Step 2: Volume and moment

$$V(a) = 2\pi \int_0^a (a^2 - r^2)r \, dr = \frac{\pi a^4}{2},$$

$$M_z(a) = 2\pi \int_0^a \int_{r^2}^{a^2} zr \, dz \, dr = \frac{\pi a^6}{3}.$$

Therefore

$$z_{\text{avg}} = \frac{M_z(a)}{V(a)} = \frac{2a^2}{3}.$$

Step 3: Recover the parameter The condition $2a^2/3 = 6$ gives $a^2 = 9$. Since $a > 0$,

$$\boxed{a = 3}, \quad \boxed{V(3) = \frac{81\pi}{2}}.$$

Verification For $a = 3$, the solid has $0 \leq z \leq 9$, and the average 6 lies inside that range. Substitution returns $2(3^2)/3 = 6$ exactly.

Question 1

Let E be the part of the spherical shell

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 3$$

in the first octant. Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV$$

using spherical coordinates.

Tasks

1. Convert the region, integrand, and volume element.
2. Evaluate the integral.
3. Check the result using the integrand range and region volume.

Question 1 – Solution

Strategy. A first-octant spherical shell has constant bounds in all three spherical variables, and the radial integrand cancels one power of the Jacobian.

Step 1: Convert Using

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,$$

the bounds are

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 1 \leq \rho \leq 3,$$

and $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.

Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^3 \frac{1}{\rho} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left(\frac{\pi}{2}\right) (1) \left[\frac{\rho^2}{2}\right]_1^3 = \boxed{2\pi}. \end{aligned}$$

Verification The shell-octant volume is $\frac{1}{8} \frac{4\pi}{3} (3^3 - 1^3) = 13\pi/3$. Hence the average integrand is $I/V = 6/13$, which lies between its endpoint values $1/3$ and 1 .

Question 2

Find the volume inside the sphere $x^2 + y^2 + z^2 \leq 16$ and above the cone

$$z = \sqrt{3}\sqrt{x^2 + y^2}.$$

Tasks

1. Convert the cone to a constant ϕ -boundary.
2. Set up and evaluate the spherical integral.
3. Compare with the volume of the containing ball.

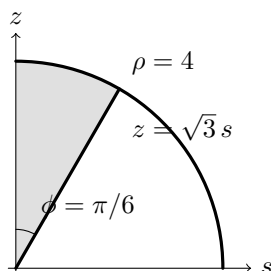
Question 2 – Solution

Strategy. In a meridian plane, the cone fixes the polar angle while the sphere fixes the radial endpoint.

Step 1: Angular boundary Since $z = \rho \cos \phi$ and $\sqrt{x^2 + y^2} = \rho \sin \phi$,

$$\rho \cos \phi = \sqrt{3} \rho \sin \phi \implies \tan \phi = \frac{1}{\sqrt{3}}.$$

The region above the cone therefore has $0 \leq \phi \leq \pi/6$.



Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/6} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 2\pi \left(1 - \frac{\sqrt{3}}{2}\right) \frac{64}{3} = \boxed{\frac{64\pi}{3}(2 - \sqrt{3})}. \end{aligned}$$

Verification The angular factor $1 - \sqrt{3}/2$ is between 0 and 1, so this positive sector is smaller than the upper half-ball and therefore smaller than the entire ball of volume $256\pi/3$.

Question 3

The upper half of the spherical shell $1 \leq \rho \leq 2$ has density $\delta = \rho$.

Tasks

1. Find its mass in spherical coordinates.
2. Find its average density.
3. Verify the average against the density range.

Question 3 – Solution

Strategy. The hemisphere contributes the full azimuthal interval but only $0 \leq \phi \leq \pi/2$; the radial density adds one power of ρ .

Step 1: Mass

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= (2\pi)(1) \left[\frac{\rho^4}{4} \right]_1^2 = \boxed{\frac{15\pi}{2}}. \end{aligned}$$

Step 2: Average density The region volume is half the difference of two ball volumes:

$$V = \frac{1}{2} \frac{4\pi}{3} (2^3 - 1^3) = \frac{14\pi}{3}.$$

Therefore

$$\boxed{\delta_{\text{avg}} = \frac{M}{V} = \frac{45}{28}}.$$

Verification Since $1 \leq \delta = \rho \leq 2$ throughout the shell, the value $45/28 \approx 1.61$ is admissible. The larger outer spherical layers correctly pull it above the midpoint $3/2$.

Question 4

Let E be the upper half of the ball $x^2 + y^2 + z^2 \leq 9$.

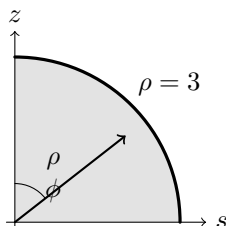
Tasks

1. Use spherical coordinates to compute $\iiint_E z \, dV$.
2. Find the average height z_{avg} .
3. Check its geometric range.

Question 4 – Solution

Strategy. Write $z = \rho \cos \phi$ and use the upper-hemisphere bounds $0 \leq \phi \leq \pi/2$.

Step 1: Geometry and integral



$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= (2\pi) \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} \left[\frac{\rho^4}{4} \right]_0^3 = \boxed{\frac{81\pi}{4}}. \end{aligned}$$

Step 2: Average height The hemisphere volume is $V = \frac{1}{2} \frac{4\pi}{3} (3^3) = 18\pi$. Thus

$$\boxed{z_{\text{avg}} = \frac{I}{V} = \frac{9}{8}}.$$

Verification Every point has $0 \leq z \leq 3$, and $9/8$ lies in that interval. The value is below the mid-height $3/2$ because cross-sectional area is greatest near $z = 0$.

Question 5

Inside a ball of radius R , consider the cone sector

$$0 \leq \rho \leq R, \quad 0 \leq \phi \leq \alpha, \quad 0 \leq \theta \leq 2\pi,$$

where $0 < \alpha < \pi/2$. Its volume is one-third of the upper half-ball.

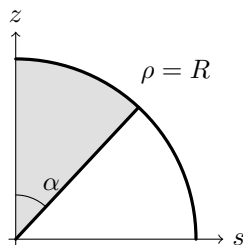
Tasks

1. Determine α .
2. Write the boundary cone in Cartesian coordinates.
3. Verify the stated volume ratio.

Question 5 – Solution

Strategy. The radial and azimuthal factors cancel from the ratio, leaving only the polar-angle factor $1 - \cos \alpha$.

Step 1: Sector volume



$$V_{\text{sector}} = \int_0^{2\pi} \int_0^\alpha \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{2\pi R^3}{3} (1 - \cos \alpha).$$

The upper half-ball has volume $2\pi R^3/3$. Hence

$$1 - \cos \alpha = \frac{1}{3}, \quad \boxed{\alpha = \arccos\left(\frac{2}{3}\right)}.$$

Step 2: Cartesian cone On $\phi = \alpha$, $z = s \cot \alpha$, where $s = \sqrt{x^2 + y^2}$. Since $\sin \alpha = \sqrt{5}/3$, we have $\cot \alpha = 2/\sqrt{5}$. Thus

$$\boxed{z = \frac{2}{\sqrt{5}} \sqrt{x^2 + y^2}}.$$

Verification Substituting $\cos \alpha = 2/3$ makes the angular factor $1 - \cos \alpha = 1/3$, so the sector volume is exactly one-third of the upper half-ball volume.

Question 6

Use spherical coordinates to find the volume inside

$$x^2 + y^2 + z^2 \leq 2z.$$

Tasks

1. Convert the boundary to a radial spherical bound.
2. Evaluate the volume.
3. Identify the Cartesian center and radius as a check.

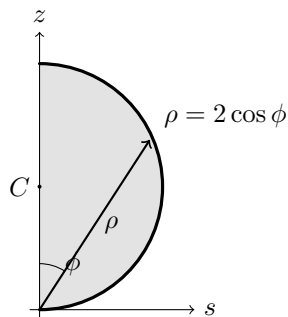
Question 6 – Solution

Strategy. The shifted sphere becomes $\rho \leq 2 \cos \phi$, so its nonconstant radial bound encodes the displacement from the origin.

Step 1: Bounds Substitution gives

$$\rho^2 \leq 2\rho \cos \phi.$$

For $\rho \geq 0$, the nontrivial radial interval is $0 \leq \rho \leq 2 \cos \phi$, which requires $0 \leq \phi \leq \pi/2$. Also $0 \leq \theta \leq 2\pi$.



Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \frac{16\pi}{3} \int_0^{\pi/2} \cos^3 \phi \sin \phi \, d\phi = \boxed{\frac{4\pi}{3}}. \end{aligned}$$

Verification Completing the square gives $x^2 + y^2 + (z - 1)^2 \leq 1$, a sphere of radius 1 centered at $(0, 0, 1)$. Its ordinary volume is $4\pi/3$, confirming the integral.

Question 7

Let

$$E = \{(x, y, z) : 2 \leq \sqrt{x^2 + y^2 + z^2} \leq 3, x \geq 0, y \geq x, z \leq 0\}.$$

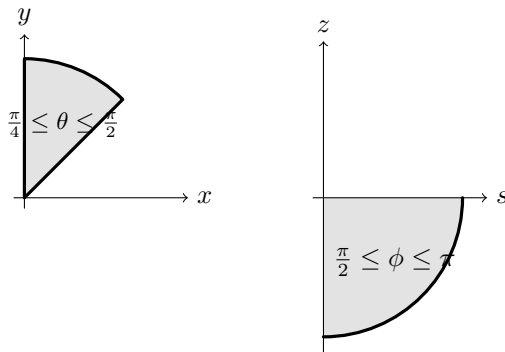
Tasks

1. Determine exact spherical bounds for E .
2. Find its volume.
3. Verify the angular fractions geometrically.

Question 7 – Solution

Strategy. Resolve the horizontal inequalities into a θ -wedge and the vertical inequality into a ϕ -hemisphere.

Step 1: Angular bounds The conditions $x \geq 0$ and $y \geq x$ select $\pi/4 \leq \theta \leq \pi/2$. The condition $z \leq 0$ selects $\pi/2 \leq \phi \leq \pi$, while the shell gives $2 \leq \rho \leq 3$.



Step 2: Volume

$$\begin{aligned} V &= \int_{\pi/4}^{\pi/2} \int_{\pi/2}^{\pi} \int_2^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left(\frac{\pi}{4}\right) (1) \left(\frac{27-8}{3}\right) = \boxed{\frac{19\pi}{12}}. \end{aligned}$$

Verification The azimuthal wedge is $(\pi/4)/(2\pi) = 1/8$ of a full turn, and $z \leq 0$ supplies half the polar solid angle. Thus E occupies $1/16$ of the full shell; indeed $\frac{1}{16} \frac{4\pi}{3} (27-8) = 19\pi/12$.

Question 8

A full spherical shell $1 \leq \rho \leq 2$ has density $\delta = \rho$. Find its moment of inertia about the z -axis.

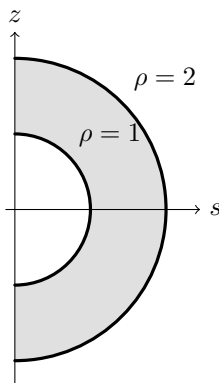
Tasks

1. Express the squared distance to the z -axis in spherical coordinates.
2. Compute the mass and I_z .
3. Find and check the radius of gyration $k_z = \sqrt{I_z/M}$.

Question 8 – Solution

Strategy. The distance to the z -axis is $\rho \sin \phi$; combine its square with the radial density and spherical Jacobian.

Step 1: Region and mass



$$M = \int_0^{2\pi} \int_0^\pi \int_1^2 \rho \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = (2\pi)(2) \frac{15}{4} = \boxed{15\pi}.$$

Step 2: Moment of inertia

$$\begin{aligned} I_z &= \iiint_E (\rho \sin \phi)^2 \delta \, dV \\ &= \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^5 \sin^3 \phi \, d\rho \, d\phi \, d\theta \\ &= (2\pi) \left(\frac{4}{3} \right) \left(\frac{2^6 - 1}{6} \right) = \boxed{28\pi}. \end{aligned}$$

Step 3: Radius of gyration

$$\boxed{k_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{28}{15}}}.$$

It lies between the minimum and maximum distances 0 and 2 from the axis, as required.

Question 9

Explain why spherical coordinates use

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta,$$

then evaluate $\iiint_E \rho^{-2} \, dV$ on the shell $1 \leq \rho \leq 3$.

Tasks

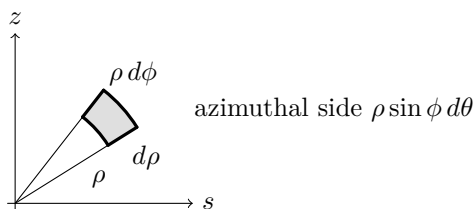
1. Derive the three local scale factors geometrically.
2. Evaluate the integral over the full shell.
3. Check the dimensional scaling and behavior near the origin.

Question 9 – Solution

Strategy. Approximate a small spherical cell by a box with one radial side and two angular arc-length sides.

Step 1: Volume element The local side lengths are

$$d\rho, \quad \rho d\phi, \quad \rho \sin \phi d\theta.$$



Their product is

$$(d\rho)(\rho d\phi)(\rho \sin \phi d\theta) = \rho^2 \sin \phi d\rho d\phi d\theta.$$

Step 2: Apply

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^\pi \int_1^3 \rho^{-2} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= (2\pi)(2)(3-1) = \boxed{8\pi}. \end{aligned}$$

Verification The shell excludes $\rho = 0$, so the integrand is bounded. Its units are length^{-2} ; multiplying by volume produces length, matching the radial factor $3 - 1$ in the result. Even if the inner radius tends to 0, the radial integral remains finite: $4\pi \int_0^3 d\rho = 12\pi$. Thus the singularity is integrable.

Question 10

For $a > 0$, a uniform spherical shell occupies

$$a \leq \rho \leq 2a.$$

Its average distance from the origin is $45/7$.

Tasks

1. Derive the shell volume as a function of a .
2. Derive the average value of ρ .
3. Recover a , the outer radius, and the shell volume.

Question 10 – Solution

Strategy. Divide the first radial moment by volume; the common angular factors cancel, leaving a linear function of a .

Step 1: Volume

$$\begin{aligned} V(a) &= \int_0^{2\pi} \int_0^\pi \int_a^{2a} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 4\pi \left[\frac{\rho^3}{3} \right]_a^{2a} = \frac{28\pi a^3}{3}. \end{aligned}$$

Step 2: Average radius The first radial moment is

$$M_\rho(a) = 4\pi \int_a^{2a} \rho^3 \, d\rho = 15\pi a^4.$$

Hence

$$\rho_{\text{avg}} = \frac{M_\rho(a)}{V(a)} = \frac{45a}{28}.$$

Step 3: Recover the shell Setting $45a/28 = 45/7$ gives $\boxed{a = 4}$. Therefore

$$\boxed{\text{outer radius} = 8}, \quad \boxed{V = \frac{1792\pi}{3}}.$$

Verification The average $45/7 \approx 6.43$ lies strictly between the recovered radii 4 and 8. Substituting $a = 4$ into $28\pi a^3/3$ gives the stated volume.

Question 1

Let

$$R = \{(x, y) : |x + y| \leq 2, |x - y| \leq 1\}.$$

Use $u = x + y$ and $v = x - y$ to evaluate

$$\iint_R (x + y)^2 dA.$$

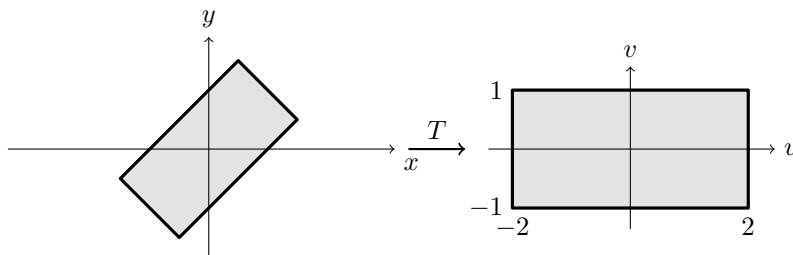
Tasks

1. Describe the image of R in the uv -plane.
2. Compute the inverse Jacobian and evaluate the integral.
3. Verify the sign and scale of the answer.

Question 1 – Solution

Strategy. The boundary expressions themselves become coordinates, turning the tilted rectangle into an axis-aligned rectangle.

Step 1: Transform the region



The inequalities become $-2 \leq u \leq 2$ and $-1 \leq v \leq 1$. Solving $u = x + y$, $v = x - y$ gives

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}.$$

Step 2: Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}, \quad dA = \frac{1}{2} du dv.$$

Step 3: Evaluate

$$\begin{aligned} I &= \int_{-2}^2 \int_{-1}^1 u^2 \left(\frac{1}{2} \right) dv du \\ &= \frac{1}{2}(2) \left[\frac{u^3}{3} \right]_{-2}^2 = \boxed{\frac{16}{3}}. \end{aligned}$$

Verification The integrand is nonnegative. The transformed rectangle has area 8, scaled by $1/2$, so $\text{area}(R) = 4$; the result corresponds to average value $4/3$, which is below the maximum $u^2 = 4$.

Question 2

In the first quadrant, let R be bounded by

$$xy = 1, \quad xy = 4, \quad \frac{x}{y} = 1, \quad \frac{x}{y} = 9.$$

Use $u = xy$ and $v = x/y$ to evaluate $\iint_R (x/y) \, dA$.

Tasks

1. Show that the transformation is one-to-one on R .
2. Find the inverse Jacobian.
3. Evaluate and check the result against the integrand range.

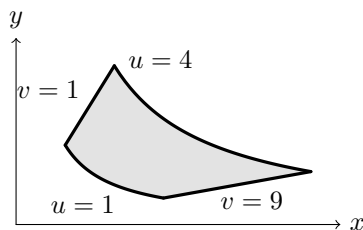
Question 2 – Solution

Strategy. Product and ratio coordinates straighten the hyperbolas and rays into a rectangle.

Step 1: Region and inverse In the first quadrant,

$$x = \sqrt{uv}, \quad y = \sqrt{\frac{u}{v}},$$

so positive (u, v) determine a unique (x, y) . The image is $1 \leq u \leq 4$, $1 \leq v \leq 9$.



Step 2: Jacobian Differentiating the inverse formulas gives

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{2v}.$$

Therefore the factor $x/y = v$ cancels the v in the denominator.

Step 3: Evaluate

$$I = \int_1^4 \int_1^9 v \left(\frac{1}{2v} \right) dv du = \frac{1}{2}(3)(8) = \boxed{12}.$$

Verification The region area is $\int_1^4 \int_1^9 (2v)^{-1} dv du = 3 \ln 3$. Since $1 \leq x/y \leq 9$, the bounds $3 \ln 3 \leq I \leq 27 \ln 3$ hold; $I = 12$ lies between them.

Question 3

Let

$$E = \left\{ (x, y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \right\}.$$

Use $x = 3u$, $y = 2v$ and then polar coordinates in the uv -plane to evaluate

$$\iint_E \left(\frac{x^2}{9} + \frac{y^2}{4} \right) dA.$$

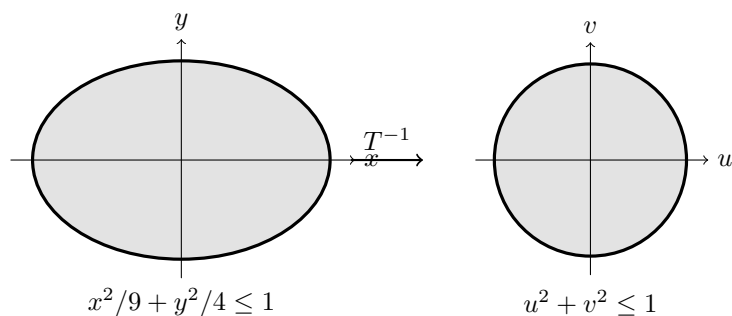
Tasks

1. Map the ellipse to a unit disk.
2. Track both Jacobian factors and evaluate.
3. Verify by comparing the integral with the ellipse area.

Question 3 – Solution

Strategy. First remove the unequal axis scales, then exploit radial symmetry on the resulting unit disk.

Step 1: Linear scaling



The map $x = 3u$, $y = 2v$ sends the unit disk D to E , with

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = 6, \quad \frac{x^2}{9} + \frac{y^2}{4} = u^2 + v^2.$$

Step 2: Polar coordinates in D Write $u = r \cos \theta$, $v = r \sin \theta$ and $du dv = r dr d\theta$. Then

$$\begin{aligned} I &= 6 \int_0^{2\pi} \int_0^1 r^2 (r dr d\theta) \\ &= 6(2\pi) \left[\frac{r^4}{4} \right]_0^1 = \boxed{3\pi}. \end{aligned}$$

Verification The ellipse area is 6π , and the transformed radial-square function has average $1/2$ on the unit disk. Thus the integral must be $(6\pi)(1/2) = 3\pi$.

Question 4

Let R be bounded by the four lines

$$x + 2y = 0, \quad x + 2y = 2, \quad 3x - y = 1, \quad 3x - y = 4.$$

Use $u = x + 2y$, $v = 3x - y$ to evaluate

$$\iint_R (x + 2y)(3x - y) \, dA.$$

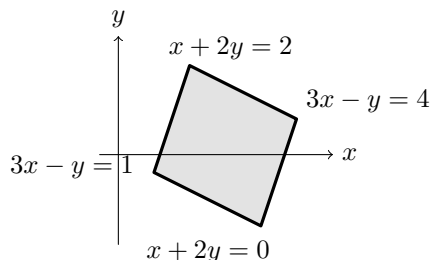
Tasks

1. Identify the transformed rectangle.
2. Compute the Jacobian and integral.
3. Verify that the transformation preserves one-to-one coverage.

Question 4 – Solution

Strategy. Each pair of parallel boundary lines becomes a pair of constant-coordinate sides.

Step 1: Geometry



The image is the rectangle $0 \leq u \leq 2$, $1 \leq v \leq 4$. The coefficient matrix has determinant

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7.$$

Since this determinant is nonzero, the linear map is globally one-to-one.

Step 2: Change variables We have $dA = \frac{1}{7} du dv$ and the integrand is uv . Hence

$$\begin{aligned} I &= \frac{1}{7} \int_0^2 \int_1^4 uv \, dv \, du \\ &= \frac{1}{7} \left[\frac{u^2}{2} \right]_0^2 \left[\frac{v^2}{2} \right]_1^4 = \boxed{\frac{15}{7}}. \end{aligned}$$

Verification On the transformed rectangle, $u \geq 0$ and $v > 0$, so the positive answer has the required sign. The constant area scale $1/7$ also matches the inverse determinant.

Question 5

Let

$$R = \{(x, y) : 1 \leq x \leq 3, x \leq y \leq 2x\}.$$

Use $u = x$ and $v = y/x$ to evaluate

$$\iint_R \frac{1}{x} dA.$$

Tasks

1. Find the inverse map and transformed bounds.
2. Compute the Jacobian and evaluate.
3. Verify directly in Cartesian coordinates.

Question 5 – Solution

Strategy. The ratio $v = y/x$ turns the two sloping sides into constant-coordinate boundaries.

Step 1: Inverse and bounds Since $x > 0$ on R ,

$$x = u, \quad y = uv.$$

The region becomes the rectangle

$$1 \leq u \leq 3, \quad 1 \leq v \leq 2.$$

The inverse map is one-to-one because $u = x$ and $v = y/x$ recover each point uniquely.

Step 2: Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u.$$

Thus

$$I = \int_1^3 \int_1^2 \frac{1}{u}(u) \, dv \, du = \int_1^3 \int_1^2 1 \, dv \, du = \boxed{2}.$$

Verification Direct Cartesian integration gives

$$\int_1^3 \int_x^{2x} \frac{1}{x} \, dy \, dx = \int_1^3 \frac{2x - x}{x} \, dx = 2.$$

The two methods agree exactly, including the variable area scale u .

Question 6

Consider the map

$$T(u, v) = (u^2, v), \quad -1 \leq u \leq 1, \quad 0 \leq v \leq 1.$$

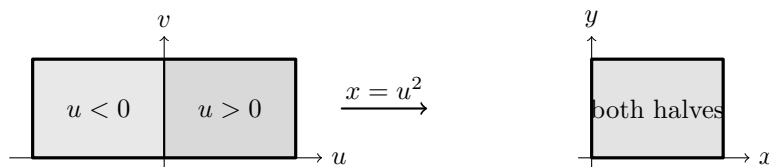
Tasks

1. Describe the image and determine whether T is one-to-one.
2. Explain why integrating $|J_T|$ over the full source gives the wrong image area.
3. Repair the change of variables calculation.

Question 6 – Solution

Strategy. Track multiplicity: the two halves of the source fold onto the same target rectangle.

Step 1: Image and folding



The image is $0 \leq x \leq 1$, $0 \leq y \leq 1$. Except along $u = 0$, the points (u, v) and $(-u, v)$ have the same image, so T is two-to-one.

Step 2: Diagnose the overcount Since

$$J_T = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u & 0 \\ 0 & 1 \end{vmatrix} = 2u,$$

integrating its absolute value over the full source gives

$$\int_{-1}^1 \int_0^1 |2u| \, dv \, du = 2,$$

twice the actual image area 1. The absolute value corrects orientation, not multiplicity.

Step 3: Repair Restrict to the one-to-one half $0 \leq u \leq 1$:

$$\text{area}(T(S)) = \int_0^1 \int_0^1 2u \, dv \, du = \boxed{1}.$$

Using $-1 \leq u \leq 0$ alone gives the same result after taking $|J_T|$.

Question 7

Let

$$R = \{(x, y) : 0 \leq x \leq 2, x^2 \leq y \leq x^2 + 3\}.$$

Use $u = x$ and $v = y - x^2$ to evaluate

$$\iint_R (y - x^2) dA.$$

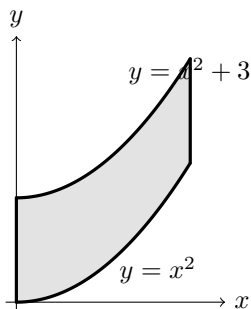
Tasks

1. Show how the change of variables straightens the parabolic strip.
2. Compute the Jacobian and integral.
3. Verify directly from the vertical thickness.

Question 7 – Solution

Strategy. Measure vertical displacement from the lower parabola; the curved strip then becomes a rectangle.

Step 1: Straighten the region



The inverse map is

$$x = u, \quad y = u^2 + v,$$

and the bounds become $0 \leq u \leq 2$, $0 \leq v \leq 3$.

Step 2: Jacobian and integral

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ 2u & 1 \end{vmatrix} = 1.$$

Because $y - x^2 = v$,

$$I = \int_0^2 \int_0^3 v \, dv \, du = (2) \left[\frac{v^2}{2} \right]_0^3 = \boxed{9}.$$

Verification For each fixed x , the displacement $t = y - x^2$ ranges from 0 to 3. Thus direct integration gives $\int_0^2 \int_0^3 t \, dt \, dx = 9$. The unit Jacobian reflects a vertical shear with no area scaling.

Question 8

The unit cube $0 \leq u, v, w \leq 1$ is mapped into xyz -space by

$$x = u + v, \quad y = v + w, \quad z = w + u.$$

Evaluate $\iiint_E (x + y + z) dV$, where E is the image parallelepiped.

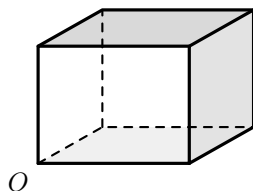
Tasks

1. Compute the three-dimensional Jacobian and establish invertibility.
2. Evaluate the transformed integral.
3. Verify using the image volume and centroid.

Question 8 – Solution

Strategy. A nonsingular linear map sends the unit cube to a parallelepiped with constant volume scale.

Step 1: Jacobian and geometry



$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2.$$

Because $J \neq 0$, the linear map is invertible and covers its image once.

Step 2: Transform the integrand We have $x + y + z = 2(u + v + w)$ and $dV = 2 du dv dw$. Therefore

$$\begin{aligned} I &= \int_0^1 \int_0^1 \int_0^1 4(u + v + w) dw dv du \\ &= 4 \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = \boxed{6}. \end{aligned}$$

Verification The image volume is $|J|$ times the cube volume, hence 2. The cube centroid $(1/2, 1/2, 1/2)$ maps to $(1, 1, 1)$, where $x + y + z = 3$. A linear function averages to its centroid value, so $I = (2)(3) = 6$.

Question 9

For $a, b > 0$, let

$$E_{a,b} = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

Suppose

$$\text{area}(E_{a,b}) = 6\pi, \quad \iint_{E_{a,b}} x^2 \, dA = \frac{27\pi}{2}.$$

Tasks

1. Use $x = au$, $y = bv$ to translate both measurements.
2. Recover a and b .
3. Verify both original conditions.

Question 9 – Solution

Strategy. Scale the ellipse to the unit disk; area determines ab , while the x^2 moment determines one additional power of a^2 .

Step 1: Area condition The map $x = au$, $y = bv$ has $dA = ab \, du \, dv$. Since the unit disk has area π ,

$$\pi ab = 6\pi, \quad \text{so} \quad ab = 6.$$

Step 2: Moment condition On the unit disk,

$$\iint_D u^2 \, du \, dv = \frac{\pi}{4}$$

by symmetry or a polar calculation. Therefore

$$\iint_{E_{a,b}} x^2 \, dA = a^2(ab) \frac{\pi}{4} = \frac{3\pi a^2}{2}.$$

Equating this with $27\pi/2$ gives $a^2 = 9$. Positivity then yields

$$\boxed{a = 3}, \quad \boxed{b = 2}.$$

Verification The recovered ellipse has area $\pi(3)(2) = 6\pi$. Its moment is $a^2(ab)\pi/4 = (9)(6)\pi/4 = 27\pi/2$, so both measurements are satisfied without sign ambiguity.

Question 10

Let

$$R = \{(x, y) : x \geq 0, y \geq 0, 1 \leq x + y \leq 3\}.$$

Use

$$u = x + y, \quad v = \frac{x}{x + y}$$

to evaluate $\iint_R (x + y)^{-1} dA$.

Tasks

1. Derive the inverse map and transformed rectangle.
2. Compute the Jacobian and integral.
3. Explain geometrically what the coordinate v measures.

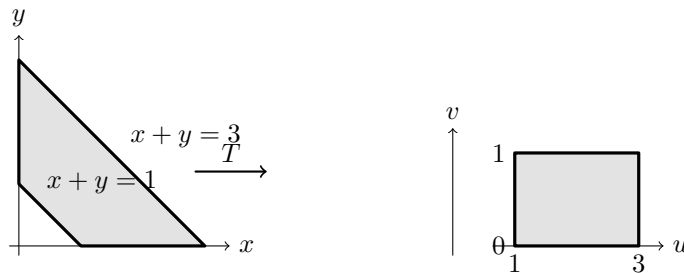
Question 10 – Solution

Strategy. Use total distance along the coordinate-sum direction and the fraction allocated to x .

Step 1: Inverse and geometry Since $u = x + y > 0$,

$$x = uv, \quad y = u(1 - v).$$

Nonnegativity gives $0 \leq v \leq 1$, while the two diagonal boundaries give $1 \leq u \leq 3$.



Step 2: Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u, \quad dA = u \, du \, dv.$$

Hence the integrand $1/(x+y) = 1/u$ cancels the area scale:

$$I = \int_1^3 \int_0^1 \frac{1}{u}(u) \, dv \, du = \boxed{2}.$$

Step 3: Interpretation and check On a segment $x+y=u$, the quantity $v=x/u$ is the fraction of the total coordinate sum assigned to x ; it runs once from the y -axis to the x -axis. Thus the inverse is one-to-one in the interior, and $u > 0$ confirms the absolute Jacobian used above.

Question 1

Find the area of the part of the plane

$$z = 2x - y$$

lying above the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 2$.

Tasks

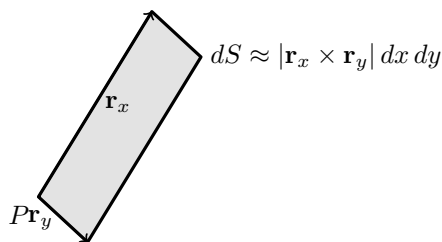
1. Derive the graph surface-area factor from tangent vectors.
2. Evaluate the surface area.
3. Verify by interpreting the constant area scale.

Question 1 – Solution

Strategy. Parametrize the graph by its projection and use the tangent-parallelogram area scale.

Step 1: Derive the element For a graph $z = f(x, y)$,

$$\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle, \quad \mathbf{r}_x = \langle 1, 0, f_x \rangle, \quad \mathbf{r}_y = \langle 0, 1, f_y \rangle.$$



Thus

$$|\mathbf{r}_x \times \mathbf{r}_y| = |\langle -f_x, -f_y, 1 \rangle| = \sqrt{1 + f_x^2 + f_y^2}.$$

Step 2: Apply to the plane Here $f_x = 2$ and $f_y = -1$, so

$$S = \int_0^3 \int_0^2 \sqrt{1 + 2^2 + (-1)^2} dy dx = \sqrt{6}(3)(2) = \boxed{6\sqrt{6}}.$$

Verification Every small projected area is enlarged by the same factor $\sqrt{6}$. Since the rectangle has area 6, the surface area must be $6\sqrt{6}$.

Question 2

Find the area of the paraboloid

$$z = x^2 + y^2$$

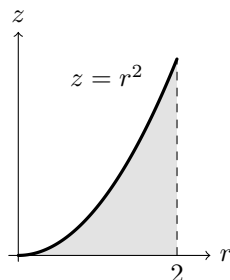
above the disk $x^2 + y^2 \leq 4$.

Tasks

1. Form the graph surface-area integrand.
2. Use polar coordinates to evaluate the integral.
3. Check the answer against the projected area.

Question 2 – Solution

Strategy. The squared gradient depends only on r , so polar coordinates reduce the area integral to one variable.

Step 1: Geometry and gradient

For $f = x^2 + y^2$,

$$f_x = 2x, \quad f_y = 2y, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 4r^2}.$$

Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta \\ &= 2\pi \left[\frac{(1 + 4r^2)^{3/2}}{12} \right]_0^2 = \boxed{\frac{\pi}{6} (17\sqrt{17} - 1)}. \end{aligned}$$

Verification The area factor is at least 1, so S must exceed the projected disk area 4π . The exact result does. Its integrand also increases with r , matching the increasing steepness of the paraboloid.

Question 3

Find the area of the conical surface

$$z = \sqrt{x^2 + y^2}$$

lying above the annulus $1 \leq x^2 + y^2 \leq 9$.

Tasks

1. Explain why the graph formula is valid on the annulus.
2. Evaluate the area.
3. Verify with the lateral-area formula for a conical frustum.

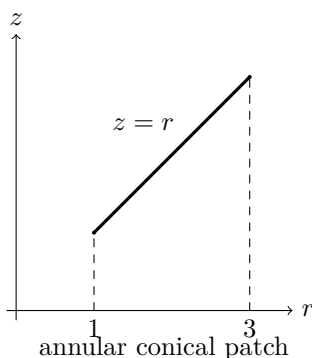
Question 3 – Solution

Strategy. The annulus avoids the cone's nondifferentiable vertex, and the cone has constant slope there.

Step 1: Gradient Write $r = \sqrt{x^2 + y^2}$. For $r > 0$,

$$f_x = \frac{x}{r}, \quad f_y = \frac{y}{r}, \quad f_x^2 + f_y^2 = 1.$$

The entire projection has $r \geq 1$, so no derivative singularity occurs.



Step 2: Area

$$S = \iint_D \sqrt{1 + 1} \, dA = \sqrt{2} \pi (3^2 - 1^2) = \boxed{8\sqrt{2} \pi}.$$

Verification The frustum has radii 1 and 3. Its slant height is $\sqrt{(3-1)^2 + (3-1)^2} = 2\sqrt{2}$, so the classical lateral area $\pi(R+r)\ell = \pi(3+1)(2\sqrt{2}) = 8\sqrt{2}\pi$ agrees.

Question 4

Use a surface-area integral to find the area of the upper hemisphere

$$z = \sqrt{9 - x^2 - y^2}.$$

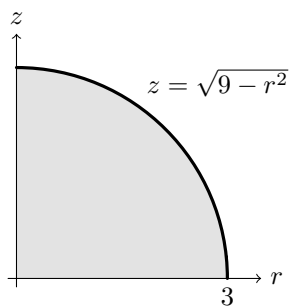
Tasks

1. Derive the surface-area factor over the open disk $x^2 + y^2 < 9$.
2. Evaluate the resulting improper integral.
3. Explain why the boundary singularity is integrable.

Question 4 – Solution

Strategy. Integrate over disks of radius $b < 3$, then take the limit as $b \rightarrow 3^-$.

Step 1: Area factor



With $r^2 = x^2 + y^2$,

$$f_x = -\frac{x}{\sqrt{9-r^2}}, \quad f_y = -\frac{y}{\sqrt{9-r^2}},$$

so

$$\sqrt{1 + f_x^2 + f_y^2} = \frac{3}{\sqrt{9-r^2}}.$$

Step 2: Improper integral

$$\begin{aligned} S &= \lim_{b \rightarrow 3} \int_0^{2\pi} \int_0^b \frac{3r}{\sqrt{9-r^2}} dr d\theta \\ &= \lim_{b \rightarrow 3} 6\pi \left(3 - \sqrt{9-b^2} \right) = \boxed{18\pi}. \end{aligned}$$

Verification The factor diverges as $r \rightarrow 3^-$ because the sphere becomes vertical, but its antiderivative has a finite limit. The result also equals half the full sphere area $4\pi(3^2) = 36\pi$.

Question 5

Find the area of the sideways paraboloid

$$x = y^2 + z^2$$

over the disk $y^2 + z^2 \leq 1$ in the yz -plane.

Tasks

1. State the surface-area formula for a graph $x = g(y, z)$.
2. Evaluate using polar coordinates in the yz -plane.
3. Check the result against the projected disk area.

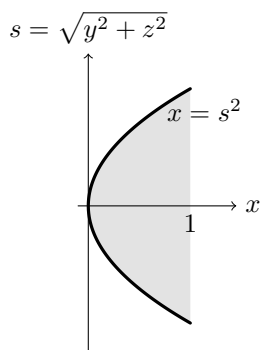
Question 5 – Solution

Strategy. Project onto the yz -plane, where both the region and squared gradient are radial.

Step 1: Projection formula For $x = g(y, z)$,

$$dS = \sqrt{1 + g_y^2 + g_z^2} dy dz.$$

Here $g = y^2 + z^2$, so with $s^2 = y^2 + z^2$ the factor is $\sqrt{1 + 4s^2}$.



Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4s^2} s ds d\theta \\ &= 2\pi \left[\frac{(1 + 4s^2)^{3/2}}{12} \right]_0^1 = \boxed{\frac{\pi}{6} (5\sqrt{5} - 1)}. \end{aligned}$$

Verification The graph projects one-to-one onto a disk of area π , and the area factor exceeds 1 except at its center. Accordingly, the computed surface area is strictly larger than π .

Question 6

Find the area of the portion of

$$z = 4 - x - y$$

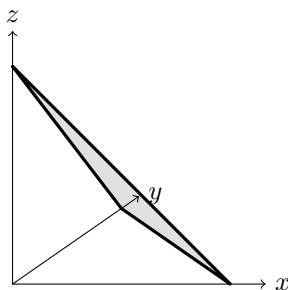
in the first octant.

Tasks

1. Determine the projection in the xy -plane.
2. Evaluate the graph surface-area integral.
3. Verify using the three intercept vertices.

Question 6 – Solution

Strategy. The coordinate-plane restrictions cut a triangular patch from a plane with constant slope.

Step 1: Region

In the first octant, $x \geq 0$, $y \geq 0$, and $z = 4 - x - y \geq 0$. Thus the projection is

$$D = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq 4 - x\},$$

a right triangle of area 8.

Step 2: Surface integral Since $f_x = f_y = -1$,

$$S = \iint_D \sqrt{1 + (-1)^2 + (-1)^2} dA = \sqrt{3}(8) = \boxed{8\sqrt{3}}.$$

Verification From $(4, 0, 0)$, use edge vectors $\mathbf{a} = \langle -4, 4, 0 \rangle$ and $\mathbf{b} = \langle -4, 0, 4 \rangle$. Then $|\mathbf{a} \times \mathbf{b}| = 16\sqrt{3}$, so the triangle area is $\frac{1}{2}(16\sqrt{3}) = 8\sqrt{3}$.

Question 7

Find the area of the surface

$$z = xy$$

above the quarter-disk $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.

Tasks

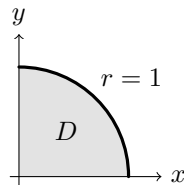
1. Show that the surface-area factor is radial even though $z = xy$ is not.
2. Evaluate the polar integral.
3. Verify the angular and radial bounds.

Question 7 – Solution

Strategy. Differentiate first: the two gradient components exchange x and y , so their squares combine to r^2 .

Step 1: Gradient and projection

$$f_x = y, \quad f_y = x, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + x^2 + y^2} = \sqrt{1 + r^2}.$$



The quarter-disk has $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq 1$.

Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{\pi/2} \int_0^1 \sqrt{1 + r^2} r \, dr \, d\theta \\ &= \frac{\pi}{2} \left[\frac{(1 + r^2)^{3/2}}{3} \right]_0^1 = \boxed{\frac{\pi}{6}(2\sqrt{2} - 1)}. \end{aligned}$$

Verification The angular width is one quarter of a full turn. The factor ranges from 1 to $\sqrt{2}$, so the result must lie between the projection area $\pi/4$ and $\sqrt{2}\pi/4$; it does.

Question 8

For constants a, b, c , the plane

$$z = ax + by + c$$

lies above the unit disk $x^2 + y^2 \leq 1$. Its surface area there is 3π .

Tasks

1. Determine all possible gradient pairs (a, b) .
2. Explain the role of c .
3. Give two distinct planes satisfying the condition and verify them.

Question 8 – Solution

Strategy. A plane has a constant area-enlargement factor determined only by the magnitude of its gradient.

Step 1: Area equation Since $f_x = a$ and $f_y = b$,

$$S = \iint_D \sqrt{1 + a^2 + b^2} \, dA = \pi \sqrt{1 + a^2 + b^2}.$$

The condition $S = 3\pi$ gives

$$1 + a^2 + b^2 = 9, \quad \boxed{a^2 + b^2 = 8}.$$

Thus all possible gradients form the circle of radius $2\sqrt{2}$ in the ab -plane.

Step 2: Vertical translation The constant c does not appear in either derivative, so

$$\boxed{c \text{ is arbitrary}}.$$

Changing c translates the patch vertically without stretching it.

Step 3: Examples and verification Two choices are

$$\boxed{z = 2\sqrt{2}x} \quad \text{and} \quad \boxed{z = 2x + 2y + 5}.$$

Their gradient-square sums are respectively $8 + 0$ and $4 + 4$. Each therefore has area factor $\sqrt{1 + 8} = 3$ and area 3π above the unit disk.

Question 9

Find the area of the saddle

$$z = \frac{x^2 - y^2}{2}$$

above the disk $x^2 + y^2 \leq 3$.

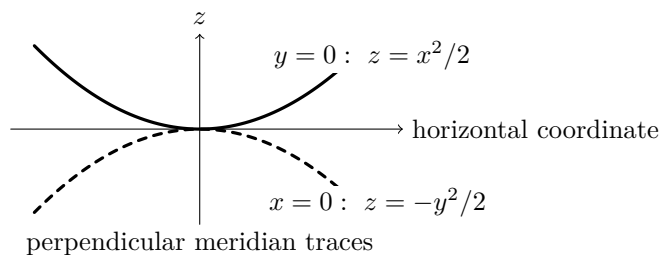
Tasks

1. Relate the two perpendicular meridian traces to the saddle shape.
2. Show that the surface-area factor is radial.
3. Evaluate the area exactly.

Question 9 – Solution

Strategy. Although the height changes sign by direction, the squared gradient depends only on distance from the origin.

Step 1: Geometry



Along $y = 0$ the trace opens upward, while along $x = 0$ it opens downward. Differentiation gives

$$f_x = x, \quad f_y = -y, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + r^2}.$$

Step 2: Polar integral

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + r^2} r \, dr \, d\theta \\ &= 2\pi \left[\frac{(1 + r^2)^{3/2}}{3} \right]_0^{\sqrt{3}} \\ &= \frac{2\pi}{3} (8 - 1) = \boxed{\frac{14\pi}{3}}. \end{aligned}$$

Verification The projected disk has area 3π , while the area factor ranges from 1 to 2. Hence $3\pi < S < 6\pi$, and $14\pi/3$ lies in that interval.

Question 10

Let f be continuously differentiable on a region D of area 12, and suppose

$$1 \leq |\nabla f(x, y)| \leq 2 \quad \text{throughout } D.$$

Let S be the area of the graph $z = f(x, y)$ above D .

Tasks

1. Derive the sharpest bounds guaranteed for S .
2. Prove the bounds from the surface-area integral.
3. Show that both endpoints can occur.

Question 10 – Solution

Strategy. Apply the monotonicity of $\sqrt{1+t^2}$ to the pointwise gradient bounds, then integrate.

Step 1: Pointwise comparison From $1 \leq |\nabla f| \leq 2$,

$$\sqrt{2} = \sqrt{1+1^2} \leq \sqrt{1+|\nabla f|^2} \leq \sqrt{1+2^2} = \sqrt{5}.$$

Step 2: Integrate The graph area is

$$S = \iint_D \sqrt{1+|\nabla f|^2} dA.$$

Integrating the pointwise inequalities and using $\text{area}(D) = 12$ yields

$$\boxed{12\sqrt{2} \leq S \leq 12\sqrt{5}}.$$

Step 3: Sharpness The lower endpoint occurs for any plane with constant gradient magnitude 1, for example $f(x, y) = x$. The upper endpoint occurs for a plane with constant gradient magnitude 2, for example $f(x, y) = 2x$. Indeed,

$$S_{f=x} = 12\sqrt{1+1} = 12\sqrt{2}, \quad S_{f=2x} = 12\sqrt{1+4} = 12\sqrt{5}.$$

Therefore no narrower interval can be guaranteed from the supplied data.

Question 1

Let $a > 0$. Find the area enclosed by the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

using the change of variables $x = u^3$, $y = v^3$.

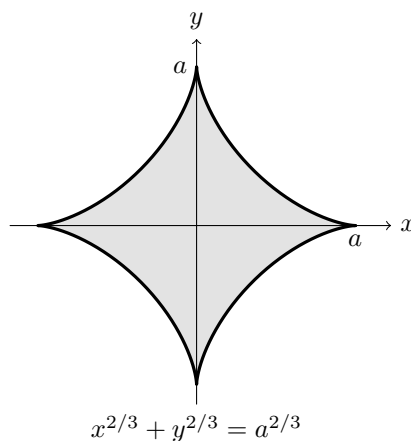
Tasks

1. Describe the transformed region and justify the substitution.
2. Compute the Jacobian and evaluate the area.
3. Check the scaling of the answer with respect to a .

Question 1 – Solution

Strategy. The cubic substitution converts the astroid into a disk, after which polar coordinates handle the transformed integral.

Step 1: Transform the region



Because the real cube function is one-to-one,

$$x^{2/3} = u^2, \quad y^{2/3} = v^2.$$

Thus the interior maps one-to-one to $u^2 + v^2 \leq a^{2/3}$, a disk of radius $a^{1/3}$.

Step 2: Jacobian and integral The absolute Jacobian is

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = |(3u^2)(3v^2)| = 9u^2v^2.$$

Writing $u = r \cos \theta$, $v = r \sin \theta$ gives

$$\begin{aligned} A &= 9 \int_0^{2\pi} \int_0^{a^{1/3}} r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta \\ &= 9 \left(\frac{a^2}{6} \right) \left(\frac{\pi}{4} \right) = \boxed{\frac{3\pi a^2}{8}}. \end{aligned}$$

Verification Replacing a by ka enlarges every length by k , so area must scale by k^2 . The result is proportional to a^2 , as required.

Question 2

Find the area common to the disks

$$x^2 + y^2 \leq 2x, \quad x^2 + y^2 \leq 2y.$$

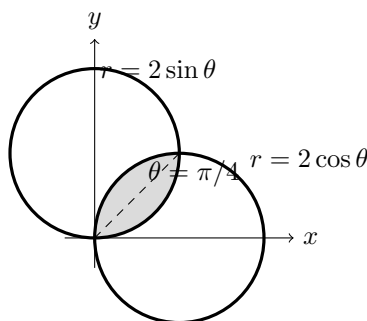
Tasks

1. Translate both inequalities into polar coordinates.
2. Identify where the radial boundary changes and evaluate the area.
3. Verify the symmetry used in the calculation.

Question 2 – Solution

Strategy. In the first quadrant the smaller of the two polar bounds switches at the line $\theta = \pi/4$.

Step 1: Polar description



For $r \geq 0$, the disks become

$$r \leq 2 \cos \theta, \quad r \leq 2 \sin \theta.$$

Their common region lies in $0 \leq \theta \leq \pi/2$. The bounds are equal at $\theta = \pi/4$.

Step 2: Evaluate Reflection across $y = x$ makes the two halves congruent, so

$$\begin{aligned} A &= 2 \int_0^{\pi/4} \int_0^{2 \sin \theta} r \, dr \, d\theta = 4 \int_0^{\pi/4} \sin^2 \theta \, d\theta \\ &= 4 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/4} = \boxed{\frac{\pi}{2} - 1}. \end{aligned}$$

Verification The circles have equal radius and centers $(1, 0)$ and $(0, 1)$, which are interchanged by reflecting across $y = x$. Their intersections, $(0, 0)$ and $(1, 1)$, also lie on that line.

Question 3

Find the volume enclosed between the paraboloids

$$z = x^2 + y^2 \quad \text{and} \quad z = 6 - x^2 - y^2.$$

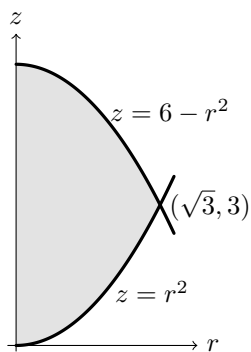
Tasks

1. Find the curve where the surfaces meet.
2. Set up and evaluate a cylindrical-coordinate integral.
3. Check that the height is nonnegative throughout the base.

Question 3 – Solution

Strategy. Subtract the lower paraboloid from the upper one over their circular projection.

Step 1: Intersection



Setting the surfaces equal gives

$$r^2 = 6 - r^2, \quad r = \sqrt{3}, \quad z = 3.$$

Hence the projection is the disk $0 \leq r \leq \sqrt{3}$.

Step 2: Volume The vertical height is $(6 - r^2) - r^2 = 6 - 2r^2$. Therefore

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\sqrt{3}} (6 - 2r^2)r \, dr \, d\theta \\ &= 2\pi \left[3r^2 - \frac{r^4}{2} \right]_0^{\sqrt{3}} = 2\pi \left(9 - \frac{9}{2} \right) = \boxed{9\pi}. \end{aligned}$$

Verification On $0 \leq r \leq \sqrt{3}$, the height is $2(3 - r^2) \geq 0$; it decreases from 6 at the axis to 0 at the intersection circle, matching the geometry.

Question 4

Let $a > 0$. Two perpendicular circular cylinders form the solid

$$x^2 + z^2 \leq a^2, \quad y^2 + z^2 \leq a^2.$$

Find its volume.

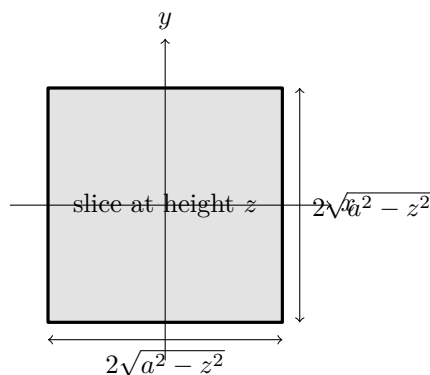
Tasks

1. Describe a horizontal cross-section at a fixed height z .
2. Integrate the cross-sectional area.
3. Check symmetry and dimensional scaling.

Question 4 – Solution

Strategy. At each fixed z , the two cylinder inequalities give independent bounds on x and y , producing a square.

Step 1: Cross-section



For $-a \leq z \leq a$,

$$|x| \leq \sqrt{a^2 - z^2}, \quad |y| \leq \sqrt{a^2 - z^2}.$$

Thus the slice is a square with area

$$A(z) = \left(2\sqrt{a^2 - z^2}\right)^2 = 4(a^2 - z^2).$$

Step 2: Integrate

$$\begin{aligned} V &= \int_{-a}^a 4(a^2 - z^2) dz = 8 \int_0^a (a^2 - z^2) dz \\ &= 8 \left[a^2 z - \frac{z^3}{3} \right]_0^a = \boxed{\frac{16a^3}{3}}. \end{aligned}$$

Verification The cross-sectional area is even in z , justifying the factor 2. The answer is proportional to a^3 , the correct scaling for a volume.

Question 5

A cylindrical hole of radius a is drilled through the center of a sphere of radius R , where $0 < a < R$. The remaining solid has height

$$h = 2\sqrt{R^2 - a^2}.$$

Find its volume and express the result using only h .

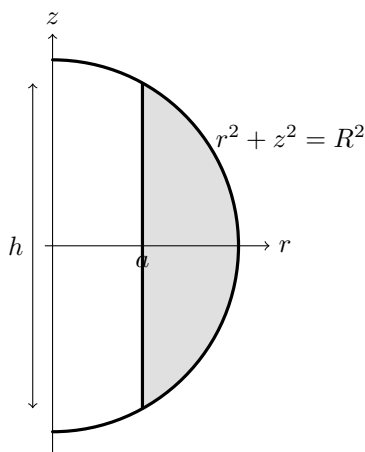
Tasks

1. Set up the volume in cylindrical coordinates.
2. Evaluate it and eliminate R and a in favor of h .
3. Explain the resulting “napkin-ring” invariance.

Question 5 – Solution

Strategy. Use cylindrical shells with $a \leq r \leq R$; the sphere supplies the upper and lower z -bounds.

Step 1: Bounds



The sphere is $r^2 + z^2 \leq R^2$. Hence

$$0 \leq \theta \leq 2\pi, \quad a \leq r \leq R, \quad -\sqrt{R^2 - r^2} \leq z \leq \sqrt{R^2 - r^2}.$$

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_a^R 2\sqrt{R^2 - r^2} r \, dr \, d\theta \\ &= 4\pi \left[-\frac{(R^2 - r^2)^{3/2}}{3} \right]_a^R = \frac{4\pi}{3} (R^2 - a^2)^{3/2}. \end{aligned}$$

Since $R^2 - a^2 = h^2/4$,

$$\boxed{V = \frac{\pi h^3}{6}}.$$

Verification The final formula contains only the remaining height. Thus every drilled sphere with the same height h leaves the same volume, regardless of its original R and hole radius a .

Question 6

Find the volume of the solid torus

$$\left(\sqrt{x^2 + y^2} - 3\right)^2 + z^2 \leq 1$$

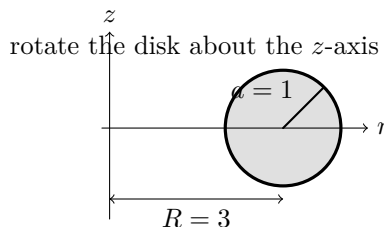
using cylindrical coordinates.

Tasks

1. Describe the bounds in the meridian (r, z) -plane.
2. Evaluate the triple integral, using symmetry where helpful.
3. Give a geometric check of the answer.

Question 6 – Solution

Strategy. In cylindrical coordinates the torus is obtained by rotating a radius-1 disk centered at $(r, z) = (3, 0)$.

Step 1: Bounds

The radial variable ranges from 2 to 4, and

$$-\sqrt{1 - (r - 3)^2} \leq z \leq \sqrt{1 - (r - 3)^2}, \quad 0 \leq \theta \leq 2\pi.$$

Step 2: Evaluate With $u = r - 3$,

$$\begin{aligned} V &= 2\pi \int_2^4 2r \sqrt{1 - (r - 3)^2} dr \\ &= 4\pi \int_{-1}^1 (u + 3) \sqrt{1 - u^2} du. \end{aligned}$$

The u -term is odd, while $\int_{-1}^1 \sqrt{1 - u^2} du = \pi/2$. Therefore

$$V = 4\pi(3) \frac{\pi}{2} = \boxed{6\pi^2}.$$

Verification The generating disk has area π , and its center travels a circle of circumference $2\pi(3) = 6\pi$. Their product is $6\pi^2$, agreeing with the integral.

Question 7

Consider the solids

$$S_1 = \{(x, y, z) : 0 \leq z \leq 1, x^2 + y^2 \leq z^2\}$$

and

$$S_2 = \{(x, y, z) : 0 \leq z \leq 1, 0 \leq x \leq z^2, 0 \leq y \leq \pi\}.$$

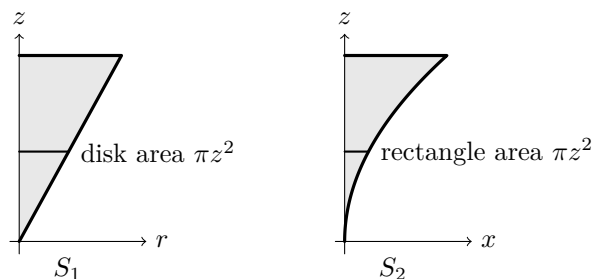
Tasks

1. Find the area of each horizontal cross-section.
2. Use Cavalieri's principle to compare the volumes.
3. Compute their common volume.

Question 7 – Solution

Strategy. Although the slices have different shapes, compare their areas at the same height.

Step 1: Slice areas



At height z , the slice of S_1 is a disk of radius z , so

$$A_1(z) = \pi z^2.$$

The slice of S_2 is a rectangle of side lengths z^2 and π , so

$$A_2(z) = z^2 \pi = \pi z^2.$$

Step 2: Compare and integrate Both solids occupy $0 \leq z \leq 1$ and have equal cross-sectional area at every height. Cavalieri's principle therefore gives equal volumes, and

$$V_1 = V_2 = \int_0^1 \pi z^2 dz = \pi \left[\frac{z^3}{3} \right]_0^1 = \boxed{\frac{\pi}{3}}.$$

Verification The conclusion depends on slice areas, not slice shapes: circular slices in S_1 and rectangular slices in S_2 accumulate the same volume height by height.

Question 8

For $A > 0$, let E_A be the solid bounded below by

$$z = x^2 + y^2$$

and above by the plane $z = A$. If E_A has volume 18π , determine A , the base radius, and the height.

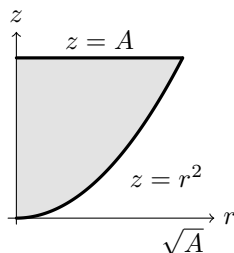
Tasks

1. Derive a formula for $V(A)$.
2. Solve the inverse volume condition.
3. Verify the dimensions and positivity of the result.

Question 8 – Solution

Strategy. First express the volume as a function of the unknown cutting height A , then solve the resulting equation.

Step 1: Geometry and volume



The plane and paraboloid meet when $r^2 = A$, so $0 \leq r \leq \sqrt{A}$. Thus

$$\begin{aligned} V(A) &= \int_0^{2\pi} \int_0^{\sqrt{A}} (A - r^2)r \, dr \, d\theta \\ &= 2\pi \left[\frac{Ar^2}{2} - \frac{r^4}{4} \right]_0^{\sqrt{A}} = \frac{\pi A^2}{2}. \end{aligned}$$

Step 2: Recover the dimensions The condition $V(A) = 18\pi$ yields

$$\frac{\pi A^2}{2} = 18\pi, \quad A^2 = 36.$$

Since $A > 0$,

$$\boxed{A = 6}, \quad \boxed{\text{base radius} = \sqrt{6}}, \quad \boxed{\text{height} = 6}.$$

Verification Substitution gives $V(6) = \pi(36)/2 = 18\pi$. The positive root is required because A is a height, and $r = \sqrt{A}$ follows from the intersection equation.

Question 9

Find the volume of the ellipsoidal shell

$$1 \leq \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \leq 4.$$

Use the change of variables

$$x = 2u, \quad y = 3v, \quad z = 4w.$$

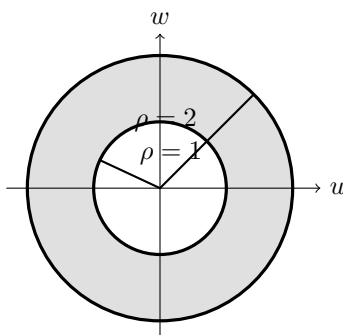
Tasks

1. Describe the transformed shell.
2. Compute the three-dimensional Jacobian and the volume.
3. Check the scaling factor geometrically.

Question 9 – Solution

Strategy. The coordinate scaling turns the ellipsoids into concentric spheres and multiplies every volume by a constant Jacobian.

Step 1: Transform the shell



scaled spherical shell

The inequality becomes

$$1 \leq u^2 + v^2 + w^2 \leq 4,$$

so the transformed region is the spherical shell $1 \leq \rho \leq 2$.

Step 2: Jacobian and volume The derivative matrix is diagonal, so

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = |(2)(3)(4)| = 24.$$

Consequently,

$$V = 24 \left[\frac{4\pi}{3} (2^3 - 1^3) \right] = 24 \left(\frac{28\pi}{3} \right) = \boxed{224\pi}.$$

Verification The map stretches lengths by factors 2, 3, and 4 along the coordinate axes, so it must stretch volume by their product 24, exactly the Jacobian used above.

Question 10

A sphere of radius R has surface-area-to-volume ratio

$$\frac{S}{V} = \frac{3}{2}.$$

Determine its radius, surface area, and volume.

Tasks

1. Derive V with a spherical-coordinate triple integral.
2. Derive S by viewing the two hemispheres as graphs.
3. Use the given ratio and verify its units.

Question 10 – Solution

Strategy. Rebuild both standard sphere formulas from integration, then use their ratio to recover the radius.

Step 1: Volume In spherical coordinates,

$$V = \int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi(2) \frac{R^3}{3} = \frac{4\pi R^3}{3}.$$

Step 2: Surface area For the upper graph $z = \sqrt{R^2 - r^2}$, the area factor is $R/\sqrt{R^2 - r^2}$. Its area is

$$S_{\text{upper}} = \int_0^{2\pi} \int_0^R \frac{Rr}{\sqrt{R^2 - r^2}} \, dr \, d\theta = 2\pi R^2.$$

Doubling for both hemispheres gives $S = 4\pi R^2$.

Step 3: Use the ratio

$$\frac{S}{V} = \frac{4\pi R^2}{4\pi R^3/3} = \frac{3}{R} = \frac{3}{2}, \quad \boxed{R = 2}.$$

Therefore

$$\boxed{S = 16\pi}, \quad \boxed{V = \frac{32\pi}{3}}.$$

Verification Surface area divided by volume has units of inverse length, and $3/R$ has those units. Substituting $R = 2$ reproduces the stated ratio.

Question 1

Consider the planar vector field

$$\mathbf{F}(x, y) = \langle x, -y \rangle.$$

Tasks

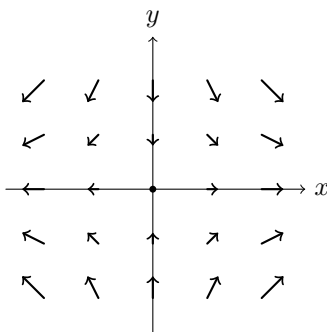
1. Evaluate $\mathbf{F}$ at $(1, 2)$, $(-2, 1)$, and $(0, 0)$.
2. Find the magnitude and direction of $\mathbf{F}(1, 2)$.
3. Describe the field on the coordinate axes and identify every zero vector.

Question 1 – Solution

Strategy. Evaluate the two components pointwise, then use the component signs to understand the field globally.

Step 1: Sample vectors

$$\mathbf{F}(1, 2) = \langle 1, -2 \rangle, \quad \mathbf{F}(-2, 1) = \langle -2, -1 \rangle, \quad \mathbf{F}(0, 0) = \langle 0, 0 \rangle.$$



$$\mathbf{F}(x, y) = \langle x, -y \rangle$$

Step 2: Magnitude and direction At $(1, 2)$,

$$|\mathbf{F}| = \sqrt{1^2 + (-2)^2} = \boxed{\sqrt{5}}, \quad \hat{\mathbf{F}} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle.$$

Step 3: Axis behavior and zeros On the x -axis, $\mathbf{F}(x, 0) = \langle x, 0 \rangle$, so arrows point away from the origin. On the y -axis, $\mathbf{F}(0, y) = \langle 0, -y \rangle$, so arrows point toward the origin. Finally,

$$\mathbf{F}(x, y) = \mathbf{0} \iff x = 0 \text{ and } y = 0.$$

Thus the origin is the field's only zero.

Verification The plotted arrows reverse horizontally when x changes sign and vertically when y changes sign, exactly matching the formula.

Question 2

Determine the domain of the vector field

$$\mathbf{F}(x, y) = \left\langle \ln(x - y), \sqrt{4 - x^2 - y^2} \right\rangle.$$

Tasks

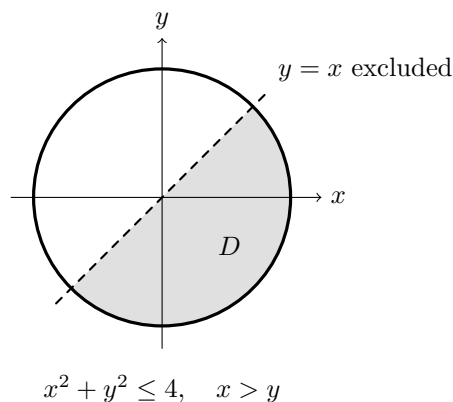
1. Express the domain as simultaneous inequalities.
2. Decide which parts of the line $y = x$ and circle $x^2 + y^2 = 4$ are included.
3. State where the field is continuous and where both components are differentiable.

Question 2 – Solution

Strategy. Intersect the natural domains of the logarithm and square root, keeping track of strict versus non-strict inequalities.

Step 1: Component restrictions The logarithm requires $x - y > 0$, while the square root requires $4 - x^2 - y^2 \geq 0$. Therefore

$$D = \{(x, y) : x > y, x^2 + y^2 \leq 4\}.$$



Step 2: Boundary decisions No point of $y = x$ is included because $\ln 0$ is undefined. The circular boundary is included only where $x > y$; there the second component equals 0 and the first remains defined.

Step 3: Regularity Both component functions are continuous wherever they are defined, so $\mathbf{F}$ is continuous on all of D . Both are differentiable where their defining inequalities are strict:

$$x > y, \quad x^2 + y^2 < 4.$$

At the included circular arc, the square-root component is continuous but its derivatives become singular.

Verification The shaded half-disk excludes its straight diagonal edge but includes the appropriate circular arc, reflecting $>$ versus $\leq$.

Question 3

For

$$\mathbf{F}(x, y) = \frac{\langle x, y \rangle}{x^2 + y^2},$$

analyze the field geometrically.

Tasks

1. State its domain and find its magnitude at a point of radius r .
2. Determine its direction and compare vectors at $(1, 0)$, $(2, 0)$, and $(0, -2)$.
3. Describe what happens as r approaches 0 and as r grows.

Question 3 – Solution

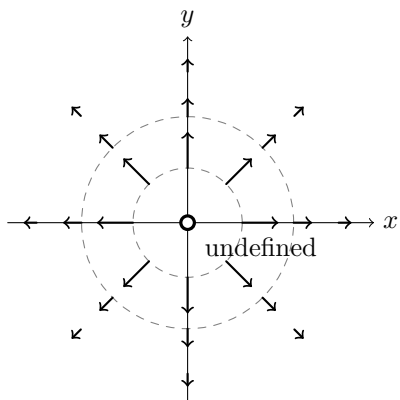
Strategy. Factor the position vector into its length and radial unit vector.

Step 1: Polar form With $r = \sqrt{x^2 + y^2}$ and $\mathbf{e}_r = \langle x/r, y/r \rangle$,

$$\mathbf{F} = \frac{r\mathbf{e}_r}{r^2} = \frac{1}{r}\mathbf{e}_r.$$

Hence the domain is $\mathbb{R}^2 \setminus \{(0, 0)\}$, the direction is radially outward, and

$$|\mathbf{F}| = \frac{1}{r}.$$



outward direction; magnitude $1/r$

Step 2: Samples

$$\mathbf{F}(1, 0) = \langle 1, 0 \rangle, \quad \mathbf{F}(2, 0) = \left\langle \frac{1}{2}, 0 \right\rangle, \quad \mathbf{F}(0, -2) = \left\langle 0, -\frac{1}{2} \right\rangle.$$

Doubling the radius halves the arrow length while preserving radial direction.

Step 3: Limiting behavior As $r \rightarrow 0^+$, $|\mathbf{F}| = 1/r \rightarrow \infty$, explaining the missing origin. As $r \rightarrow \infty$, the magnitude tends to 0.

Verification Multiplying any vector by its radius gives $\mathbf{e}_r$, a unit vector, so both the direction and inverse-radius magnitude are confirmed.

Question 4

A three-dimensional gravitational field is modeled by

$$\mathbf{G}(x, y, z) = -12 \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Evaluate $\mathbf{G}$ at $P = (2, -1, 2)$.
2. Find its magnitude and a unit vector in its direction.
3. Explain the minus sign and state the field's domain.

Question 4 – Solution

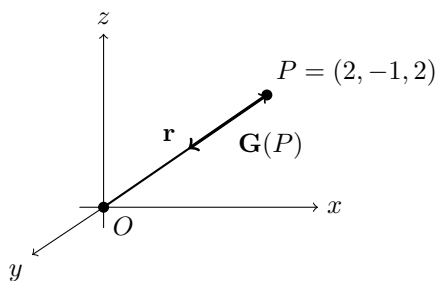
Strategy. Compute the distance from the origin first; it controls both the denominator and the inverse-square magnitude.

Step 1: Evaluate at the point At $P = (2, -1, 2)$,

$$r = \sqrt{2^2 + (-1)^2 + 2^2} = 3, \quad r^3 = 27.$$

Therefore

$$\mathbf{G}(P) = -\frac{12}{27}\langle 2, -1, 2 \rangle = \boxed{\left\langle -\frac{8}{9}, \frac{4}{9}, -\frac{8}{9} \right\rangle}.$$



Step 2: Magnitude and unit direction In general, $|\mathbf{G}| = 12r/r^3 = 12/r^2$. Thus

$$|\mathbf{G}(P)| = \boxed{\frac{4}{3}}, \quad \widehat{\mathbf{G}(P)} = \boxed{\left\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle}.$$

Step 3: Interpretation The negative sign reverses the outward position direction, so gravity points toward the origin. The denominator vanishes only at the origin; hence the domain is

$$\boxed{\mathbb{R}^3 \setminus \{(0, 0, 0)\}}.$$

Verification The displayed unit vector has length 1, and multiplying it by $4/3$ reproduces $\mathbf{G}(P)$.

Question 5

Let

$$\mathbf{F}(x, y) = \langle -y, x \rangle.$$

Tasks

1. Prove that $\mathbf{F}$ is tangent to every circle $x^2 + y^2 = c^2$ centered at the origin.
2. Find its magnitude and orientation.
3. At $P = (3, 4)$, find the vector and the corresponding unit tangent direction.

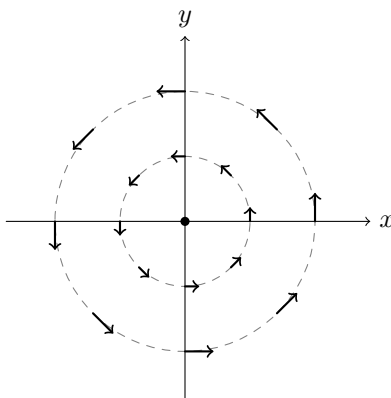
Question 5 – Solution

Strategy. A circle's radial vector is normal to the circle, so orthogonality to that vector proves tangency.

Step 1: Tangency At (x, y) the radial normal is $\mathbf{r} = \langle x, y \rangle$. Since

$$\mathbf{F} \cdot \mathbf{r} = \langle -y, x \rangle \cdot \langle x, y \rangle = -xy + xy = 0,$$

$\mathbf{F}$ is tangent to every centered circle.



vectors are tangent to centered circles

Step 2: Magnitude and orientation

$$|\mathbf{F}| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = r.$$

At $(r, 0)$ the vector is $\langle 0, r \rangle$, which points upward; therefore the orientation is counterclockwise.

Step 3: The point $(3, 4)$

$$\mathbf{F}(3, 4) = \langle -4, 3 \rangle, \quad |\mathbf{F}(3, 4)| = 5,$$

so the counterclockwise unit tangent is

$$\left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle.$$

Verification The dot product $\langle -4, 3 \rangle \cdot \langle 3, 4 \rangle = -12 + 12 = 0$, confirming tangency at P .

Question 6

A temperature distribution is

$$T(x, y) = 100 - x^2 - 4y^2.$$

The heat-flux vector field is $\mathbf{H} = -\frac{1}{2}\nabla T$.

Tasks

1. Find a formula for $\mathbf{H}(x, y)$.
2. Evaluate its magnitude and unit direction at $P = (2, 1)$.
3. Show that the heat flux is normal to the isotherm through P and identify that isotherm.

Question 6 – Solution

Strategy. The gradient is normal to level curves; multiplying it by a negative constant reverses direction but preserves normality.

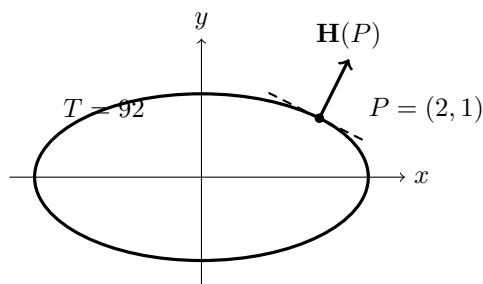
Step 1: Construct the field

$$\nabla T = \langle -2x, -8y \rangle, \quad \mathbf{H} = -\frac{1}{2}\nabla T = \boxed{\langle x, 4y \rangle}.$$

Step 2: Evaluate at P

$$\mathbf{H}(2, 1) = \langle 2, 4 \rangle, \quad |\mathbf{H}(2, 1)| = \boxed{2\sqrt{5}},$$

$$\widehat{\mathbf{H}}(2, 1) = \boxed{\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle}.$$



heat flux is normal to the isotherm

Step 3: Isotherm and normality Since $T(2, 1) = 100 - 4 - 4 = 92$, the isotherm is

$$x^2 + 4y^2 = 8.$$

Its normal direction is $\nabla(x^2 + 4y^2) = \langle 2x, 8y \rangle$. At P this is $\langle 4, 8 \rangle = 2\mathbf{H}(P)$, so the flux is normal to the ellipse and points toward lower temperature.

Verification Moving a short distance along $\mathbf{H}(P)$ increases $x^2 + 4y^2$, so it decreases $T = 100 - (x^2 + 4y^2)$, as heat flow should.

Question 7

A linear planar vector field has the form

$$\mathbf{F}(x, y) = \langle ax + by, cx + dy \rangle.$$

Measurements give

$$\mathbf{F}(1, 0) = \langle 2, -1 \rangle, \quad \mathbf{F}(0, 1) = \langle 3, 4 \rangle.$$

Tasks

1. Determine a, b, c, d and write the field explicitly.
2. Evaluate $\mathbf{F}(-1, 2)$.
3. Find every point where $\mathbf{F} = \mathbf{0}$ and explain uniqueness.

Question 7 – Solution

Strategy. The values on the coordinate basis vectors are exactly the columns of the matrix representing a linear field.

Step 1: Recover the coefficients Substitution at $(1, 0)$ gives $\langle a, c \rangle = \langle 2, -1 \rangle$, while substitution at $(0, 1)$ gives $\langle b, d \rangle = \langle 3, 4 \rangle$. Hence

$$\boxed{\mathbf{F}(x, y) = \langle 2x + 3y, -x + 4y \rangle}.$$

Equivalently,

$$\mathbf{F}(x, y) = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Step 2: Evaluate

$$\mathbf{F}(-1, 2) = \langle -2 + 6, 1 + 8 \rangle = \boxed{\langle 4, 9 \rangle}.$$

Step 3: Find zero vectors Solve

$$2x + 3y = 0, \quad -x + 4y = 0.$$

The second equation gives $x = 4y$; the first then gives $11y = 0$. Thus

$$\boxed{(x, y) = (0, 0)}$$

is the only zero. Equivalently, the coefficient determinant is $2(4) - 3(-1) = 11 \neq 0$, so the linear map has a trivial nullspace.

Verification The recovered formula returns both measured vectors when the two basis points are substituted, so all four coefficients satisfy the data.

Question 8

For a real parameter λ , consider

$$\mathbf{F}_\lambda(x, y) = \langle x - y, \lambda x + y \rangle.$$

Tasks

1. Find all equilibrium points satisfying $\mathbf{F}_\lambda = \mathbf{0}$.
2. Identify the exceptional value of λ and describe its equilibrium set.
3. For that exceptional value, describe the field on either side of the equilibrium set.

Question 8 – Solution

Strategy. Solve the two component equations and separate the parameter value that makes them dependent.

Step 1: Solve generally The first component gives $x = y$. Substituting into the second gives

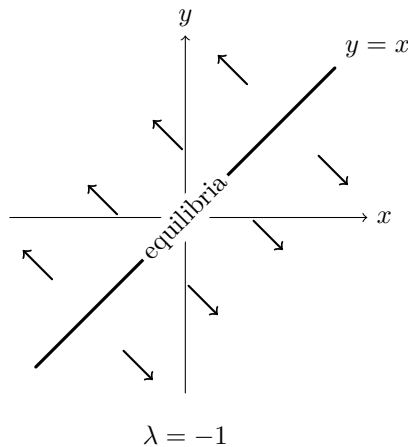
$$\lambda x + y = (\lambda + 1)x = 0.$$

If $\lambda \neq -1$, then $x = 0$ and $y = 0$, so

$$\lambda \neq -1 : \text{the origin is the only equilibrium}.$$

Step 2: Exceptional value If $\lambda = -1$, the second component is $-x + y$, the negative of the first. Every point with $x = y$ is then a zero:

$$\lambda = -1 : \{(x, y) : y = x\} \text{ is the equilibrium set}.$$



Step 3: Behavior off the line For $\lambda = -1$, set $d = x - y$. Then

$$\mathbf{F}_{-1} = \langle d, -d \rangle = d\langle 1, -1 \rangle.$$

This vector is normal to $y = x$. When $x > y$, it points southeast; when $x < y$, it points northwest. Both directions lead away from the equilibrium line.

Verification The coefficient determinant is $1(1) - (-1)\lambda = 1 + \lambda$, which vanishes exactly at $\lambda = -1$, confirming the case split.

Question 9

Let

$$\mathbf{F}(x, y, z) = \langle y, z, x \rangle$$

and let the helix be

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle.$$

Tasks

1. Express the field along the helix as $\mathbf{F}(\mathbf{r}(t))$.
2. Find the unit tangent $\mathbf{T}(t)$.
3. At $t = 0$, decompose $\mathbf{F}(\mathbf{r}(0))$ into components parallel and perpendicular to the helix.

Question 9 – Solution

Strategy. Substitute the curve into the field, then use vector projection onto the unit tangent.

Step 1: Restrict the field to the curve

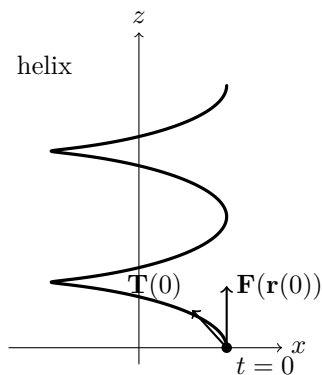
$$\mathbf{F}(\mathbf{r}(t)) = \langle \sin t, t, \cos t \rangle.$$

Also,

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle, \quad |\mathbf{r}'(t)| = \sqrt{2},$$

so

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle.$$



Step 2: Data at $t = 0$

$$\mathbf{F}(\mathbf{r}(0)) = \langle 0, 0, 1 \rangle, \quad \mathbf{T}(0) = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle.$$

The scalar tangential component is

$$\mathbf{F} \cdot \mathbf{T} = \frac{1}{\sqrt{2}}.$$

Step 3: Vector decomposition

$$\mathbf{F}_{\parallel} = (\mathbf{F} \cdot \mathbf{T})\mathbf{T} = \left\langle 0, \frac{1}{2}, \frac{1}{2} \right\rangle,$$

$$\mathbf{F}_{\perp} = \mathbf{F} - \mathbf{F}_{\parallel} = \left\langle 0, -\frac{1}{2}, \frac{1}{2} \right\rangle.$$

Verification The two boxed vectors sum to $\langle 0, 0, 1 \rangle$, and $\mathbf{F}_{\perp} \cdot \mathbf{T}(0) = 0$.

Question 10

On the unit circle $x^2 + y^2 = 1$, consider

$$\mathbf{F}(x, y) = \langle 2x - y, x + 2y \rangle.$$

Tasks

1. Decompose $\mathbf{F}$ into outward radial and counterclockwise tangential components.
2. Find its magnitude and its angle from the outward radial direction.
3. Evaluate the decomposition at $(0, 1)$ and determine whether the field has any zero away from the origin.

Question 10 – Solution

Strategy. Recognize the field as a fixed combination of the radial vector and its 90° counterclockwise rotation.

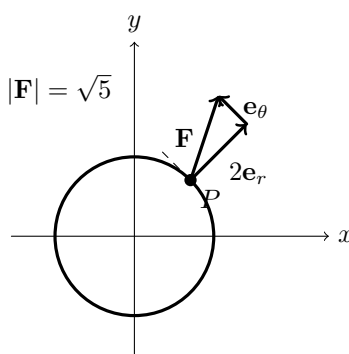
Step 1: Decompose On the unit circle,

$$\mathbf{e}_r = \langle x, y \rangle, \quad \mathbf{e}_\theta = \langle -y, x \rangle.$$

Therefore

$$\mathbf{F} = 2\langle x, y \rangle + \langle -y, x \rangle = \boxed{2\mathbf{e}_r + \mathbf{e}_\theta}.$$

The radial scalar component is 2 and the counterclockwise tangential scalar component is 1.



Step 2: Magnitude and angle Because the two unit vectors are perpendicular,

$$\boxed{|\mathbf{F}| = \sqrt{2^2 + 1^2} = \sqrt{5}}.$$

If α is measured counterclockwise from $\mathbf{e}_r$, then

$$\boxed{\alpha = \tan^{-1}\left(\frac{1}{2}\right)}.$$

Step 3: Point check and zeros At $(0, 1)$,

$$\mathbf{e}_r = \langle 0, 1 \rangle, \quad \mathbf{e}_\theta = \langle -1, 0 \rangle, \quad \mathbf{F} = \boxed{\langle -1, 2 \rangle}.$$

For the full plane, $\mathbf{F} = \mathbf{0}$ gives $2x - y = 0$ and $x + 2y = 0$. Their coefficient determinant is $5 \neq 0$, so only the origin is a zero; in particular, there are none on the unit circle.

Verification The sample vector $\langle -1, 2 \rangle$ has length $\sqrt{5}$ and equals $2\mathbf{e}_r + \mathbf{e}_\theta$, confirming all parts.

Question 1

Evaluate the scalar line integral

$$\int_C (x + y) \, ds,$$

where C is the line segment from $(0, 0)$ to $(3, 4)$.

Tasks

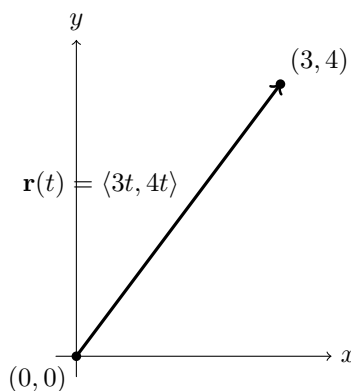
1. Parametrize C on $0 \leq t \leq 1$.
2. Compute ds and evaluate the integral.
3. Verify the constant speed and total length of the segment.

Question 1 – Solution

Strategy. Use linear interpolation between the endpoints, then replace ds by $|\mathbf{r}'(t)| dt$.

Step 1: Parametrize the segment

$$\mathbf{r}(t) = \langle 3t, 4t \rangle, \quad 0 \leq t \leq 1.$$



Along the curve,

$$x + y = 3t + 4t = 7t.$$

Step 2: Arc-length element

$$\mathbf{r}'(t) = \langle 3, 4 \rangle, \quad |\mathbf{r}'(t)| = 5, \quad ds = 5 dt.$$

Therefore

$$\int_C (x + y) ds = \int_0^1 (7t)(5) dt = 35 \left[\frac{t^2}{2} \right]_0^1 = \boxed{\frac{35}{2}}.$$

Verification The speed is constant and the parameter interval has length 1, so the curve length is 5, agreeing with the distance formula $\sqrt{3^2 + 4^2} = 5$. The integrand rises linearly from 0 to 7, whose average is $7/2$; multiplying by length 5 again gives $35/2$.

Question 2

Evaluate

$$\int_C x \, ds,$$

where C is the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Tasks

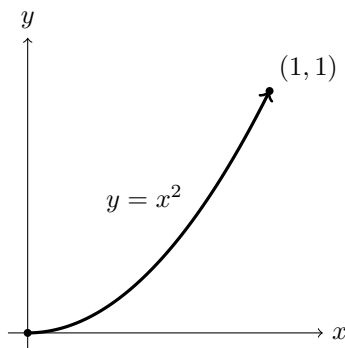
1. Use x itself as a parameter.
2. Derive the arc-length element and evaluate the integral.
3. Check the sign and size of the result.

Question 2 – Solution

Strategy. Parametrize the graph by $\mathbf{r}(x) = \langle x, x^2 \rangle$ so the given scalar function is already the parameter.

Step 1: Parameter and speed

$$\begin{aligned}\mathbf{r}(x) &= \langle x, x^2 \rangle, & 0 \leq x \leq 1, \\ \mathbf{r}'(x) &= \langle 1, 2x \rangle, & ds = \sqrt{1 + 4x^2} \, dx.\end{aligned}$$



Step 2: Evaluate Let $u = 1 + 4x^2$, so $du = 8x \, dx$. Then

$$\begin{aligned}\int_C x \, ds &= \int_0^1 x \sqrt{1 + 4x^2} \, dx \\ &= \frac{1}{8} \int_1^5 u^{1/2} \, du = \frac{1}{12} \left[u^{3/2} \right]_1^5 = \boxed{\frac{5\sqrt{5} - 1}{12}}.\end{aligned}$$

Verification Both x and ds are nonnegative, so the answer must be positive. Also $0 \leq x \leq 1$ implies the integral is no larger than the curve length; numerically the result is about 0.848, consistent with that bound.

Question 3

Let C be the upper semicircle

$$x^2 + y^2 = 4, \quad y \geq 0,$$

oriented from $(2, 0)$ to $(-2, 0)$. Evaluate

$$\int_C (x^2 + y^2) \, ds.$$

Tasks

1. Parametrize the semicircle with its stated orientation.
2. Evaluate the line integral.
3. Give a geometric verification without integration.

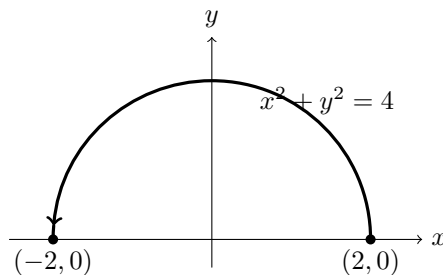
Question 3 – Solution

Strategy. On a circle, the scalar integrand $x^2 + y^2$ is constant, so the integral is that constant times arc length.

Step 1: Parametrization

$$\mathbf{r}(\theta) = \langle 2 \cos \theta, 2 \sin \theta \rangle, \quad 0 \leq \theta \leq \pi.$$

This begins at $(2, 0)$ and ends at $(-2, 0)$, as required.

**Step 2: Evaluate** Along C ,

$$x^2 + y^2 = 4, \quad |\mathbf{r}'(\theta)| = | \langle -2 \sin \theta, 2 \cos \theta \rangle | = 2.$$

Thus

$$\int_C (x^2 + y^2) ds = \int_0^\pi (4)(2) d\theta = \boxed{8\pi}.$$

Verification The upper semicircle has length $\frac{1}{2}(2\pi \cdot 2) = 2\pi$. Since the integrand is constantly 4, the integral must be $4(2\pi) = 8\pi$.

Question 4

Let C be the helix

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle, \quad 0 \leq t \leq \pi.$$

Evaluate

$$\int_C (x^2 + y^2 + z) \, ds.$$

Tasks

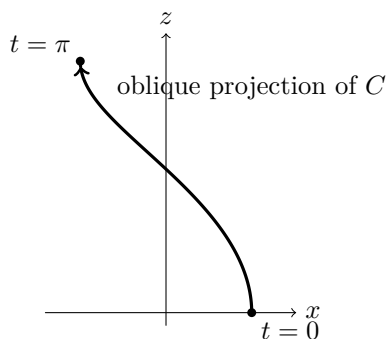
1. Express the scalar function along the helix.
2. Compute the speed and evaluate the integral.
3. Verify the result by separating the constant and variable parts.

Question 4 – Solution

Strategy. Substitute the parametrization into both the scalar function and the arc-length factor.

Step 1: Restrict the integrand Along C ,

$$x^2 + y^2 + z = 4 \cos^2 t + 4 \sin^2 t + t = 4 + t.$$



Step 2: Arc length and integral

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 1 \rangle, \quad |\mathbf{r}'(t)| = \sqrt{4 + 1} = \sqrt{5}.$$

Therefore

$$\begin{aligned} \int_C (x^2 + y^2 + z) ds &= \sqrt{5} \int_0^\pi (4 + t) dt \\ &= \sqrt{5} \left[4t + \frac{t^2}{2} \right]_0^\pi = \boxed{\sqrt{5} \left(4\pi + \frac{\pi^2}{2} \right)}. \end{aligned}$$

Verification The constant part 4 contributes $4L = 4\pi\sqrt{5}$ because $L = \pi\sqrt{5}$. The $z = t$ part has average $\pi/2$ and contributes $(\pi/2)L = \pi^2\sqrt{5}/2$, matching the two terms above.

Question 5

The piecewise smooth curve $C = C_1 \cup C_2$ travels from

$$A = (0, 0) \quad \text{to} \quad B = (1, 0) \quad \text{to} \quad D = (1, 2)$$

along straight segments. Evaluate

$$\int_C (xy + 1) \, ds.$$

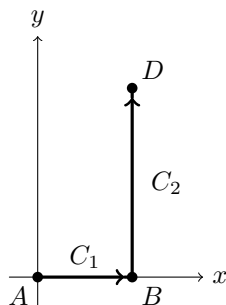
Tasks

1. Parametrize each segment.
2. Evaluate the integral on each piece and add the results.
3. Check the contribution using the integrand along each segment.

Question 5 – Solution

Strategy. A scalar line integral over a piecewise curve is the sum of the integrals over its smooth pieces.

Step 1: First segment



For C_1 , use $\mathbf{r}_1(t) = \langle t, 0 \rangle$, $0 \leq t \leq 1$. Then $ds = dt$ and $xy + 1 = 1$, so

$$\int_{C_1} (xy + 1) ds = \int_0^1 1 dt = 1.$$

Step 2: Second segment For C_2 , use $\mathbf{r}_2(u) = \langle 1, u \rangle$, $0 \leq u \leq 2$. Then $ds = du$ and $xy + 1 = u + 1$, giving

$$\int_{C_2} (xy + 1) ds = \int_0^2 (u + 1) du = \left[\frac{u^2}{2} + u \right]_0^2 = 4.$$

Hence

$$\boxed{\int_C (xy + 1) ds = 1 + 4 = 5}.$$

Verification On C_1 the constant value 1 times length 1 gives 1. On C_2 the integrand rises linearly from 1 to 3, so its average 2 times length 2 gives 4.

Question 6

Let C be the unit circle. Compute

$$\int_C (x + 2) \, ds$$

once with a counterclockwise parametrization and once with a clockwise parametrization.

Tasks

1. Write parametrizations for both orientations.
2. Show directly that both integrals have the same value.
3. Explain why every scalar line integral with respect to ds is orientation-independent.

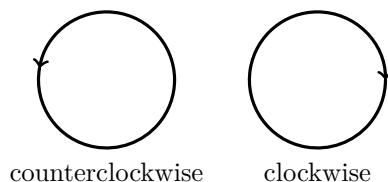
Question 6 – Solution

Strategy. Reversing direction changes the parameter derivative's sign but not its magnitude, so ds is unchanged.

Step 1: Counterclockwise Let $\mathbf{r}_+(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Since $|\mathbf{r}'_+(t)| = 1$,

$$I_+ = \int_0^{2\pi} (\cos t + 2) dt = 0 + 4\pi = 4\pi.$$

same unit circle, opposite orientations



Step 2: Clockwise Let $\mathbf{r}_-(t) = \langle \cos t, -\sin t \rangle$, $0 \leq t \leq 2\pi$. Again the speed is 1, and $x = \cos t$, so

$$I_- = \int_0^{2\pi} (\cos t + 2) dt = \boxed{4\pi}.$$

Step 3: General reason Under a reversed parametrization, $\tilde{\mathbf{r}}(t) = \mathbf{r}(a+b-t)$, the derivative is $-\mathbf{r}'(a+b-t)$, but

$$|\tilde{\mathbf{r}}'(t)| = |\mathbf{r}'(a+b-t)|.$$

After substitution, the scalar values and arc-length factors are identical. Therefore $\int_C f ds$ does not depend on orientation.

Verification Here the x -term cancels by symmetry, while the constant 2 contributes 2 times the circumference 2π , confirming 4π .

Question 7

A wire follows the first-quadrant astroid arc

$$\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad 0 \leq t \leq \frac{\pi}{2},$$

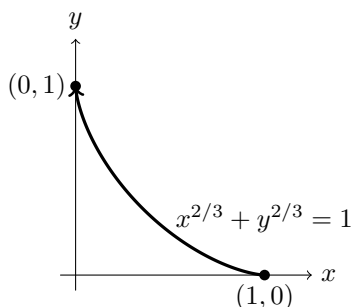
and has linear density $\delta(x, y) = x + y$. Find its mass.

Tasks

1. Derive and simplify the speed on the given interval.
2. Express the density in terms of t and evaluate the mass.
3. Check positivity and endpoint behavior.

Question 7 – Solution

Strategy. Differentiate the astroid parametrization; in the first quadrant the absolute values in the speed simplify.

Step 1: Speed

$$\mathbf{r}'(t) = \langle -3 \cos^2 t \sin t, 3 \sin^2 t \cos t \rangle.$$

Because $\sin t, \cos t \geq 0$,

$$\begin{aligned} |\mathbf{r}'(t)| &= 3 \sin t \cos t \sqrt{\cos^2 t + \sin^2 t} \\ &= 3 \sin t \cos t. \end{aligned}$$

Step 2: Mass Along the wire, $\delta = \cos^3 t + \sin^3 t$. Hence

$$\begin{aligned} M &= \int_0^{\pi/2} 3(\cos^3 t + \sin^3 t) \sin t \cos t \, dt \\ &= 3 \left(\int_0^{\pi/2} \cos^4 t \sin t \, dt + \int_0^{\pi/2} \sin^4 t \cos t \, dt \right) \\ &= 3 \left(\frac{1}{5} + \frac{1}{5} \right) = \boxed{\frac{6}{5}}. \end{aligned}$$

Verification Density and speed are nonnegative. The speed vanishes only at the two cusps, which does not affect the integral; the finite positive mass $6/5$ is therefore consistent.

Question 8

Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$. Find the average value of

$$f(x, y) = x^2$$

along C with respect to arc length.

Tasks

1. Compute $\int_C x^2 ds$ and the length of C .
2. Find the arc-length average value.
3. Identify all points on C where x^2 equals that average.

Question 8 – Solution

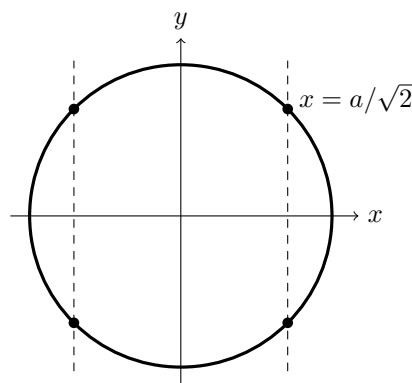
Strategy. Parametrize the circle and divide the scalar line integral by total arc length.

Step 1: Integral and length Use

$$\mathbf{r}(\theta) = \langle a \cos \theta, a \sin \theta \rangle, \quad 0 \leq \theta \leq 2\pi.$$

Then $ds = a d\theta$, so

$$\int_C x^2 ds = a^3 \int_0^{2\pi} \cos^2 \theta d\theta = \pi a^3, \quad L = 2\pi a.$$



$$x^2 + y^2 = a^2 \text{ (scaled sketch)}$$

Step 2: Average value

$$f_{\text{avg}} = \frac{1}{L} \int_C f ds = \frac{\pi a^3}{2\pi a} = \boxed{\frac{a^2}{2}}.$$

Step 3: Points attaining the average We need $x^2 = a^2/2$, so $x = \pm a/\sqrt{2}$. The circle equation then gives $y = \pm a/\sqrt{2}$, independently. Thus the four points are

$$\boxed{\left(\frac{\pm a}{\sqrt{2}}, \frac{\pm a}{\sqrt{2}} \right)},$$

with all four sign combinations.

Verification Symmetry gives equal averages for x^2 and y^2 . Their sum is constantly a^2 on C , so each average must be $a^2/2$.

Question 9

For $a > 0$, a straight wire runs from $(0, 0)$ to (a, a) and has density

$$\delta(x, y) = x + y.$$

Its mass is $8\sqrt{2}$. Determine a and the wire's length.

Tasks

1. Parametrize the wire and derive its mass as a function of a .
2. Solve the inverse condition, respecting $a > 0$.
3. Compute the resulting length and verify the mass.

Question 9 – Solution

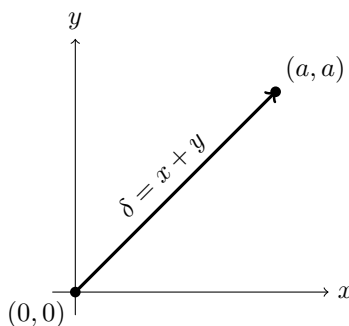
Strategy. Express the line integral entirely in terms of the unknown endpoint coordinate a , then solve the resulting quadratic equation.

Step 1: Mass formula Use

$$\mathbf{r}(t) = \langle at, at \rangle, \quad 0 \leq t \leq 1.$$

Then $\delta = 2at$ and $ds = |\langle a, a \rangle| dt = a\sqrt{2} dt$. Therefore

$$M(a) = \int_0^1 (2at)(a\sqrt{2}) dt = 2a^2\sqrt{2} \left[\frac{t^2}{2} \right]_0^1 = a^2\sqrt{2}.$$



Step 2: Recover a The given mass implies

$$a^2\sqrt{2} = 8\sqrt{2}, \quad a^2 = 8.$$

Because $a > 0$,

$$\boxed{a = 2\sqrt{2}}.$$

Step 3: Length and verification The diagonal length is

$$L = \sqrt{a^2 + a^2} = a\sqrt{2} = (2\sqrt{2})(\sqrt{2}) = \boxed{4}.$$

Substitution into the mass formula gives $M = (2\sqrt{2})^2\sqrt{2} = 8\sqrt{2}$, exactly the specified mass.

Question 10

A wire occupies the upper semicircle

$$x^2 + y^2 = R^2, \quad y \geq 0,$$

and has density $\delta(x, y) = 1 + y/R$, where $R > 0$.

Tasks

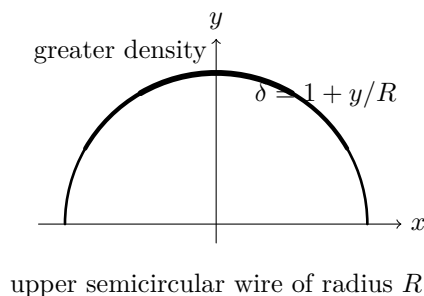
1. Find the wire's mass.
2. Compute its moments about the x - and y -axes.
3. Determine its center of mass and check the result by symmetry and bounds.

Question 10 – Solution

Strategy. Parametrize by polar angle; symmetry immediately predicts that the center of mass lies on the y -axis.

Step 1: Mass Let $x = R \cos \theta$, $y = R \sin \theta$, $0 \leq \theta \leq \pi$, so $ds = R d\theta$ and $\delta = 1 + \sin \theta$. Then

$$M = R \int_0^\pi (1 + \sin \theta) d\theta = \boxed{R(\pi + 2)}.$$



Step 2: Moments The moment about the x -axis is

$$\begin{aligned} M_x &= \int_C y \delta ds = R^2 \int_0^\pi \sin \theta (1 + \sin \theta) d\theta \\ &= R^2 \left(2 + \frac{\pi}{2} \right). \end{aligned}$$

For the y -axis,

$$M_y = R^2 \int_0^\pi \cos \theta (1 + \sin \theta) d\theta = 0.$$

Step 3: Center of mass

$$\bar{x} = \frac{M_y}{M} = 0, \quad \bar{y} = \frac{M_x}{M} = \boxed{\frac{R(\pi + 4)}{2(\pi + 2)}}.$$

Thus

$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{R(\pi + 4)}{2(\pi + 2)} \right)}.$$

Verification Left-right symmetry forces $\bar{x} = 0$. Also $0 < (\pi + 4)/[2(\pi + 2)] < 1$, so $0 < \bar{y} < R$, placing the center of mass inside the semicircle's vertical range.

Question 1

Let C be the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$. Evaluate

$$\int_C y \, dx.$$

Tasks

1. Parametrize the curve using x as the parameter.
2. Replace dx correctly and evaluate the integral.
3. Explain why no arc-length factor appears.

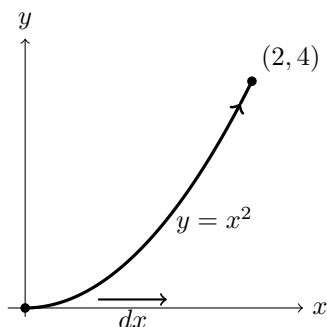
Question 1 – Solution

Strategy. For an integral with respect to x , use $dx = x'(t) dt$ rather than the arc-length element.

Step 1: Parametrize

$$\mathbf{r}(t) = \langle t, t^2 \rangle, \quad 0 \leq t \leq 2.$$

Thus $x = t$, $y = t^2$, and $dx = dt$.



Step 2: Evaluate

$$\int_C y \, dx = \int_0^2 t^2 \, dt = \left[\frac{t^3}{3} \right]_0^2 = \boxed{\frac{8}{3}}.$$

Step 3: Distinguish from ds The differential is $dx = x'(t) dt = dt$. The speed $\sqrt{1 + 4t^2}$ would be used only for an integral with respect to ds ; inserting it here would change the problem.

Verification Since x increases from 0 to 2 and $0 \leq y \leq 4$, the integral must lie between 0 and $4(2) = 8$. The value $8/3$ satisfies this bound.

Question 2

On the same parabola $C : y = x^2$ from $(0, 0)$ to $(2, 4)$, evaluate

$$\int_C x \, dy.$$

Tasks

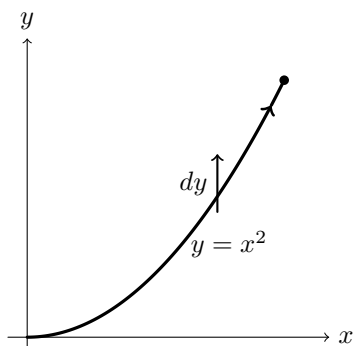
1. Express dy in terms of dx .
2. Evaluate the integral.
3. Compare the answer with $\int_C y \, dx = 8/3$ and explain why they differ.

Question 2 – Solution

Strategy. Use the graph relation to write $dy = y'(x) dx$.

Step 1: Convert the differential Along $y = x^2$,

$$dy = 2x dx, \quad 0 \leq x \leq 2.$$



Step 2: Evaluate

$$\int_C x dy = \int_0^2 x(2x) dx = 2 \left[\frac{x^3}{3} \right]_0^2 = \boxed{\frac{16}{3}}.$$

Step 3: Compare Although both integrals use the same curve, they measure change in different coordinates:

$$y dx = x^2 dx, \quad x dy = x(2x dx) = 2x^2 dx.$$

The second integrand is exactly twice the first, so its value is twice $8/3$.

Verification Here y increases from 0 to 4 and $0 \leq x \leq 2$, so the value must lie between 0 and $2(4) = 8$. The result $16/3$ does.

Question 3

Let C be the straight segment from $A = (0, 0)$ to $B = (2, 1)$. Evaluate

$$\int_C (x + y) dx + (x - y) dy.$$

Tasks

1. Parametrize the segment with the stated orientation.
2. Convert both differentials to the parameter.
3. Evaluate and check the endpoint orientation.

Question 3 – Solution

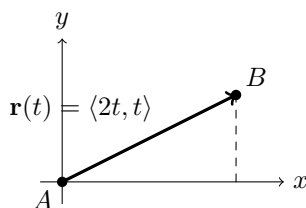
Strategy. A combined differential-form integral becomes one ordinary integral after substituting $dx = x'(t)dt$ and $dy = y'(t)dt$.

Step 1: Parameter data

$$\mathbf{r}(t) = \langle 2t, t \rangle, \quad 0 \leq t \leq 1,$$

so

$$x + y = 3t, \quad x - y = t, \quad dx = 2 dt, \quad dy = dt.$$



Step 2: Evaluate

$$\begin{aligned} \int_C (x + y) dx + (x - y) dy &= \int_0^1 [(3t)(2) + (t)(1)] dt \\ &= \int_0^1 7t dt = \boxed{\frac{7}{2}}. \end{aligned}$$

Verification At $t = 0$ the parametrization gives A , and at $t = 1$ it gives B , so the orientation is correct. Directly grouping the transformed integrand gives $[2(x + y) + (x - y)]dt = (3x + y)dt = 7t dt$, confirming the algebra.

Question 4

Let C_1 be the upper unit semicircle oriented from $(1, 0)$ to $(-1, 0)$, and let C_2 be the same arc with the reverse orientation. Compute

$$\int_C y \, dx$$

for both orientations.

Tasks

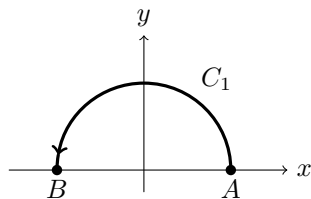
1. Parametrize C_1 and evaluate the integral.
2. Obtain the value on C_2 .
3. Explain why orientation matters for dx but not for ds .

Question 4 – Solution

Strategy. Compute once in the counterclockwise direction, then use the sign change caused by reversing dx .

Step 1: Forward orientation For C_1 , use

$$x = \cos \theta, \quad y = \sin \theta, \quad 0 \leq \theta \leq \pi, \quad dx = -\sin \theta \, d\theta.$$



reversal changes the sign

Therefore

$$\int_{C_1} y \, dx = - \int_0^\pi \sin^2 \theta \, d\theta = \boxed{-\frac{\pi}{2}}.$$

Step 2: Reverse orientation Reversing the same curve negates the coordinate differential, so

$$\boxed{\int_{C_2} y \, dx = \frac{\pi}{2}}.$$

Step 3: Why the rules differ Under reversal, $dx = x'(t)dt$ changes sign because the derivative reverses. By contrast, $ds = |\mathbf{r}'(t)|dt$ uses a magnitude and remains nonnegative.

Verification On C_1 , $y \geq 0$ while x decreases, so $dx \leq 0$ and the integral must be nonpositive. Its reverse must be nonnegative, matching the two answers.

Question 5

Let C be the boundary of the rectangle

$$0 \leq x \leq 3, \quad 0 \leq y \leq 2,$$

traversed counterclockwise. Evaluate

$$\oint_C y \, dx$$

by splitting the boundary into four segments.

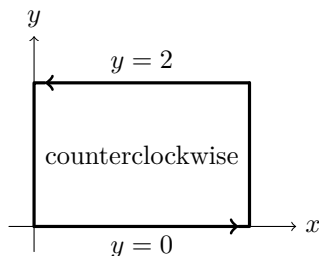
Tasks

1. Determine which edges contribute zero.
2. Evaluate the contribution from each horizontal edge.
3. Add the pieces and check the sign from the orientation.

Question 5 – Solution

Strategy. On vertical edges $dx = 0$, while y is constant on each horizontal edge.

Step 1: Traverse the edges



Along the bottom edge, $y = 0$, so its contribution is 0. On the right and left edges, x is constant, so $dx = 0$ and both contributions are also 0.

Step 2: Top edge Counterclockwise travel moves along the top from $(3, 2)$ to $(0, 2)$. Thus

$$\int_{\text{top}} y \, dx = \int_3^0 2 \, dx = 2[x]_3^0 = -6.$$

Step 3: Sum

$$\oint_C y \, dx = 0 + 0 - 6 + 0 = -6.$$

Verification The only nonzero edge has $y > 0$ and moves left, so $dx < 0$; a negative answer is necessary. Reversing the rectangle would change the answer to $+6$.

Question 6

Let C be the graph $y = x^3$ from $(0, 0)$ to $(1, 1)$. Evaluate

$$\int_C x^2 dy.$$

Tasks

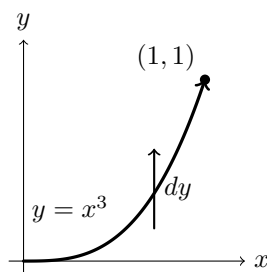
1. Convert dy using the graph equation.
2. Evaluate the resulting single-variable integral.
3. Verify the result by using y as the parameter instead.

Question 6 – Solution

Strategy. First calculate with x as parameter, then reparametrize by y to confirm invariance under a valid change of parameter.

Step 1: Use x

$$y = x^3, \quad dy = 3x^2 dx, \quad 0 \leq x \leq 1.$$



Therefore

$$\int_C x^2 dy = 3 \int_0^1 x^4 dx = 3 \left[\frac{x^5}{5} \right]_0^1 = \boxed{\frac{3}{5}}.$$

Step 2: Use y instead On this interval, $x = y^{1/3}$, so $x^2 = y^{2/3}$. Since the differential is already dy ,

$$\int_C x^2 dy = \int_0^1 y^{2/3} dy = \left[\frac{3}{5} y^{5/3} \right]_0^1 = \frac{3}{5}.$$

Verification Both parametrizations preserve the motion from $(0,0)$ to $(1,1)$ and give the same value, confirming the differential conversion.

Question 7

Let C be the vertical segment $x = 2$ from $(2, 0)$ to $(2, 3)$. Evaluate both

$$\int_C (x + y) \, dx \quad \text{and} \quad \int_C (x + y) \, dy.$$

Tasks

1. Parametrize the vertical segment.
2. Evaluate both integrals.
3. Explain why treating y as a function of x is inappropriate here.

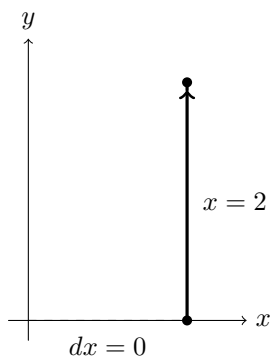
Question 7 – Solution

Strategy. Parametrize by the changing coordinate y ; the constant coordinate has zero differential.

Step 1: Parametrize

$$\mathbf{r}(t) = \langle 2, t \rangle, \quad 0 \leq t \leq 3.$$

Thus $dx = 0$, $dy = dt$, and $x + y = 2 + t$.

**Step 2: Evaluate**

$$\int_C (x + y) dx = 0$$

because $dx = 0$ everywhere. Meanwhile,

$$\int_C (x + y) dy = \int_0^3 (2 + t) dt = \left[2t + \frac{t^2}{2} \right]_0^3 = \frac{21}{2}.$$

Step 3: Representation issue A vertical segment assigns many y -values to the single value $x = 2$, so it cannot be written as a single-valued graph $y = g(x)$. Parametrization avoids this limitation.

Verification The second integrand rises linearly from 2 to 5; its average $7/2$ times the y -interval length 3 gives $21/2$.

Question 8

Let

$$C : \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Evaluate

$$\int_C z \, dx + x \, dy + y \, dz.$$

Tasks

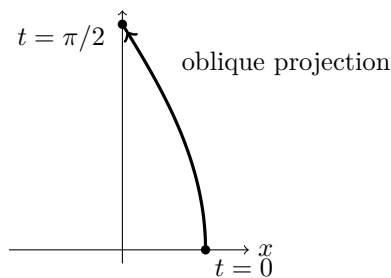
1. Compute dx , dy , and dz .
2. Convert the line integral to one parameter and evaluate it.
3. Verify the cancellation among the three contributions.

Question 8 – Solution

Strategy. Treat each coordinate differential separately, then combine all three ordinary integrals.

Step 1: Differential data

$$\begin{aligned}x &= \cos t, & y &= \sin t, & z &= t, \\dx &= -\sin t \, dt, & dy &= \cos t \, dt, & dz &= dt.\end{aligned}$$



Step 2: Evaluate

$$\int_C z \, dx + x \, dy + y \, dz = \int_0^{\pi/2} (-t \sin t + \cos^2 t + \sin t) \, dt.$$

The three contributions are

$$\int_0^{\pi/2} -t \sin t \, dt = -1, \quad \int_0^{\pi/2} \cos^2 t \, dt = \frac{\pi}{4}, \quad \int_0^{\pi/2} \sin t \, dt = 1.$$

Hence

$$\boxed{\int_C z \, dx + x \, dy + y \, dz = \frac{\pi}{4}}.$$

Verification The first and third contributions cancel exactly, leaving only $\int_0^{\pi/2} \cos^2 t \, dt = \pi/4$.

Question 9

On the parabola $C : y = x^2$ from $(0, 0)$ to $(1, 1)$, a constant a satisfies

$$\int_C ax \, dx + y \, dy = 3.$$

Determine a .

Tasks

1. Convert the entire integral to the parameter x .
2. Solve the resulting equation for a .
3. Substitute the result to verify the specified value.

Question 9 – Solution

Strategy. Use $dy = 2x \, dx$ to make the unknown coefficient appear in a simple algebraic equation.

Step 1: Convert Since $y = x^2$ and $dy = 2x \, dx$,

$$\int_C ax \, dx + y \, dy = \int_0^1 (ax + x^2(2x)) \, dx.$$

Therefore

$$\int_0^1 (ax + 2x^3) \, dx = \frac{a}{2} + \frac{1}{2}.$$

Step 2: Solve

$$\frac{a}{2} + \frac{1}{2} = 3, \quad \frac{a}{2} = \frac{5}{2}, \quad \boxed{a = 5}.$$

Step 3: Verify With $a = 5$,

$$\int_0^1 (5x + 2x^3) \, dx = \left[\frac{5x^2}{2} + \frac{x^4}{2} \right]_0^1 = \frac{5}{2} + \frac{1}{2} = 3.$$

Verification The computed value satisfies the original integral exactly. Because the coefficient of a is $\int_0^1 x \, dx = 1/2 \neq 0$, the solution is unique.

Question 10

Two curves connect $A = (0, 0)$ to $B = (1, 1)$:

$$C_1 : y = x, \quad C_2 : y = x^2, \quad 0 \leq x \leq 1.$$

Compare

$$\int_C y \, dx$$

along the two paths.

Tasks

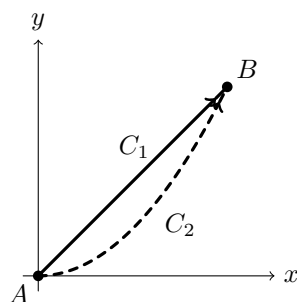
1. Evaluate the integral on C_1 .
2. Evaluate it on C_2 .
3. Explain what the unequal values demonstrate about path dependence.

Question 10 – Solution

Strategy. Use the shared coordinate parameter x on both paths, then compare the scalar functions traced between the same endpoints.

Step 1: Straight path On C_1 , $y = x$, so

$$\int_{C_1} y \, dx = \int_0^1 x \, dx = \boxed{\frac{1}{2}}.$$



Step 2: Parabolic path On C_2 , $y = x^2$, so

$$\int_{C_2} y \, dx = \int_0^1 x^2 \, dx = \boxed{\frac{1}{3}}.$$

Step 3: Interpretation Both curves have the same initial point, terminal point, and orientation, yet

$$\frac{1}{2} \neq \frac{1}{3}.$$

Therefore the value of $\int_C y \, dx$ is not determined by endpoints alone; it depends on the path between them.

Verification For $0 < x < 1$, the line $y = x$ lies above the parabola $y = x^2$. With $dx > 0$, its integral must be larger, consistent with $1/2 > 1/3$.

Question 1

Let

$$\mathbf{F}(x, y) = \langle y, x^2 \rangle, \quad \mathbf{r}(t) = \langle t, t^2 \rangle, \quad 0 \leq t \leq 1.$$

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

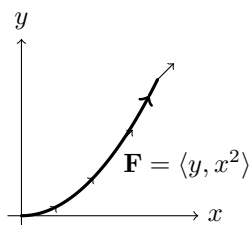
1. Evaluate $\mathbf{F}$ along the curve and compute $\mathbf{r}'(t)$.
2. Form the dot product and evaluate the integral.
3. State how the sign of the integrand agrees with the geometry.

Question 1 – Solution

Strategy. Use $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$.

Step 1: Restrict the field to the curve Since $x = t$ and $y = t^2$,

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^2, t^2 \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t \rangle.$$



Step 2: Form the scalar integrand

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t^2 + 2t^3.$$

Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^3) dt = \left[\frac{t^3}{3} + \frac{t^4}{2} \right]_0^1 = \boxed{\frac{5}{6}}.$$

Verification For $0 \leq t \leq 1$, both components of the field and both components of the velocity are nonnegative. Their dot product must be nonnegative, consistent with the positive answer.

Question 2

A particle follows the helix

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \leq t \leq 2\pi,$$

through the constant force field $\mathbf{F} = \langle 2, -1, 3 \rangle$.

Tasks

1. Compute the work done by the force.
2. Identify the contributions from the horizontal and vertical motion.
3. Check the result using the net displacement.

Question 2 – Solution

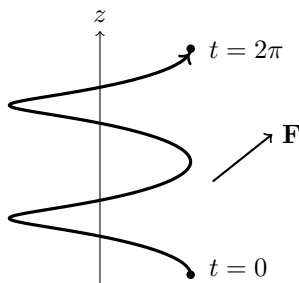
Strategy. Dot the constant force with the velocity, then use the displacement as an independent check.

Step 1: Velocity and dot product

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle,$$

so

$$\mathbf{F} \cdot \mathbf{r}'(t) = -2 \sin t - \cos t + 3.$$

**Step 2: Work**

$$\begin{aligned} W &= \int_0^{2\pi} (-2 \sin t - \cos t + 3) dt \\ &= 0 + 0 + 6\pi = \boxed{6\pi}. \end{aligned}$$

The horizontal coordinates complete a full turn, so their signed contributions cancel. The vertical rise is 2π , producing work $3(2\pi)$.

Verification The net displacement is $\mathbf{r}(2\pi) - \mathbf{r}(0) = \langle 0, 0, 2\pi \rangle$. For a constant force, its dot product with this displacement is $\langle 2, -1, 3 \rangle \cdot \langle 0, 0, 2\pi \rangle = 6\pi$.

Question 3

Let C travel from $A = (0, 0)$ to $B = (2, 0)$ and then from B to $C = (2, 1)$. For

$$\mathbf{F}(x, y) = \langle -y, x \rangle,$$

evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

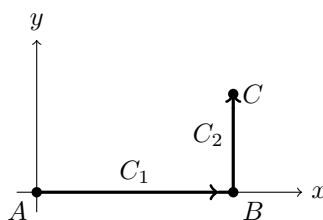
1. Parametrize each segment with the stated orientation.
2. Compute the contribution from each segment.
3. Explain geometrically why the first contribution vanishes.

Question 3 – Solution

Strategy. Split the piecewise path and add the two oriented line integrals.

Step 1: Horizontal segment Use $\mathbf{r}_1(t) = \langle t, 0 \rangle$, $0 \leq t \leq 2$. Then

$$\mathbf{F}(\mathbf{r}_1(t)) = \langle 0, t \rangle, \quad \mathbf{r}'_1(t) = \langle 1, 0 \rangle, \quad \mathbf{F} \cdot \mathbf{r}'_1 = 0.$$



The field is vertical along this segment while the motion is horizontal, so the vectors are perpendicular.

Step 2: Vertical segment Use $\mathbf{r}_2(s) = \langle 2, s \rangle$, $0 \leq s \leq 1$. Thus

$$\mathbf{F}(\mathbf{r}_2(s)) = \langle -s, 2 \rangle, \quad \mathbf{r}'_2(s) = \langle 0, 1 \rangle,$$

and

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2 \, ds = 2.$$

Step 3: Add Hence

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 2 = 2}.$$

Verification Only the upward component on the vertical segment contributes, and that component is the constant 2 over a distance of 1.

Question 4

Let C be the unit circle and let

$$\mathbf{F}(x, y) = \langle -y, x \rangle.$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ when C is traversed (a) counterclockwise and (b) clockwise.

Tasks

1. Parametrize the counterclockwise circle and evaluate the integral.
2. Use orientation reversal to obtain the clockwise value.
3. Interpret the signs using the direction of the field.

Question 4 – Solution

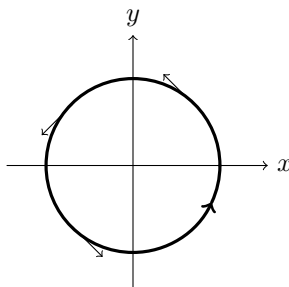
Strategy. On the unit circle this field is the counterclockwise unit tangent, making the dot product especially simple.

Step 1: Counterclockwise traversal Take

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq 2\pi.$$

Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle = \mathbf{r}'(t).$$



field and traversal are counterclockwise

Consequently,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{r}'(t) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} 1 dt = \boxed{2\pi}.$$

Step 2: Reverse the orientation Reversing a vector-field line integral changes its sign, so the clockwise value is

$$\boxed{-2\pi}.$$

Verification The field assists counterclockwise motion at every point and opposes clockwise motion at every point, so the respective signs must be positive and negative.

Question 5

Let

$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle, \quad \mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \quad 0 \leq t \leq 1.$$

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Evaluate the field along the space curve.
2. Compute and integrate the dot product.
3. Verify the calculation by expanding $z \, dx + x \, dy + y \, dz$.

Question 5 – Solution

Strategy. Substitute all three coordinates before taking the dot product with the three-dimensional velocity.

Step 1: Field and velocity

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^3, t, t^2 \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle.$$

Step 2: Dot and integrate

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= t^3 + 2t^2 + 3t^4, \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \left[\frac{t^4}{4} + \frac{2t^3}{3} + \frac{3t^5}{5} \right]_0^1 \\ &= \frac{1}{4} + \frac{2}{3} + \frac{3}{5} = \boxed{\frac{91}{60}}. \end{aligned}$$

Step 3: Differential-form check Along the curve,

$$z \, dx = t^3 \, dt, \quad x \, dy = t(2t \, dt) = 2t^2 \, dt, \quad y \, dz = t^2(3t^2 \, dt) = 3t^4 \, dt.$$

These are exactly the three terms above.

Verification Every term is nonnegative on $[0, 1]$, and the common-denominator sum $(15 + 40 + 36)/60 = 91/60$ confirms the arithmetic.

Question 6

For a constant a , let

$$\mathbf{F}_a(x, y) = \langle ay, x \rangle.$$

Along the parabola $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$, the work done is 2. Determine a .

Tasks

1. Express the work in terms of a .
2. Solve for a .
3. Substitute the result and verify the specified work.

Question 6 – Solution

Strategy. Restrict the one-parameter family of fields to the given curve; the work condition becomes a linear equation in a .

Step 1: Build the integrand

$$\mathbf{F}_a(\mathbf{r}(t)) = \langle at^2, t \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t \rangle.$$

Thus

$$\mathbf{F}_a(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = at^2 + 2t^2 = (a + 2)t^2.$$

Step 2: Impose the work condition

$$2 = \int_0^1 (a + 2)t^2 dt = \frac{a + 2}{3}.$$

Therefore $a + 2 = 6$, and

$$\boxed{a = 4}.$$

Step 3: Verify With $a = 4$,

$$\int_0^1 (4 + 2)t^2 dt = 6 \left[\frac{t^3}{3} \right]_0^1 = 2,$$

which is the required work.

Uniqueness check The coefficient of a in the work is $\int_0^1 t^2 dt = 1/3 \neq 0$, so no second value of a can satisfy the condition.

Question 7

The field $\mathbf{F}(x, y) = \langle -y, x \rangle$ acts between $A = (0, 0)$ and $B = (1, 1)$. Compare the work along

$$C_1 : \mathbf{r}(t) = \langle t, t \rangle, \quad 0 \leq t \leq 1,$$

and the broken path C_2 that goes from $(0, 0)$ to $(1, 0)$ to $(1, 1)$.

Tasks

1. Compute the work along C_1 .
2. Compute the work along both pieces of C_2 .
3. Explain what the comparison demonstrates.

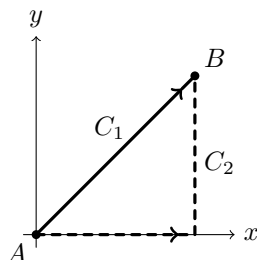
Question 7 – Solution

Strategy. Evaluate the same vector-field line integral on two explicitly parametrized paths with common endpoints.

Step 1: Diagonal path On C_1 ,

$$\mathbf{F}(\mathbf{r}(t)) = \langle -t, t \rangle, \quad \mathbf{r}'(t) = \langle 1, 1 \rangle.$$

The dot product is $-t + t = 0$, so $W_1 = \boxed{0}$.



Step 2: Broken path On the horizontal piece $\langle t, 0 \rangle$, the dot product $\langle 0, t \rangle \cdot \langle 1, 0 \rangle$ is zero. On the vertical piece $\langle 1, s \rangle$,

$$\langle -s, 1 \rangle \cdot \langle 0, 1 \rangle = 1, \quad 0 \leq s \leq 1.$$

Thus

$$W_2 = 0 + \int_0^1 1 \, ds = \boxed{1}.$$

Step 3: Compare Since $W_1 \neq W_2$ despite identical endpoints, the work for this field depends on the path taken.

Verification Along the diagonal the field is perpendicular to the motion. Along the final vertical piece of C_2 , its upward component is 1, explaining the extra unit of work.

Question 8

Let C be the counterclockwise boundary of the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$. For

$$\mathbf{F}(x, y) = \langle y^2, x \rangle,$$

compute the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly from the four edges.

Tasks

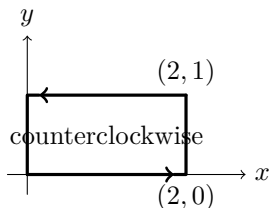
1. Compute each oriented edge contribution.
2. Add the four contributions.
3. Check which signs are forced by the traversal direction.

Question 8 – Solution

Strategy. Write $\mathbf{F} \cdot d\mathbf{r} = y^2 dx + x dy$ and exploit the constant coordinate on each edge.

Step 1: Bottom and right Along the bottom, $y = 0$ and $dy = 0$, so the contribution is 0. Along the right edge, $x = 2$ and $y : 0 \rightarrow 1$, giving

$$\int_0^1 2 dy = 2.$$



Step 2: Top and left Along the top, $y = 1$ and counterclockwise travel has $x : 2 \rightarrow 0$:

$$\int_2^0 1 dx = -2.$$

On the left, $x = 0$, and $dx = 0$, so the contribution is 0.

Step 3: Sum

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 + 2 - 2 + 0 = 0.$$

Verification The right-edge contribution is positive because both the field's vertical component and dy are positive. The top contribution is negative because $y^2 > 0$ while $dx < 0$; their equal magnitudes cancel.

Question 9

Let C be the line segment from $(0,0)$ to $(1,2)$ in the field

$$\mathbf{F}(x,y) = \langle x-y, x+y \rangle.$$

Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ using both

$$\mathbf{r}_1(t) = \langle t, 2t \rangle, \ 0 \leq t \leq 1, \quad \text{and} \quad \mathbf{r}_2(u) = \langle u^2, 2u^2 \rangle, \ 0 \leq u \leq 1.$$

Tasks

1. Evaluate the integral with each parametrization.
2. Explain why the zero velocity of $\mathbf{r}_2$ at $u = 0$ causes no problem.
3. State the invariance illustrated by the equal answers.

Question 9 – Solution

Strategy. Carry the parameter derivative through both computations; the slower start must be included in dr .

Step 1: First parametrization Here

$$\mathbf{F}(\mathbf{r}_1(t)) = \langle -t, 3t \rangle, \quad \mathbf{r}'_1(t) = \langle 1, 2 \rangle.$$

Thus

$$\int_0^1 (-t + 6t) dt = \int_0^1 5t dt = \boxed{\frac{5}{2}}.$$

Step 2: Nonuniform parametrization Now

$$\mathbf{F}(\mathbf{r}_2(u)) = \langle -u^2, 3u^2 \rangle, \quad \mathbf{r}'_2(u) = \langle 2u, 4u \rangle.$$

Therefore

$$\int_0^1 (-2u^3 + 12u^3) du = \int_0^1 10u^3 du = \boxed{\frac{5}{2}}.$$

Step 3: Interpret Although $\mathbf{r}'_2(0) = \mathbf{0}$, the curve pauses only at one endpoint and then traces the segment once with the same orientation. The derivative factor compensates for its varying speed.

Verification Both maps begin at $(0, 0)$, end at $(1, 2)$, and move monotonically along the same segment, so parameter independence requires the matching values obtained above.

Question 10

On the counterclockwise circle $C : x^2 + y^2 = 9$, consider

$$\mathbf{F}(x, y) = \langle x - 2y, 2x + y \rangle.$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by separating $\mathbf{F}$ into radial and tangential parts.

Tasks

1. Decompose the field into radial and tangential vectors.
2. Determine the work contributed by each part.
3. Verify the answer by direct parametrization.

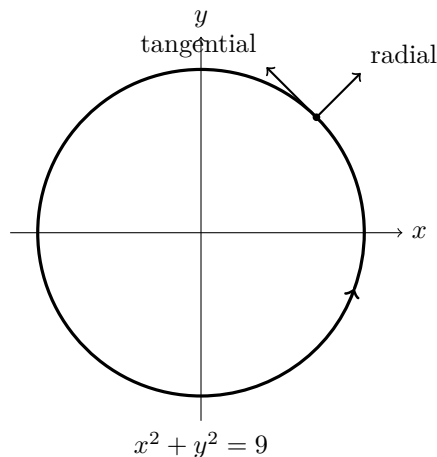
Question 10 – Solution

Strategy. Write the field as $\langle x, y \rangle + 2\langle -y, x \rangle$; on a circle the first part is normal and the second is tangent.

Step 1: Decompose

$$\mathbf{F} = \underbrace{\langle x, y \rangle}_{\text{radial}} + \underbrace{2\langle -y, x \rangle}_{\text{counterclockwise tangential}}.$$

The radial part is perpendicular to the circle's tangent, so it contributes zero work.



On the radius-3 circle, $\langle -y, x \rangle$ has magnitude 3 in the positive tangent direction. Hence the doubled part has tangential magnitude 6. Multiplying by the circumference 6π gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 + 6(6\pi) = \boxed{36\pi}.$$

Step 2: Direct verification Set $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t \rangle$, $0 \leq t \leq 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 18,$$

because the radial terms cancel and the tangential terms sum to $18(\sin^2 t + \cos^2 t)$. Therefore

$$\int_0^{2\pi} 18 \, dt = 36\pi,$$

confirming both the magnitude and the positive orientation.

Question 1

Let

$$f(x, y, z) = x^2y + e^z,$$

and let C be any smooth curve from $A = (1, 0, 0)$ to $B = (2, 1, \ln 2)$. Evaluate

$$\int_C \nabla f \cdot d\mathbf{r}.$$

Tasks

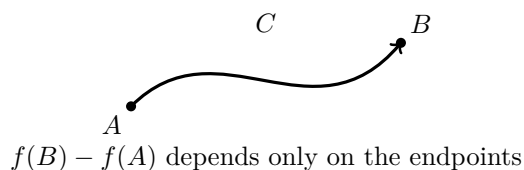
1. State the Fundamental Theorem for Line Integrals.
2. Evaluate the integral without parametrizing C .
3. Explain why the unspecified shape of C is irrelevant.

Question 1 – Solution

Strategy. Since the field is explicitly ∇f , apply the endpoint formula $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$.

Step 1: Evaluate the endpoint values

$$f(B) = 2^2(1) + e^{\ln 2} = 4 + 2 = 6, \quad f(A) = 1^2(0) + e^0 = 1.$$



Step 2: Apply the theorem

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 6 - 1 = \boxed{5}.$$

Step 3: Explain path independence Along any smooth parametrization $\mathbf{r}(t)$, $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ is the derivative of $f(\mathbf{r}(t))$. Its integral therefore records only the net change in f .

Verification The logarithmic coordinate was chosen so that $e^{\ln 2} = 2$; using $\ln 2$ itself in $f(B)$ would be an endpoint-substitution error.

Question 2

Let $f(x, y) = x^2 + y^2$. A smooth curve C travels from $A = (3, 4)$ to $B = (0, 5)$.

Tasks

1. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$.
2. Interpret the result using level curves of f .
3. Decide whether a different smooth path from A to B could change the value.

Question 2 – Solution

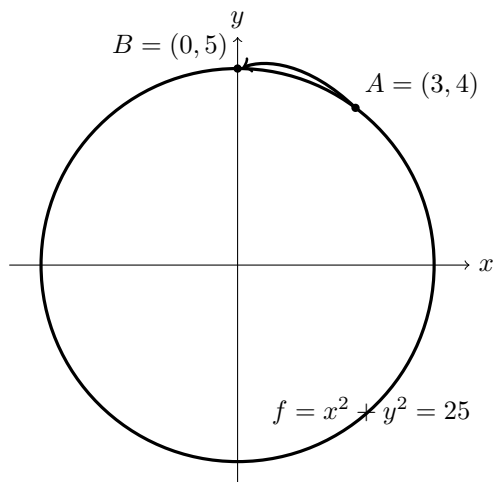
Strategy. Compare the potential values at the endpoints before doing any curve calculation.

Step 1: Endpoint values

$$f(A) = 3^2 + 4^2 = 25, \quad f(B) = 0^2 + 5^2 = 25.$$

Hence

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 0.}$$



Step 2: Interpret Both endpoints lie on the level circle $x^2 + y^2 = 25$. The curve between them need not stay on that circle: f may rise and fall along the way, but its total change is still zero.

Step 3: Compare paths Any other smooth path with the same orientation from A to B gives the same endpoint difference, so it cannot change the integral.

Verification The result is not claiming that ∇f vanishes. Indeed $\nabla f = \langle 2x, 2y \rangle$ is nonzero at both endpoints; only the net change in f is zero.

Question 3

Let

$$f(x, y, z) = xy + z^2,$$

and let C be any piecewise smooth closed curve contained in the domain of f . Evaluate

$$\oint_C \nabla f \cdot d\mathbf{r}.$$

Tasks

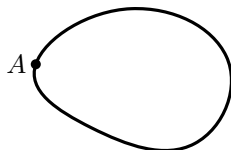
1. Apply the endpoint form of the theorem to the closed curve.
2. Explain the role of the curve's initial and terminal points.
3. State whether self-intersections affect the result.

Question 3 – Solution

Strategy. Regard a closed curve as an oriented path whose terminal point equals its initial point.

Step 1: Name the common endpoint Let A be the point at which the parametrization starts and ends. The theorem gives

$$\oint_C \nabla f \cdot d\mathbf{r} = f(A) - f(A) = \boxed{0}.$$



initial point = terminal point

Step 2: Interpret The integral measures the accumulated change of f during one traversal. Returning to the starting point returns to the same value of f , so all increases and decreases cancel.

Step 3: Self-intersections Self-intersections do not alter the endpoint calculation. Each piece contributes the change in f across that piece, and the intermediate values telescope when the pieces are added.

Verification Reversing the closed curve would negate its integral, but the negative of zero is still zero, consistent with the answer.

Question 4

Let $f(x, y) = xe^y$. A curve C is oriented from $A = (0, 0)$ to $B = (2, \ln 3)$, and $-C$ denotes the same curve with the reverse orientation.

Tasks

1. Compute $\int_C \nabla f \cdot d\mathbf{r}$.
2. Compute $\int_{-C} \nabla f \cdot d\mathbf{r}$.
3. Explain the sign change using both endpoints and orientation reversal.

Question 4 – Solution

Strategy. Evaluate the same endpoint difference in the two opposite orders.

Step 1: Forward orientation

$$f(A) = 0e^0 = 0, \quad f(B) = 2e^{\ln 3} = 6.$$

Thus

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = \boxed{6}.$$

Step 2: Reverse orientation The initial point of $-C$ is B and its terminal point is A , so

$$\int_{-C} \nabla f \cdot d\mathbf{r} = f(A) - f(B) = \boxed{-6}.$$

Step 3: Connect the rules Swapping the endpoints reverses the subtraction. This is exactly the general orientation rule

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Verification Adding the forward and reverse traversals makes a closed trip, and $6 + (-6) = 0$, as the theorem requires for a gradient field.

Question 5

Let $f(x, y) = x^2y$ and let C be the quarter unit circle

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ in two ways.

Tasks

1. Use the Fundamental Theorem for Line Integrals.
2. Evaluate directly from $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$.
3. Identify the chain-rule antiderivative in the direct calculation.

Question 5 – Solution

Strategy. First use the endpoints, then verify that the parameter integrand is the derivative of $f(\mathbf{r}(t))$.

Step 1: Endpoint method The curve begins at $(1, 0)$ and ends at $(0, 1)$. Both endpoint values are zero:

$$f(1, 0) = 0, \quad f(0, 1) = 0.$$

Therefore the theorem gives $\boxed{0}$.

Step 2: Direct method Since $\nabla f = \langle 2xy, x^2 \rangle$,

$$\begin{aligned}\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \langle 2 \cos t \sin t, \cos^2 t \rangle \cdot \langle -\sin t, \cos t \rangle \\ &= -2 \cos t \sin^2 t + \cos^3 t.\end{aligned}$$

This is

$$\frac{d}{dt}(\cos^2 t \sin t) = \frac{d}{dt}f(\mathbf{r}(t)).$$

Hence

$$\int_0^{\pi/2} \frac{d}{dt}(\cos^2 t \sin t) dt = [\cos^2 t \sin t]_0^{\pi/2} = \boxed{0}.$$

Verification The two methods agree, and the displayed antiderivative explicitly exhibits the chain rule behind the theorem.

Question 6

Let $f(x, y) = x^2 + y^2$. A curve begins at $A = (1, 2)$ and ends at a first-quadrant point B on the line $y = x + 1$. Suppose

$$\int_C \nabla f \cdot d\mathbf{r} = 20.$$

Determine B .

Tasks

1. Convert the work condition into a level-curve condition on B .
2. Solve the line-circle intersection equation.
3. Use the quadrant restriction to select and verify the endpoint.

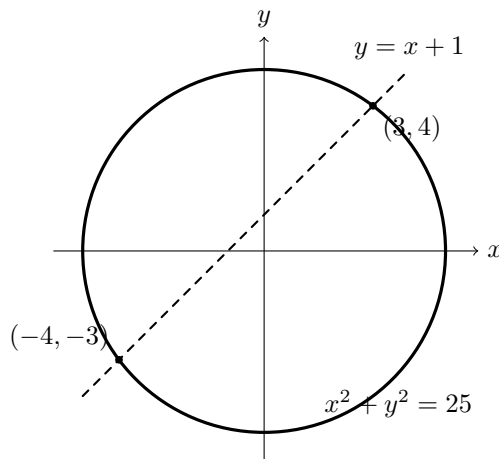
Question 6 – Solution

Strategy. Use the theorem in reverse: the known integral determines the terminal potential value.

Step 1: Determine the level Since $f(A) = 1^2 + 2^2 = 5$,

$$20 = f(B) - 5, \quad f(B) = 25.$$

Thus B lies on $x^2 + y^2 = 25$ as well as $y = x + 1$.



Step 2: Solve the intersection Substitute $y = x + 1$:

$$x^2 + (x + 1)^2 = 25 \implies 2x^2 + 2x - 24 = 0 \implies (x - 3)(x + 4) = 0.$$

The candidates are $(3, 4)$ and $(-4, -3)$.

Step 3: Apply the restriction Only $(3, 4)$ lies in the first quadrant, so

$$\boxed{B = (3, 4)}.$$

Verification The line condition gives $4 = 3 + 1$, and $f(B) - f(A) = 25 - 5 = 20$, exactly the specified integral.

Question 7

Let

$$f(x, y) = x^3 - 3xy + y^2.$$

A piecewise smooth path C travels from $A = (0, 0)$ to $(1, 0)$, then to $(1, 1)$, and finally to $B = (2, 1)$.

Tasks

1. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ using the theorem.
2. Show how the endpoint changes telescope across the three pieces.
3. Explain why the corners do not invalidate the theorem.

Question 7 – Solution

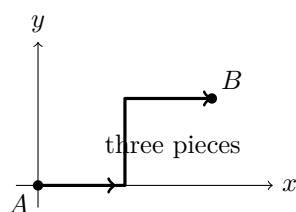
Strategy. Apply the theorem to the whole piecewise smooth path, then expose the same cancellation piece by piece.

Step 1: Whole-path calculation

$$f(A) = 0, \quad f(B) = 2^3 - 3(2)(1) + 1^2 = 8 - 6 + 1 = 3.$$

Therefore

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = 3}.$$



Step 2: Telescope Let $P = (1, 0)$ and $Q = (1, 1)$. Then

$$\begin{aligned} \int_C \nabla f \cdot d\mathbf{r} &= [f(P) - f(A)] + [f(Q) - f(P)] + [f(B) - f(Q)] \\ &= f(B) - f(A) = 3. \end{aligned}$$

Every intermediate potential value appears once positively and once negatively.

Step 3: Corners Smoothness is required on each segment, not at the finitely many joining corners. Adding the valid integral on each piece yields the integral on the piecewise smooth path.

Verification Direct values $f(P) = 1$ and $f(Q) = -1$ give contributions 1, -2 , 4, whose sum is 3.

Question 8

Let

$$f(x, y, z) = xyz + \frac{1}{2}(x^2 + y^2 + z^2).$$

Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ for any smooth space curve from $A = (1, -1, 0)$ to $B = (2, 1, 3)$.

Tasks

1. Compute $f(A)$ and $f(B)$ carefully.
2. Apply the Fundamental Theorem for Line Integrals.
3. Verify that the associated vector field is three-dimensional by computing ∇f .

Question 8 – Solution

Strategy. The spatial geometry of C is unnecessary because the integrand is the gradient of the supplied scalar function.

Step 1: Initial value

$$f(1, -1, 0) = (1)(-1)(0) + \frac{1}{2}(1 + 1 + 0) = 1.$$

Step 2: Terminal value

$$f(2, 1, 3) = (2)(1)(3) + \frac{1}{2}(4 + 1 + 9) = 6 + 7 = 13.$$

Thus

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 13 - 1 = 12}.$$

Step 3: Display the field Differentiating componentwise gives

$$\nabla f = \langle yz + x, xz + y, xy + z \rangle.$$

This confirms that the line integral uses all three coordinate components, even though no parametrization is needed.

Verification The quadratic contribution changes from 1 to 7, a gain of 6, and the product contribution changes from 0 to 6, another gain of 6. The total change is 12.

Question 9

A force is derived from the potential energy

$$U(x, y) = x^2 + 2y^2$$

by $\mathbf{F} = -\nabla U$. A particle moves along any smooth curve from $A = (2, 1)$ to $B = (0, 3)$.

Tasks

1. Compute the work $W = \int_C \mathbf{F} \cdot d\mathbf{r}$.
2. Relate the work to the change in potential energy.
3. Explain the sign of the answer.

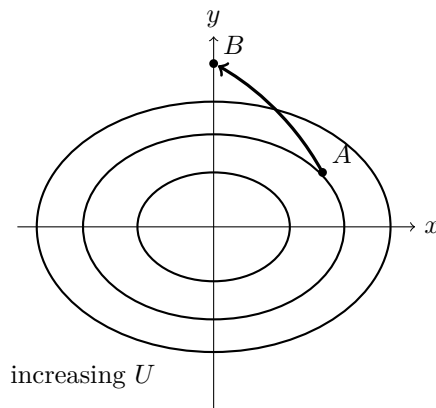
Question 9 – Solution

Strategy. Apply the theorem to $-U$: work by $-\nabla U$ is the negative of the change in U .

Step 1: Potential-energy change

$$U(A) = 2^2 + 2(1^2) = 6, \quad U(B) = 0^2 + 2(3^2) = 18.$$

Therefore $\Delta U = U(B) - U(A) = 12$.



Step 2: Work Since $\mathbf{F} = \nabla(-U)$,

$$\begin{aligned} W &= (-U)(B) - (-U)(A) \\ &= U(A) - U(B) = 6 - 18 = \boxed{-12}. \end{aligned}$$

Equivalently, $W = -\Delta U = -12$.

Step 3: Interpret the sign The particle ends at a point with greater potential energy. The force derived from $-\nabla U$ therefore does negative net work during the motion.

Verification The identity $W + \Delta U = -12 + 12 = 0$ checks the sign. Reversing the trip would produce work $+12$.

Question 10

Let f have continuous first partial derivatives on an open region containing a smooth curve $\mathbf{r}(t)$, $a \leq t \leq b$. Prove

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Tasks

1. Rewrite the line integral using the parametrization.
2. Apply the multivariable chain rule.
3. Finish with the one-variable Fundamental Theorem of Calculus and identify the hypotheses used.

Question 10 – Solution

Strategy. Show that the vector-field integrand is exactly the ordinary derivative of the composite function $f \circ \mathbf{r}$.

Step 1: Parameter form Write $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. By definition,

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Expanding the dot product gives

$$\int_a^b (f_x x'(t) + f_y y'(t) + f_z z'(t)) dt,$$

where the partial derivatives are evaluated at $\mathbf{r}(t)$.

Step 2: Apply the chain rule The multivariable chain rule states

$$\frac{d}{dt} f(\mathbf{r}(t)) = f_x x'(t) + f_y y'(t) + f_z z'(t).$$

Thus the line integral is $\int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt$.

Step 3: Apply the one-variable theorem

$$\int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt = \boxed{f(\mathbf{r}(b)) - f(\mathbf{r}(a))}.$$

The smoothness of $\mathbf{r}$ and continuity of the first partial derivatives make the chain-rule derivative continuous and integrable. The same proof applies piece by piece to a piecewise smooth curve, with intermediate endpoint values canceling.

Verification The right side reverses sign when a and b are interchanged, matching orientation reversal of the line integral.

Question 1

Determine whether

$$\mathbf{F}(x, y) = \langle 2xy + 3, x^2 + 4y \rangle$$

is conservative on $\mathbb{R}^2$. If it is, find a potential function f .

Tasks

1. Compare the relevant mixed partial derivatives.
2. Construct f and determine the one-variable correction term.
3. Differentiate the result to verify the field.

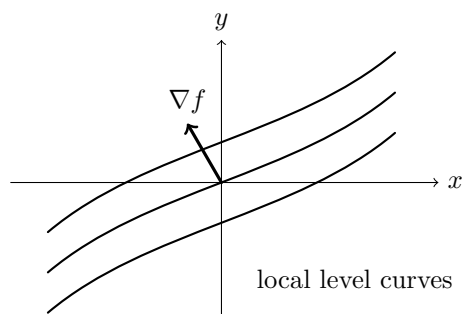
Question 1 – Solution

Strategy. Test $P_y = Q_x$ on the simply connected domain, then integrate one component and recover the missing function.

Step 1: Test the field With $P = 2xy + 3$ and $Q = x^2 + 4y$,

$$P_y = 2x, \quad Q_x = 2x.$$

The components have continuous first partial derivatives on all of $\mathbb{R}^2$, which is simply connected. Thus the equality guarantees that $\mathbf{F}$ is conservative.



Step 2: Construct a potential Integrating $f_x = P$ with respect to x gives

$$f = x^2y + 3x + g(y).$$

Then $f_y = x^2 + g'(y) = Q = x^2 + 4y$, so $g'(y) = 4y$ and $g(y) = 2y^2 + C$.

$$\boxed{f(x, y) = x^2y + 3x + 2y^2 + C}.$$

Verification Differentiation gives $\nabla f = \langle 2xy + 3, x^2 + 4y \rangle = \mathbf{F}$. The arbitrary constant has zero gradient.

Question 2

Consider

$$\mathbf{F}(x, y) = \langle y^2, 2xy + x \rangle$$

on $\mathbb{R}^2$.

Tasks

1. Test whether $\mathbf{F}$ is conservative.
2. Identify the exact obstruction.
3. Explain why no potential function can exist even locally near any point.

Question 2 – Solution

Strategy. A continuously differentiable gradient field must satisfy equality of the cross partials.

Step 1: Compute the cross partials Let $P = y^2$ and $Q = 2xy + x$. Then

$$P_y = 2y, \quad Q_x = 2y + 1.$$

Since $P_y \neq Q_x$ at every point, the necessary condition fails.

$\mathbf{F}$ is not conservative on $\mathbb{R}^2$.

Step 2: Identify the contradiction If $\mathbf{F} = \nabla f$ for a twice continuously differentiable f , then

$$f_x = P, \quad f_y = Q,$$

and hence $f_{xy} = P_y$ and $f_{yx} = Q_x$. Equality of mixed partials would require $2y = 2y + 1$, which is impossible.

Step 3: Local conclusion The mismatch is the constant 1 everywhere, not merely at an isolated point. Every open neighborhood contains points where the necessary equality fails, so no local potential exists on any such neighborhood.

Verification Integrating P gives $f = xy^2 + g(y)$, whose y -derivative is $2xy + g'(y)$. Matching Q would require $g'(y) = x$, impossible because g' cannot depend on x .

Question 3

For a real constant a , let

$$\mathbf{F}_a(x, y) = \langle ay + 2xy, 3x + x^2 \rangle.$$

Determine the value of a for which $\mathbf{F}_a$ is conservative on $\mathbb{R}^2$, and find a potential for that value.

Tasks

1. Use the mixed-partial condition to determine a .
2. Construct a potential function.
3. Verify both components and uniqueness of the parameter.

Question 3 – Solution

Strategy. Make the cross-partial identity hold for every point, not just at a selected point.

Step 1: Determine the parameter Write $P = ay + 2xy$ and $Q = 3x + x^2$. Then

$$P_y = a + 2x, \quad Q_x = 3 + 2x.$$

Equality for all x requires

$$\boxed{a = 3}.$$

No other constant can make the two affine expressions identical.

Step 2: Construct the potential For $a = 3$,

$$f_x = 3y + 2xy.$$

Integrating with respect to x gives

$$f = 3xy + x^2y + g(y).$$

Now $f_y = 3x + x^2 + g'(y)$ must equal $Q = 3x + x^2$, so $g'(y) = 0$.

$$\boxed{f(x, y) = 3xy + x^2y + C}.$$

Verification The gradient is $\langle 3y + 2xy, 3x + x^2 \rangle$, exactly $\mathbf{F}_3$. The coefficient comparison also proves the value $a = 3$ is unique.

Question 4

On the punctured plane $D = \mathbb{R}^2 \setminus \{(0, 0)\}$, consider

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

Investigate whether $\mathbf{F}$ is conservative on D .

Tasks

1. Verify that $P_y = Q_x$ on D .
2. Evaluate the line integral around the counterclockwise unit circle.
3. Reconcile the two results using the geometry of the domain.

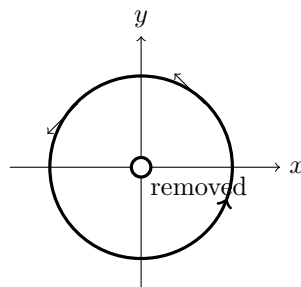
Question 4 – Solution

Strategy. The cross-partial test needs a suitable domain; use a closed-loop integral to detect the hole.

Step 1: Cross partials For $(x, y) \neq (0, 0)$,

$$P_y = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad Q_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Thus the local equality holds throughout D .



$\mathbb{R}^2 \setminus \{(0, 0)\}$ has a hole

Step 2: Test a closed circle Set $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle = \mathbf{r}'(t),$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 \, dt = \boxed{2\pi}.$$

A conservative field must have zero integral around every closed curve. Therefore

$$\boxed{\mathbf{F} \text{ is not conservative on } D}.$$

Step 3: Reconcile The punctured plane is not simply connected: the unit circle encloses the missing origin and cannot contract to a point within D . Consequently $P_y = Q_x$ is not sufficient here.

Question 5

Restrict the field

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

to the right half-plane $H = \{(x, y) : x > 0\}$.

Tasks

1. Find a potential function on H .
2. Verify its gradient.
3. Evaluate the line integral from $A = (1, 1)$ to $B = (\sqrt{3}, 1)$ along any path in H .

Question 5 – Solution

Strategy. The right half-plane permits a single-valued angle function with no branch crossing.

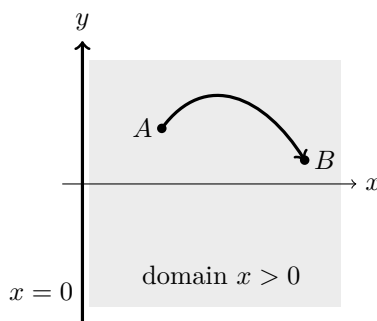
Step 1: Choose the angle potential On $x > 0$, define

$$f(x, y) = \arctan\left(\frac{y}{x}\right).$$

Using the chain rule,

$$f_x = \frac{-y}{x^2 + y^2}, \quad f_y = \frac{x}{x^2 + y^2}.$$

Hence $\nabla f = \mathbf{F}$ on H .



Step 2: Evaluate by endpoints The endpoint angles in the branch $(-\pi/2, \pi/2)$ are

$$f(A) = \arctan(1) = \frac{\pi}{4}, \quad f(B) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Therefore

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) = -\frac{\pi}{12}}.$$

Verification Both endpoints and the entire allowed path lie in $x > 0$, where the selected arctangent branch is continuous. The negative sign matches the decrease in polar angle from $\pi/4$ to $\pi/6$.

Question 6

Determine whether the three-dimensional field

$$\mathbf{F}(x, y, z) = \langle 2xy + z, x^2 + 2y, x + 2z \rangle$$

is conservative on $\mathbb{R}^3$. If so, find a potential and use it to evaluate the integral from $(0, 0, 0)$ to $(1, 2, 3)$.

Tasks

1. Check the three pairwise cross-partial relations.
2. Construct and verify a potential.
3. Evaluate the requested line integral by endpoints.

Question 6 – Solution

Strategy. Verify compatibility of all three components, then integrate successively while retaining correction functions.

Step 1: Compatibility Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$P_y = 2x = Q_x, \quad P_z = 1 = R_x, \quad Q_z = 0 = R_y.$$

On the simply connected domain $\mathbb{R}^3$, these identities show the field is conservative.

Step 2: Construct f Integrating $f_x = P$ gives

$$f = x^2y + xz + g(y, z).$$

From $f_y = x^2 + g_y = Q$, we get $g_y = 2y$, so $g = y^2 + h(z)$. Finally,

$$f_z = x + h'(z) = R = x + 2z,$$

so $h = z^2 + C$. Thus

$$\boxed{f = x^2y + xz + y^2 + z^2 + C}.$$

Step 3: Endpoint evaluation

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 3) - f(0, 0, 0) = 2 + 3 + 4 + 9 = \boxed{18}.$$

Verification Differentiating the boxed potential returns all three original components; the additive constant cancels from the endpoint difference.

Question 7

Suppose f and g are potential functions for the same continuous vector field $\mathbf{F}$ on a connected open region D .

Tasks

1. Prove that $f - g$ is constant on D .
2. Explain why connectedness is needed.
3. State the resulting uniqueness principle for potential functions.

Question 7 – Solution

Strategy. Subtract the two gradient equations and use paths inside the connected open region.

Step 1: Subtract gradients Since $\nabla f = \mathbf{F} = \nabla g$,

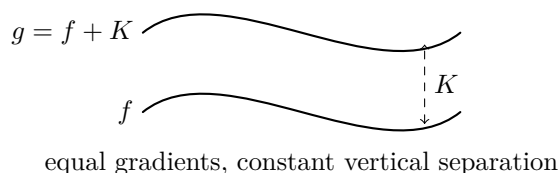
$$\nabla(f - g) = \mathbf{0}.$$

Let $A, B \in D$. An open connected region in $\mathbb{R}^n$ is path connected, so choose a piecewise smooth curve C in D from A to B . Then

$$(f - g)(B) - (f - g)(A) = \int_C \nabla(f - g) \cdot d\mathbf{r} = 0.$$

Thus $(f - g)(B) = (f - g)(A)$ for every pair, and

$$\boxed{f - g = K \text{ on } D}.$$



Step 2: Why connectedness matters On separate components, the same zero gradient permits a different constant on each component because no path in D joins them.

Step 3: Uniqueness principle A potential on a connected region is unique up to one additive constant. Adding a constant changes function values but leaves every partial derivative unchanged.

Verification If $g = f + K$, then $\nabla g = \nabla f + \nabla K = \nabla f$, confirming that every additive constant really does produce another potential.

Question 8

Let $\mathbf{F}$ be continuous on a region where the indicated curves lie. Prove that the following statements are equivalent:

1. line integrals of $\mathbf{F}$ are path independent;
2. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C .

Tasks

1. Prove path independence implies the closed-loop condition.
2. Prove the converse using two paths with common endpoints.
3. Track orientation carefully in the concatenated curve.

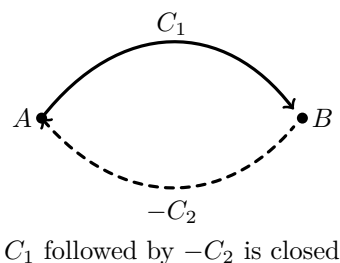
Question 8 – Solution

Strategy. A closed curve can be compared with a constant path, while two endpoint-sharing paths form a closed curve when one is reversed.

Step 1: Path independence $\Rightarrow$ zero loops Let a closed curve start and end at A . Compare it with the constant path at A , whose line integral is zero. Path independence gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Step 2: Zero loops $\Rightarrow$ path independence Let C_1 and C_2 both run from A to B . Follow C_1 from A to B , then follow $-C_2$ back to A . This concatenation is closed.



By the assumed closed-loop condition,

$$0 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Therefore

$$\boxed{\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}},$$

which is path independence.

Verification The reversal is essential: following C_2 in its original direction would begin at A , not at the terminal point B of C_1 , and would not close the loop.

Question 9

A field on $\mathbb{R}^2$ has the form

$$\mathbf{F}(x, y) = \langle 2xy + y, Q(x, y) \rangle.$$

It is conservative and satisfies $Q(0, y) = e^y$.

Tasks

1. Determine the most general Q allowed by the mixed-partial condition.
2. Use the boundary data to determine Q uniquely.
3. Find and verify a potential function.

Question 9 – Solution

Strategy. Integrate the compatibility equation for the unknown component, then use the supplied trace data to fix its arbitrary function.

Step 1: Compatibility equation With $P = 2xy + y$,

$$P_y = 2x + 1.$$

Conservativeness requires $Q_x = P_y$, so integrating with respect to x gives

$$Q(x, y) = x^2 + x + h(y).$$

Step 2: Apply the boundary data At $x = 0$,

$$Q(0, y) = h(y) = e^y.$$

Thus

$$\boxed{Q(x, y) = x^2 + x + e^y}.$$

Step 3: Find a potential Integrating $f_x = P$ yields

$$f = x^2y + xy + g(y).$$

Then

$$f_y = x^2 + x + g'(y) = Q$$

requires $g'(y) = e^y$, so

$$\boxed{f(x, y) = x^2y + xy + e^y + C}.$$

Verification Its gradient is $\langle 2xy + y, x^2 + x + e^y \rangle$, and setting $x = 0$ in the second component gives exactly e^y .

Question 10

Classify the linear vector fields

$$\mathbf{F}(x, y) = \langle ax + by, cx + dy \rangle,$$

where a, b, c, d are real constants.

Tasks

1. Find the necessary and sufficient condition for $\mathbf{F}$ to be conservative on $\mathbb{R}^2$.
2. Construct the general potential when the condition holds.
3. Under that condition, compute the work from $(0, 0)$ to (p, q) .

Question 10 – Solution

Strategy. Compare the constant cross partials, then integrate symbolically without assigning numerical coefficients.

Step 1: Classify Let $P = ax + by$ and $Q = cx + dy$. Then

$$P_y = b, \quad Q_x = c.$$

Because $\mathbb{R}^2$ is simply connected, equality is both necessary and sufficient. Hence

$$\mathbf{F} \text{ is conservative exactly when } b = c.$$

Step 2: Construct the potential Assume $b = c$. Integrating $f_x = ax + by$ gives

$$f = \frac{a}{2}x^2 + bxy + g(y).$$

Since $f_y = bx + g'(y) = bx + dy$, we have $g'(y) = dy$, so

$$f(x, y) = \frac{a}{2}x^2 + bxy + \frac{d}{2}y^2 + C.$$

Step 3: Compute the work The Fundamental Theorem for Line Integrals gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(p, q) - f(0, 0) = \frac{a}{2}p^2 + bpq + \frac{d}{2}q^2.$$

Verification Differentiating the potential gives $\langle ax + by, bx + dy \rangle$, which equals the original field precisely because $c = b$. The additive constant cancels from the work.

Question 1

Let C be the positively oriented boundary of the rectangle

$$D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 3\}.$$

Evaluate

$$\oint_C (x^2 - y) dx + (x + y^2) dy$$

using Green's Theorem.

Tasks

1. Identify P , Q , and the required derivative difference.
2. Convert the line integral to a double integral.
3. Evaluate and check the positive orientation.

Question 1 – Solution

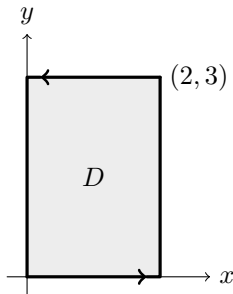
Strategy. Apply the circulation form $\oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA$.

Step 1: Differentiate Here

$$P = x^2 - y, \quad Q = x + y^2,$$

so

$$Q_x - P_y = 1 - (-1) = 2.$$



Step 2: Integrate over the region

$$\oint_C P dx + Q dy = \iint_D 2 dA = 2 \text{Area}(D) = 2(2)(3) = \boxed{12}.$$

Step 3: Orientation Positive orientation means counterclockwise traversal, with the region on the left. That is the orientation for which Green's Theorem gives the displayed positive sign.

Verification The derivative difference is the constant 2 and the rectangle has area 6, so the value must be 12; reversing C would give -12 .

Question 2

Let C be the counterclockwise ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Evaluate

$$\frac{1}{2} \oint_C (-y \, dx + x \, dy)$$

and explain its geometric meaning.

Tasks

1. Apply Green's Theorem to the given integral.
2. Compute the enclosed area.
3. Explain the effect of reversing the ellipse's orientation.

Question 2 – Solution

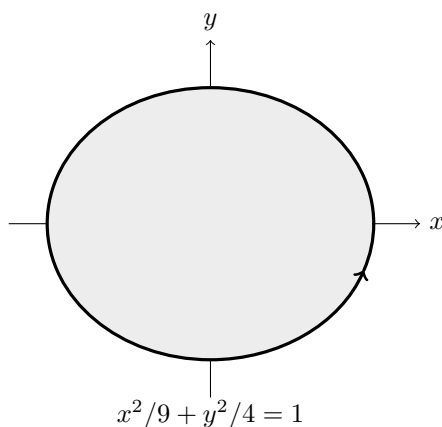
Strategy. The chosen coefficients make $Q_x - P_y = 1$, so the line integral measures area.

Step 1: Set P and Q Write

$$P = -\frac{y}{2}, \quad Q = \frac{x}{2}.$$

Then

$$Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1.$$



Step 2: Apply Green's Theorem

$$\frac{1}{2} \oint_C (-y \, dx + x \, dy) = \iint_D 1 \, dA = \text{Area}(D).$$

The ellipse has semiaxes 3 and 2, so

$$\boxed{\frac{1}{2} \oint_C (-y \, dx + x \, dy) = \pi(3)(2) = 6\pi}.$$

Step 3: Reverse orientation Clockwise traversal negates the line integral, producing -6π . Geometric area itself remains 6π ; the sign records orientation.

Verification The area formula reduces to πab , and substituting $a = 3$, $b = 2$ confirms 6π .

Question 3

The planar field

$$\mathbf{F}(x, y) = \langle x^2, xy \rangle$$

acts on the disk $D : (x - 1)^2 + y^2 \leq 1$. Let $C = \partial D$ be counterclockwise. Find the outward flux of $\mathbf{F}$ across C .

Tasks

1. State the flux form of Green's Theorem.
2. Compute the divergence and its integral over D .
3. Use symmetry or the centroid of the disk to finish.

Question 3 – Solution

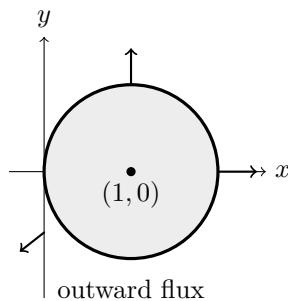
Strategy. Convert outward flux to the double integral of the planar divergence.

Step 1: Flux form For $\mathbf{F} = \langle M, N \rangle$,

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C M \, dy - N \, dx = \iint_D (M_x + N_y) \, dA.$$

Here

$$M_x + N_y = 2x + x = 3x.$$



Step 2: Integrate The disk has area π and centroid $(1, 0)$. Therefore its average x -coordinate is 1, so

$$\iint_D x \, dA = 1 \cdot \pi = \pi.$$

Consequently,

$$\boxed{\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D 3x \, dA = 3\pi}.$$

Verification The disk lies in $0 \leq x \leq 2$, so the divergence $3x$ is nonnegative. A positive outward flux is therefore consistent with the sign of the integrand.

Question 4

Let D be the annulus $1 \leq x^2 + y^2 \leq 4$ and

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

The boundary consists of the outer circle and the inner circle.

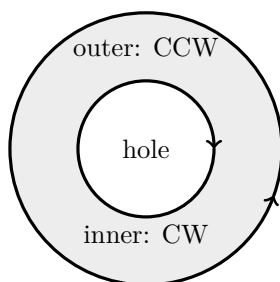
Tasks

1. Specify the positive orientation of both boundary components.
2. Apply Green's Theorem on the annulus.
3. Compute each circle's contribution and verify the cancellation.

Question 4 – Solution

Strategy. On a region with a hole, positive orientation is counterclockwise outside and clockwise around the hole.

Step 1: Orientation Traverse the radius-2 circle counterclockwise and the radius-1 circle clockwise. In both cases the annular region remains on the left.



Step 2: Apply Green's Theorem On D , the field is continuously differentiable and

$$Q_x - P_y = 0.$$

Therefore

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D 0 \, dA = \boxed{0}.$$

Step 3: Check each component On any counterclockwise circle centered at the origin, the integral is 2π , independent of radius. Hence

$$\int_{C_{\text{outer}}} \mathbf{F} \cdot d\mathbf{r} = 2\pi, \quad \int_{C_{\text{inner}}} \mathbf{F} \cdot d\mathbf{r} = -2\pi$$

because the inner circle is clockwise. Their sum is zero.

Verification If both circles were traversed counterclockwise, the total would be 4π ; that is not the positively oriented boundary of the annulus.

Question 5

Let C be the counterclockwise boundary of the region between

$$y = x^2 \quad \text{and} \quad y = x, \quad 0 \leq x \leq 1.$$

Evaluate

$$\oint_C -y \, dx + x^2 \, dy.$$

Tasks

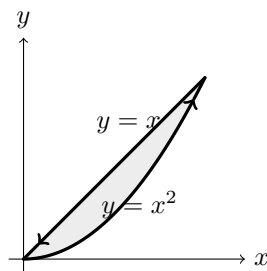
1. Describe the positively oriented boundary.
2. Set up the Green's Theorem double integral with correct bounds.
3. Evaluate and verify the sign.

Question 5 – Solution

Strategy. Replace the two curved boundary integrals with one Type I double integral over the enclosed region.

Step 1: Derivative difference Here

$$P = -y, \quad Q = x^2, \quad Q_x - P_y = 2x + 1.$$



Positive orientation travels from $(0, 0)$ to $(1, 1)$ along the lower parabola and returns along the upper line.

Step 2: Set up and evaluate

$$\begin{aligned} \oint_C P \, dx + Q \, dy &= \int_0^1 \int_{x^2}^x (2x + 1) \, dy \, dx \\ &= \int_0^1 (2x + 1)(x - x^2) \, dx \\ &= \int_0^1 (x + x^2 - 2x^3) \, dx \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = \boxed{\frac{1}{3}}. \end{aligned}$$

Verification On $0 \leq x \leq 1$, both $2x + 1$ and the vertical thickness $x - x^2$ are nonnegative, so the positive result is required.

Question 6

Use a Green's Theorem area formula to find the area enclosed by the cardioid

$$r = 1 + \cos \theta, \quad 0 \leq \theta \leq 2\pi,$$

traversed counterclockwise.

Tasks

1. Begin with $A = \frac{1}{2} \oint_C (x \, dy - y \, dx)$.
2. Show that $x \, dy - y \, dx = r^2 \, d\theta$.
3. Evaluate the resulting trigonometric integral.

Question 6 – Solution

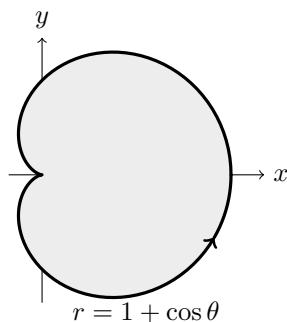
Strategy. Combine Green's area formula with polar boundary coordinates.

Step 1: Convert the differential With $x = r \cos \theta$ and $y = r \sin \theta$,

$$dx = (r' \cos \theta - r \sin \theta) d\theta, \quad dy = (r' \sin \theta + r \cos \theta) d\theta.$$

The rr' terms cancel, leaving

$$x dy - y dx = r^2(\cos^2 \theta + \sin^2 \theta) d\theta = r^2 d\theta.$$



Step 2: Evaluate the area

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{1}{2} (2\pi + 0 + \pi) = \boxed{\frac{3\pi}{2}}. \end{aligned}$$

Verification The standard polar area formula is $\frac{1}{2} \int r^2 d\theta$, exactly what Green's Theorem produced. The nonnegative integrand confirms the positive area.

Question 7

Let C be the counterclockwise triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 1)$. For

$$P(x, y) = -y^2, \quad Q(x, y) = x^2,$$

verify Green's Theorem by computing $\oint_C P \, dx + Q \, dy$ both directly and as a double integral.

Tasks

1. Evaluate all three boundary segments directly.
2. Evaluate $\iint_D (Q_x - P_y) \, dA$.
3. Compare the two results.

Question 7 – Solution

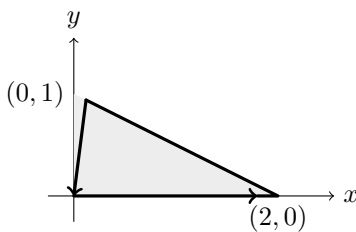
Strategy. Only the slanted edge contributes directly; then use $0 \leq y \leq 1 - x/2$ for the interior.

Step 1: Boundary calculation The horizontal and vertical coordinate-axis edges contribute zero. Parametrize the slanted edge from $(2, 0)$ to $(0, 1)$ by

$$\mathbf{r}(t) = \langle 2 - 2t, t \rangle, \quad 0 \leq t \leq 1.$$

Then $dx = -2dt$, $dy = dt$, and

$$\int_{\text{slant}} P dx + Q dy = \int_0^1 [2t^2 + 4(1 - t)^2] dt = \boxed{2}.$$



Step 2: Interior calculation Since $Q_x - P_y = 2x + 2y$,

$$\begin{aligned} \iint_D (2x + 2y) dA &= \int_0^2 \int_0^{1-x/2} (2x + 2y) dy dx \\ &= \int_0^2 \left(1 + x - \frac{3x^2}{4} \right) dx = \boxed{2}. \end{aligned}$$

Verification Both methods give 2. The slanted parametrization runs from $(2, 0)$ to $(0, 1)$, which is the required counterclockwise direction after traversing the bottom edge.

Question 8

For a constant a , let

$$\mathbf{F}_a(x, y) = \langle ay^2, x^2 \rangle.$$

The counterclockwise circulation around the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ is -2 . Determine a .

Tasks

1. Express the circulation as a double integral.
2. Solve for a .
3. Verify uniqueness and the stated circulation.

Question 8 – Solution

Strategy. Green's Theorem turns the circulation condition into a linear equation for a .

Step 1: Form the integrand With $P = ay^2$ and $Q = x^2$,

$$Q_x - P_y = 2x - 2ay.$$

Thus

$$\begin{aligned}\oint_C \mathbf{F}_a \cdot d\mathbf{r} &= \int_0^1 \int_0^1 (2x - 2ay) dy dx \\ &= \int_0^1 (2x - a) dx = 1 - a.\end{aligned}$$

Step 2: Impose the condition

$$1 - a = -2 \quad \implies \quad \boxed{a = 3}.$$

Step 3: Verify For $a = 3$,

$$\int_0^1 \int_0^1 (2x - 6y) dy dx = 1 - 3 = -2,$$

exactly the specified counterclockwise circulation.

Uniqueness check The circulation $1 - a$ has nonzero coefficient -1 on a , so the equation has exactly one solution.

Question 9

Let C be a positively oriented, simple, piecewise smooth closed curve enclosing a region D . Derive the three area formulas

$$A = \oint_C x \, dy, \quad A = - \oint_C y \, dx, \quad A = \frac{1}{2} \oint_C (x \, dy - y \, dx).$$

Tasks

1. Choose P, Q for each of the first two formulas.
2. Derive the symmetric formula from Green's Theorem or by averaging.
3. Explain the orientation requirement.

Question 9 – Solution

Strategy. Make the Green's Theorem integrand $Q_x - P_y$ equal to 1 in three different ways.

Step 1: Formula using $x \, dy$ Choose $P = 0$, $Q = x$. Then $Q_x - P_y = 1$, so

$$\text{Area}(D) = \iint_D 1 \, dA = \boxed{\oint_C x \, dy}.$$

Step 2: Formula using $-y \, dx$ Choose $P = -y$, $Q = 0$. Again $Q_x - P_y = 1$, giving

$$\text{Area}(D) = \boxed{-\oint_C y \, dx}.$$

Step 3: Symmetric formula Choose $P = -y/2$ and $Q = x/2$. Then

$$Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1,$$

and therefore

$$\boxed{\text{Area}(D) = \frac{1}{2} \oint_C (x \, dy - y \, dx)}.$$

This is also the average of the first two formulas.

Verification Green's Theorem uses positive, counterclockwise orientation. Clockwise traversal negates all three line integrals, so a minus sign would be needed to recover positive geometric area.

Question 10

Let D be the ellipse bounded counterclockwise by

$$x = 2 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq 2\pi.$$

Evaluate

$$\iint_D (3x^2 - 2y) \, dA$$

by converting it to a boundary integral.

Tasks

1. Choose P, Q so that $Q_x - P_y = 3x^2 - 2y$.
2. Parametrize the resulting boundary integral.
3. Evaluate using symmetry and a standard trigonometric integral.

Question 10 – Solution

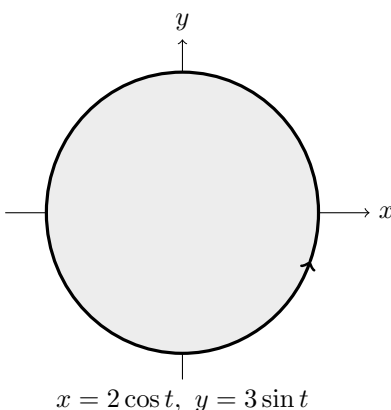
Strategy. Reverse Green's Theorem with $P = y^2$ and $Q = x^3$, then use the supplied ellipse parametrization.

Step 1: Choose a boundary field Set

$$P = y^2, \quad Q = x^3.$$

Then $Q_x - P_y = 3x^2 - 2y$, so

$$\iint_D (3x^2 - 2y) dA = \oint_C y^2 dx + x^3 dy.$$



Step 2: Substitute the boundary Since $dx = -2 \sin t dt$ and $dy = 3 \cos t dt$,

$$\oint_C y^2 dx + x^3 dy = \int_0^{2\pi} [-18 \sin^3 t + 24 \cos^4 t] dt.$$

The sine-cubed term integrates to zero over a full period, while

$$\int_0^{2\pi} \cos^4 t dt = \frac{3\pi}{4}.$$

Therefore

$$\boxed{\iint_D (3x^2 - 2y) dA = 24 \left(\frac{3\pi}{4} \right) = 18\pi}.$$

Verification The $-2y$ term is odd across the x -axis and contributes zero. Also $3x^2 \geq 0$, so the positive result has the required sign.

Question 1

For

$$\mathbf{F}(x, y, z) = \langle x^2y, yz^2, xz \rangle,$$

compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

Tasks

1. Compute the divergence term by term.
2. Compute all three components of the curl.
3. Evaluate both quantities at $(1, 2, -1)$.

Question 1 – Solution

Strategy. Label the components P, Q, R and apply the divergence and curl formulas in a fixed order.

Step 1: Divergence With $P = x^2y$, $Q = yz^2$, and $R = xz$,

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z = 2xy + z^2 + x.$$

Thus

$$\boxed{\nabla \cdot \mathbf{F} = 2xy + z^2 + x}.$$

Step 2: Curl

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle 0 - 2yz, 0 - z, 0 - x^2 \rangle.\end{aligned}$$

Therefore

$$\boxed{\nabla \times \mathbf{F} = \langle -2yz, -z, -x^2 \rangle}.$$

Step 3: Evaluate At $(1, 2, -1)$,

$$\boxed{\nabla \cdot \mathbf{F} = 6}, \quad \boxed{\nabla \times \mathbf{F} = \langle 4, 1, -1 \rangle}.$$

Verification The curl components follow the cyclic order $(R_y - Q_z, P_z - R_x, Q_x - P_y)$; recomputing from the determinant form gives the same signs.

Question 2

Consider the position field

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Tasks

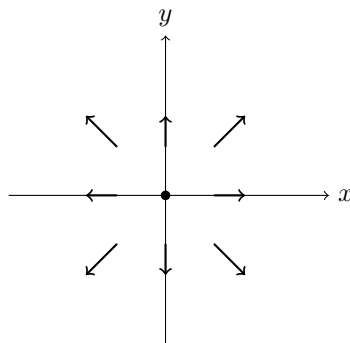
1. Compute its divergence and curl.
2. Interpret the divergence as local expansion or contraction.
3. Explain why the field has no local rotational tendency.

Question 2 – Solution

Strategy. Differentiate the matching coordinate in each divergence term and compare the cross derivatives for curl.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = \boxed{3}.$$



positive divergence: local source

The positive constant divergence describes uniform local expansion: arrows point outward and grow with distance from the origin.

Step 2: Curl Every cross derivative is zero, so

$$\nabla \times \mathbf{F} = \langle 0 - 0, 0 - 0, 0 - 0 \rangle = \boxed{\mathbf{0}}.$$

Step 3: Interpret Curl measures infinitesimal rotational tendency. This radial field expands directly away from the origin without circulating around any axis, consistent with zero curl.

Verification The field is $\nabla[(x^2 + y^2 + z^2)/2]$. A gradient field has zero curl where the second partial derivatives are continuous, confirming the calculation.

Question 3

Let

$$\mathbf{F}(x, y, z) = \langle -2y, 2x, z^2 \rangle.$$

At $P = (1, -1, 2)$, determine the local expansion rate and the curl vector.

Tasks

1. Compute and evaluate the divergence.
2. Compute and evaluate the curl.
3. Interpret the curl's axis, direction, and infinitesimal angular-speed relation.

Question 3 – Solution

Strategy. Use divergence for local volume change and curl for twice the infinitesimal angular-velocity vector.

Step 1: Expansion rate

$$\nabla \cdot \mathbf{F} = 0 + 0 + 2z = 2z.$$

At P , this is

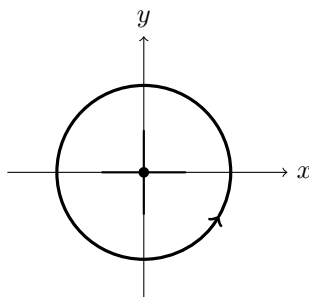
$$\boxed{(\nabla \cdot \mathbf{F})(P) = 4},$$

so the field has positive local expansion there.

Step 2: Curl

$$\nabla \times \mathbf{F} = \langle 0 - 0, 0 - 0, 2 - (-2) \rangle = \boxed{\langle 0, 0, 4 \rangle}.$$

This curl is constant, so its value at P is the same vector.



$\nabla \times \mathbf{F}$ points along $+z$

Step 3: Interpret The curl points along the positive z -axis, indicating counterclockwise rotation in horizontal planes when viewed from $+z$. The associated infinitesimal angular-velocity vector is half the curl:

$$\boxed{\vec{\omega} = \langle 0, 0, 2 \rangle}.$$

Verification The horizontal field $\langle -2y, 2x \rangle$ is rigid rotation with angular speed 2, which matches $|\nabla \times \mathbf{F}|/2 = 2$.

Question 4

For constants a, b, d , consider

$$\mathbf{F}(x, y, z) = \langle ax - by, bx - ay, dz \rangle.$$

Classify all parameter triples (a, b, d) for which the field is both divergence-free and curl-free on $\mathbb{R}^3$.

Tasks

1. Compute the divergence.
2. Compute the curl.
3. Solve the simultaneous parameter conditions and verify the resulting family.

Question 4 – Solution

Strategy. Treat the scalar divergence and all three curl components as simultaneous identities.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = a + (-a) + d = d.$$

Thus divergence-free requires $d = 0$.

Step 2: Curl With $P = ax - by$, $Q = bx - ay$, and $R = dz$,

$$\nabla \times \mathbf{F} = \langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, b - (-b) \rangle = \langle 0, 0, 2b \rangle.$$

Thus curl-free requires $b = 0$.

Step 3: Classify The parameter a is unrestricted, so

$$\boxed{(a, b, d) = (a, 0, 0), \quad a \in \mathbb{R}}.$$

The resulting field is

$$\mathbf{F} = \langle ax, -ay, 0 \rangle.$$

Verification Its divergence is $a - a + 0 = 0$, and all cross derivatives vanish. Hence every field in the boxed family satisfies both conditions, including the zero field when $a = 0$.

Question 5

Let

$$\mathbf{F}(x, y, z) = \langle xy^2, yz^2, zx^2 \rangle.$$

Verify directly that

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Tasks

1. Compute $\nabla \times \mathbf{F}$.
2. Take its divergence.
3. Explain why this is an instance of a general vector identity.

Question 5 – Solution

Strategy. Compute the curl componentwise, then differentiate each component with respect to its matching coordinate.

Step 1: Curl Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle 0 - 2yz, 0 - 2xz, 0 - 2xy \rangle \\ &= \boxed{\langle -2yz, -2xz, -2xy \rangle}.\end{aligned}$$

Step 2: Divergence of the curl

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}(-2yz) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(-2xy) \\ &= 0 + 0 + 0 = \boxed{0}.\end{aligned}$$

Step 3: General identity For a field with continuous second partial derivatives, expanding $\nabla \cdot (\nabla \times \mathbf{F})$ produces pairs of equal mixed partials with opposite signs. They cancel by equality of mixed partial derivatives.

Verification Each component of this particular curl omits the coordinate with respect to which divergence differentiates it, so all three terms vanish independently.

Question 6

For

$$f(x, y, z) = x^2y + yz^2 + xz,$$

verify directly that

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Tasks

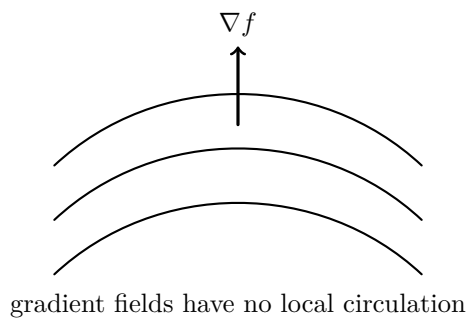
1. Compute ∇f .
2. Compute its curl and show each cancellation.
3. State the smoothness principle behind the identity.

Question 6 – Solution

Strategy. Differentiate once to obtain the gradient and again in the curl's cross-partial pairs.

Step 1: Gradient

$$\nabla f = \langle 2xy + z, x^2 + z^2, 2yz + x \rangle.$$



Step 2: Curl of the gradient Let the gradient components be P, Q, R . Then

$$R_y - Q_z = 2z - 2z = 0,$$

$$P_z - R_x = 1 - 1 = 0,$$

$$Q_x - P_y = 2x - 2x = 0.$$

Therefore

$$\boxed{\nabla \times (\nabla f) = \mathbf{0}}.$$

Step 3: General principle Each cancellation is equality of a mixed-partial pair:

$$f_{zy} = f_{yz}, \quad f_{xz} = f_{zx}, \quad f_{yx} = f_{xy}.$$

Continuous second partial derivatives guarantee these equalities.

Verification The polynomial f has continuous derivatives of every order on $\mathbb{R}^3$, so there are no domain or differentiability exceptions to the identity here.

Question 7

For the planar field

$$\mathbf{F}(x, y) = \langle x^2 - y, x + y^2 \rangle,$$

compute its planar divergence and scalar curl, then evaluate them at $(-1, 1)$.

Tasks

1. Compute $P_x + Q_y$.
2. Compute $Q_x - P_y$.
3. Interpret the two values at the specified point.

Question 7 – Solution

Strategy. In two dimensions, divergence is scalar and curl is represented by the z -component of the three-dimensional curl.

Step 1: Divergence With $P = x^2 - y$ and $Q = x + y^2$,

$$\nabla \cdot \mathbf{F} = P_x + Q_y = 2x + 2y.$$

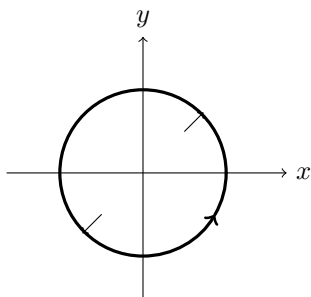
At $(-1, 1)$,

$$\boxed{\nabla \cdot \mathbf{F} = 0}.$$

Step 2: Scalar curl

$$\text{curl}_{2D} \mathbf{F} = Q_x - P_y = 1 - (-1) = \boxed{2}.$$

Equivalently, the three-dimensional embedding $\langle P, Q, 0 \rangle$ has curl $\langle 0, 0, 2 \rangle$.



zero divergence, positive scalar curl

Step 3: Interpret At the point there is no first-order net expansion or contraction, but there is positive counterclockwise rotational tendency. Zero divergence does not imply zero curl.

Verification The scalar curl is constant everywhere, while the divergence vanishes on the line $y = -x$; the point $(-1, 1)$ lies on that line.

Question 8

A linear planar field has the expansion-rotation form

$$\mathbf{F}(x, y) = \langle ax - by, bx + ay \rangle.$$

Its divergence is 4 and its scalar curl is 6 everywhere.

Tasks

1. Determine a and b .
2. Write the resulting field explicitly.
3. Separate it into pure expansion and pure rotation parts.

Question 8 – Solution

Strategy. Divergence isolates the expansion coefficient, while scalar curl isolates twice the rotation coefficient.

Step 1: Compute the invariants

$$\nabla \cdot \mathbf{F} = a + a = 2a, \quad \text{curl}_{2D} \mathbf{F} = b - (-b) = 2b.$$

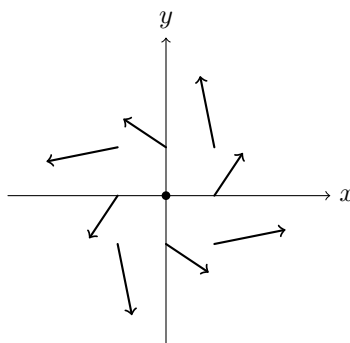
Thus

$$2a = 4, \quad 2b = 6,$$

so $\boxed{a = 2}$ and $\boxed{b = 3}$.

Step 2: Write the field

$$\boxed{\mathbf{F}(x, y) = \langle 2x - 3y, 3x + 2y \rangle}.$$



outward expansion plus counterclockwise rotation

Step 3: Decompose

$$\mathbf{F} = \underbrace{\langle 2x, 2y \rangle}_{\text{pure expansion}} + \underbrace{\langle -3y, 3x \rangle}_{\text{counterclockwise rotation}}.$$

The first part has divergence 4 and zero curl; the second has zero divergence and scalar curl 6.

Verification Recombining the two displayed parts returns both components of the boxed field and reproduces the prescribed invariants.

Question 9

On $D = \mathbb{R}^3 \setminus \{\mathbf{0}\}$, define

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Compute $\nabla \cdot \mathbf{F}$ on D .
2. Compute or otherwise justify $\nabla \times \mathbf{F}$ on D .
3. Explain why the excluded origin must be stated.

Question 9 – Solution

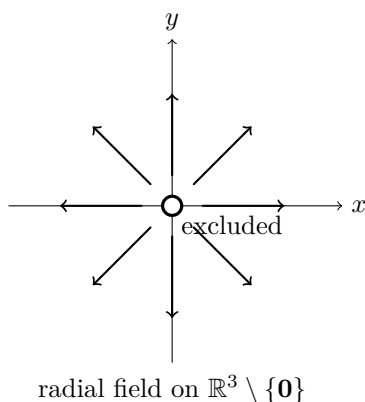
Strategy. Write $r = (x^2 + y^2 + z^2)^{1/2}$ and differentiate xr^{-3} , yr^{-3} , and zr^{-3} .

Step 1: Divergence Since

$$\frac{\partial}{\partial x}(xr^{-3}) = r^{-3} - 3x^2r^{-5},$$

with analogous formulas for y and z ,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= 3r^{-3} - 3(x^2 + y^2 + z^2)r^{-5} \\ &= 3r^{-3} - 3r^2r^{-5} = \boxed{0}, \quad r > 0.\end{aligned}$$



Step 2: Curl Because

$$\mathbf{F} = \nabla \left(-\frac{1}{r} \right)$$

on D , the curl-of-a-gradient identity gives

$$\boxed{\nabla \times \mathbf{F} = \mathbf{0}}.$$

Step 3: Domain caveat The formula is undefined at the origin, and the derivatives used above do not exist there. The conclusions are pointwise statements on D , not statements at $\mathbf{0}$.

Verification Direct cross differentiation gives, for example, $R_y - Q_z = -3yzzr^{-5} - (-3yzzr^{-5}) = 0$, and the other two components cancel similarly.

Question 10

Let

$$\mathbf{F}(x, y, z) = \langle -yz, xz, xy \rangle.$$

Tasks

1. Compute the divergence and curl everywhere.
2. Evaluate the curl at $P = (1, 2, 3)$ and give its direction.
3. Find every point at which the curl vanishes.

Question 10 – Solution

Strategy. Compute the global formulas first; then solve the component equations of the curl simultaneously.

Step 1: Divergence Each component omits its matching differentiation variable, so

$$\nabla \cdot \mathbf{F} = \frac{\partial(-yz)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z} = \boxed{0}.$$

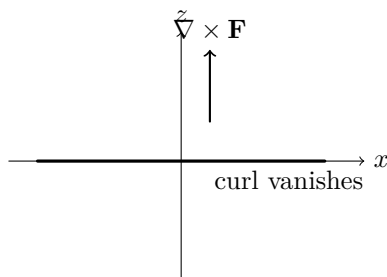
Step 2: Curl

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle x - x, -y - y, z - (-z) \rangle \\ &= \boxed{\langle 0, -2y, 2z \rangle}.\end{aligned}$$

At $P = (1, 2, 3)$,

$$(\nabla \times \mathbf{F})(P) = \boxed{\langle 0, -4, 6 \rangle},$$

which points in the direction $\boxed{\langle 0, -2, 3 \rangle / \sqrt{13}}$.



Step 3: Vanishing locus The curl is zero exactly when

$$-2y = 0, \quad 2z = 0.$$

Thus $y = z = 0$ while x is arbitrary:

$$\boxed{\nabla \times \mathbf{F} = \mathbf{0} \text{ exactly on the } x\text{-axis}.}$$

Verification Substituting $(x, 0, 0)$ into the curl formula gives $\mathbf{0}$, and any point off the x -axis has $y \neq 0$ or $z \neq 0$, making at least one curl component nonzero.

Question 1

Consider the surface patch

$$\mathbf{r}(u, v) = \langle u, v, 1 + 2u - v \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 1.$$

Tasks

1. Identify the surface and its parameter-domain image.
2. Find two tangent vectors and a normal vector.
3. List the four corner points of the patch.

Question 1 – Solution

Strategy. Read $x = u$ and $y = v$, eliminate the parameters, and map the rectangle's four corners.

Step 1: Identify the surface Since $x = u$ and $y = v$,

$$z = 1 + 2x - y.$$

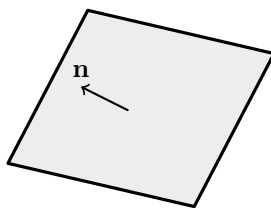
Thus the image is the portion of this plane above the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$.

Step 2: Tangent and normal vectors

$$\mathbf{r}_u = \langle 1, 0, 2 \rangle, \quad \mathbf{r}_v = \langle 0, 1, -1 \rangle.$$

Therefore

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v = \langle -2, 1, 1 \rangle}.$$



planar parameter rectangle

Step 3: Map the corners

$$\mathbf{r}(0, 0) = (0, 0, 1),$$

$$\mathbf{r}(2, 0) = (2, 0, 5),$$

$$\mathbf{r}(0, 1) = (0, 1, 0),$$

$$\mathbf{r}(2, 1) = (2, 1, 4).$$

These four points bound the parallelogram patch.

Verification The normal $\langle -2, 1, 1 \rangle$ is perpendicular to both tangent vectors, and each corner satisfies $z = 1 + 2x - y$.

Question 2

Parametrize the paraboloid $z = x^2 + y^2$ as a graph. At the point $P = (1, -1, 2)$, find a normal vector and the tangent plane.

Tasks

1. Give a two-parameter graph parametrization.
2. Compute the tangent vectors and their cross product at P .
3. Write and verify the tangent-plane equation.

Question 2 – Solution

Strategy. Use $x = u$, $y = v$, and let the defining graph supply the third component.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u, v, u^2 + v^2 \rangle, \quad (u, v) \in \mathbb{R}^2.$$

The point P corresponds to $(u, v) = (1, -1)$.

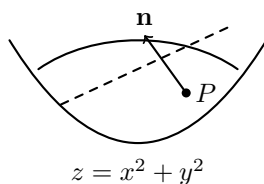
Step 2: Normal vector

$$\mathbf{r}_u = \langle 1, 0, 2u \rangle, \quad \mathbf{r}_v = \langle 0, 1, 2v \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2u, -2v, 1 \rangle.$$

At P , an upward normal is $\boxed{\langle -2, 2, 1 \rangle}$.



Step 3: Tangent plane

$$-2(x - 1) + 2(y + 1) + (z - 2) = 0,$$

or

$$\boxed{z = 2x - 2y - 2}.$$

Verification Substitution of $(1, -1, 2)$ gives $2 = 2 + 2 - 2$. The normal from the graph equation $x^2 + y^2 - z = 0$ is $\langle 2, -2, -1 \rangle$, the negative of the chosen normal.

Question 3

Parametrize the half-cylinder

$$x^2 + y^2 = 4, \quad y \geq 0, \quad -1 \leq z \leq 3,$$

and choose the orientation whose normal points outward.

Tasks

1. Give a parameter domain that covers the patch once, apart from boundary seams.
2. Compute a normal from the tangent vectors.
3. Verify that the normal is outward.

Question 3 – Solution

Strategy. Use an angular parameter for the circular direction and a height parameter for z .

Step 1: Parametrize

$$\mathbf{r}(\theta, s) = \langle 2 \cos \theta, 2 \sin \theta, s \rangle, \quad 0 \leq \theta \leq \pi, \quad -1 \leq s \leq 3.$$

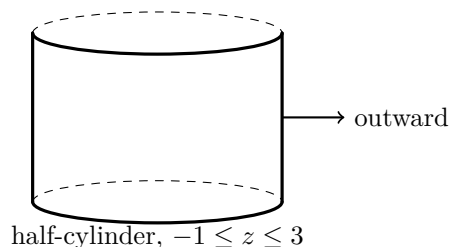
The interval $0 \leq \theta \leq \pi$ enforces $y \geq 0$.

Step 2: Tangent vectors

$$\mathbf{r}_\theta = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle, \quad \mathbf{r}_s = \langle 0, 0, 1 \rangle.$$

Their ordered cross product is

$$\mathbf{r}_\theta \times \mathbf{r}_s = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle.$$



Step 3: Check orientation At each point, the horizontal radial vector from the z -axis is $\langle x, y, 0 \rangle = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle$, exactly the cross product. Hence it points outward.

Verification Its magnitude is 2, never zero, so the parametrization is regular everywhere on the patch, including its boundary curves.

Question 4

Consider the standard spherical parametrization

$$\mathbf{r}(\phi, \theta) = \langle 3 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 3 \cos \phi \rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

Tasks

1. Show that the image is the sphere of radius 3.
2. Compute an outward normal vector from the tangent vectors.
3. Identify where the parametrization fails to be regular or one-to-one.

Question 4 – Solution

Strategy. Verify the implicit equation, then use the ordered product $\mathbf{r}_\phi \times \mathbf{r}_\theta$.

Step 1: Identify the image

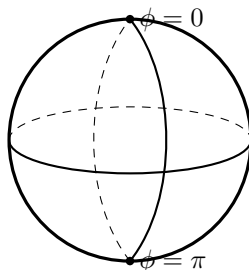
$$x^2 + y^2 + z^2 = 9 \sin^2 \phi + 9 \cos^2 \phi = 9.$$

Thus the image is $x^2 + y^2 + z^2 = 9$.

Step 2: Normal Differentiation and crossing give

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 9 \sin \phi \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle.$$

For $0 < \phi < \pi$, this is a positive multiple of the outward radial direction.



longitude collapses at the poles

Step 3: Degeneracies Its magnitude is $9 \sin \phi$, which vanishes at $\phi = 0$ and $\phi = \pi$. Every value of θ maps to the same pole there. Also $\theta = 0$ and $\theta = 2\pi$ describe the same meridian seam.

Verification Away from the poles, $9 \sin \phi > 0$, so the tangent vectors are independent and the patch is regular locally.

Question 5

Parametrize the conical frustum

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 2,$$

and find a normal pointing away from the z -axis.

Tasks

1. Choose radial and angular parameters with correct bounds.
2. Compute both possible normal orientations.
3. Select and verify the outward normal.

Question 5 – Solution

Strategy. On this cone the radius equals the height, so one parameter can serve both roles.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 1 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

Step 2: Cross the tangents

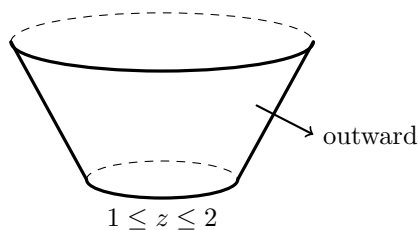
$$\mathbf{r}_u = \langle \cos v, \sin v, 1 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 0 \rangle.$$

Thus

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -u \cos v, -u \sin v, u \rangle.$$

This points inward horizontally. Reverse the order to obtain

$$\boxed{\mathbf{r}_v \times \mathbf{r}_u = \langle u \cos v, u \sin v, -u \rangle}.$$



Step 3: Verify Its horizontal component is $\langle x, y, 0 \rangle$, pointing away from the axis. It also agrees with the outward direction of $x^2 + y^2 - z^2 = 0$, whose gradient is $2\langle x, y, -z \rangle$.

Question 6

Analyze the parametrized surface

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

Tasks

1. Identify the surface geometrically.
2. Describe the coordinate curves $u = \text{constant}$ and $v = \text{constant}$.
3. Test regularity on the entire parameter domain.

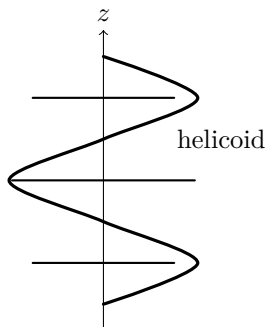
Question 6 – Solution

Strategy. Interpret radius, angle, and height directly, then test the tangent cross product.

Step 1: Identify In cylindrical terms,

$$\rho = u, \quad \theta = v, \quad z = v.$$

As the angle increases, height increases at the same rate; the image is a helicoid patch.



Step 2: Coordinate curves If $u = u_0$, then

$$\langle u_0 \cos v, u_0 \sin v, v \rangle$$

is a helix of radius u_0 ; for $u_0 = 0$ it is the z -axis. If $v = v_0$, varying u traces a radial line segment at height $z = v_0$.

Step 3: Regularity

$$\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 1 \rangle,$$

and

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \rangle}.$$

Its magnitude is $\sqrt{1 + u^2} > 0$, even at $u = 0$, so the parametrization is regular everywhere.

Verification The first two components of the cross product have squared sum 1, preventing the normal from vanishing.

Question 7

Consider

$$\mathbf{r}(u, v) = \langle u^2, v, u^3 \rangle, \quad -1 \leq u \leq 1, \quad -1 \leq v \leq 1.$$

Tasks

1. Eliminate u to find an implicit relation between x and z .
2. Determine where the parametrization is regular.
3. Explain the geometric significance of the failure set.

Question 7 – Solution

Strategy. Compare $x = u^2$ and $z = u^3$, then locate zeros of the tangent cross product.

Step 1: Eliminate the parameter

$$z^2 = u^6 = (u^2)^3 = x^3, \quad x \geq 0.$$

The image is the patch of the cuspidal cylinder $z^2 = x^3$ with $0 \leq x \leq 1$ and $-1 \leq y \leq 1$.

Step 2: Test regularity

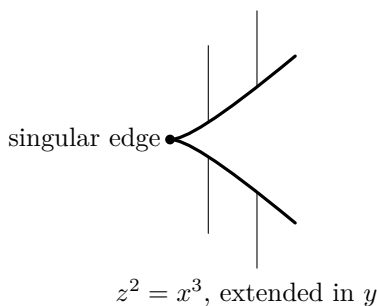
$$\mathbf{r}_u = \langle 2u, 0, 3u^2 \rangle, \quad \mathbf{r}_v = \langle 0, 1, 0 \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -3u^2, 0, 2u \rangle.$$

This vector vanishes exactly when $u = 0$. Therefore the parametrization is regular for $u \neq 0$ and singular along

$$\boxed{\mathbf{r}(0, v) = \langle 0, v, 0 \rangle}.$$



Step 3: Interpret That line is the sharp cuspidal edge. At it, the u -direction tangent collapses to zero, so the two tangent vectors cannot span a tangent plane.

Verification For $u \neq 0$, the third component $2u$ of the cross product is nonzero, proving regularity away from the edge.

Question 8

Two parametrizations are

$$\mathbf{r}(u, v) = \langle u, v, u + v \rangle, \quad \mathbf{s}(p, q) = \langle q, p, p + q \rangle.$$

Tasks

1. Show that they have the same image.
2. Compute the ordered normal from each parametrization.
3. Determine whether they induce the same or opposite orientation.

Question 8 – Solution

Strategy. Recognize that the second parametrization swaps the two parameter roles, then compare ordered cross products.

Step 1: Compare images Both satisfy

$$x + y = z.$$

Given (u, v) for $\mathbf{r}$, choosing $(p, q) = (v, u)$ gives $\mathbf{s}(v, u) = \mathbf{r}(u, v)$. Hence their images are the same plane.

Step 2: First normal

$$\mathbf{r}_u = \langle 1, 0, 1 \rangle, \quad \mathbf{r}_v = \langle 0, 1, 1 \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \boxed{\langle -1, -1, 1 \rangle}.$$

Step 3: Second normal

$$\mathbf{s}_p = \langle 0, 1, 1 \rangle, \quad \mathbf{s}_q = \langle 1, 0, 1 \rangle,$$

and therefore

$$\mathbf{s}_p \times \mathbf{s}_q = \boxed{\langle 1, 1, -1 \rangle}.$$

The normals are negatives, so the parametrizations induce opposite orientations.

Verification Swapping two parameters reverses the order in a cross product, and $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

Question 9

Let

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u^2 \rangle, \quad 1 \leq u \leq 2, \quad 0 \leq v \leq \pi.$$

Describe all boundary curves of the surface patch.

Tasks

1. Find the curves corresponding to $u = 1$ and $u = 2$.
2. Find the curves corresponding to $v = 0$ and $v = \pi$.
3. Identify the surface and verify that the four curves join correctly.

Question 9 – Solution

Strategy. Each edge of the rectangular parameter domain maps to one boundary curve.

Step 1: Constant- u edges

$$u = 1 : \langle \cos v, \sin v, 1 \rangle, \quad u = 2 : \langle 2 \cos v, 2 \sin v, 4 \rangle, \quad 0 \leq v \leq \pi.$$

These are semicircles at heights 1 and 4.

Step 2: Constant- v edges

$$v = 0 : \langle u, 0, u^2 \rangle, \quad v = \pi : \langle -u, 0, u^2 \rangle, \quad 1 \leq u \leq 2.$$

These are parabolic side curves in the plane $y = 0$.



half-annular paraboloid patch

Step 3: Identify and join Since $x^2 + y^2 = u^2$ and $z = u^2$, the patch lies on $z = x^2 + y^2$ above the half-annulus $1 \leq x^2 + y^2 \leq 4$, $y \geq 0$.

Verification The four corner images are $(1, 0, 1)$, $(-1, 0, 1)$, $(2, 0, 4)$, and $(-2, 0, 4)$; each occurs as an endpoint of exactly two adjacent boundary curves.

Question 10

A torus has major radius 3 and tube radius 1, centered about the z -axis.

Tasks

1. Construct a parametrization using two angles.
2. Compute an outward-pointing normal vector.
3. Prove the parametrization is regular everywhere and describe its seam identifications.

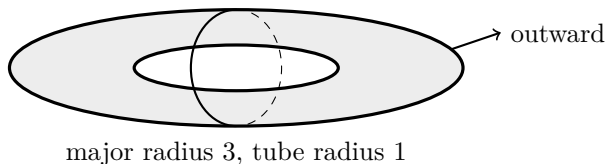
Question 10 – Solution

Strategy. First parametrize a radius-1 circle centered 3 units from the axis, then rotate it about the z -axis.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle (3 + \cos v) \cos u, (3 + \cos v) \sin u, \sin v \rangle,$$

with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.



Step 2: Compute a normal Let $A = 3 + \cos v$. Then

$$\mathbf{r}_u = \langle -A \sin u, A \cos u, 0 \rangle,$$

$$\mathbf{r}_v = \langle -\sin v \cos u, -\sin v \sin u, \cos v \rangle.$$

Their cross product is

$$\mathbf{r}_u \times \mathbf{r}_v = A \langle \cos u \cos v, \sin u \cos v, \sin v \rangle,$$

which points from the tube's center circle toward the surface, hence outward.

Step 3: Regularity and seams The unit vector in brackets has magnitude 1, while $A = 3 + \cos v \geq 2$. Thus the cross product never vanishes. The edges $u = 0, 2\pi$ are identified, as are $v = 0, 2\pi$.

Verification The distance from the z -axis is $3 + \cos v$ and the height is $\sin v$, so $(\sqrt{x^2 + y^2} - 3)^2 + z^2 = 1$, the intended torus.

Question 1

Let S be the portion of the plane $z = 2x + y$ above the rectangle

$$R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}.$$

Tasks

1. Find the scalar surface element dS .
2. Compute the area of S .
3. Evaluate $\iint_S (x + z) dS$ and find the average value of $x + z$ on S .

Question 1 – Solution

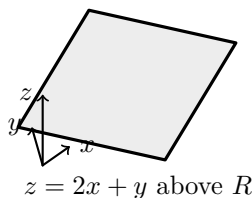
Strategy. Treat the plane as a graph, so the same constant area factor converts every integral over S into an integral over R .

Step 1: Surface element For $z = g(x, y) = 2x + y$,

$$g_x = 2, \quad g_y = 1,$$

and therefore

$$dS = \sqrt{1 + g_x^2 + g_y^2} dA = \sqrt{6} dA.$$



Step 2: Area

$$\text{area}(S) = \iint_R \sqrt{6} dA = \sqrt{6}(1)(2) = \boxed{2\sqrt{6}}.$$

Step 3: Weighted integral On S , $x + z = 3x + y$. Hence

$$\begin{aligned} \iint_S (x + z) dS &= \sqrt{6} \int_0^1 \int_0^2 (3x + y) dy dx \\ &= \sqrt{6} \int_0^1 (6x + 2) dx = \boxed{5\sqrt{6}}. \end{aligned}$$

The average value is

$$\frac{1}{\text{area}(S)} \iint_S (x + z) dS = \frac{5}{2}.$$

Verification The average $5/2$ lies between the minimum 0 and maximum 5 of $3x + y$ on R .

Question 2

Let S be the part of the paraboloid $z = x^2 + y^2$ above the unit disk $x^2 + y^2 \leq 1$.

Tasks

1. Express dS in polar coordinates.
2. Set up and evaluate $\iint_S z \, dS$.
3. Check the radial substitution and the scale of the result.

Question 2 – Solution

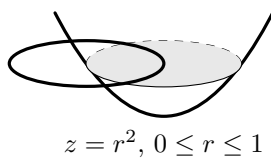
Strategy. Use the graph formula first, then exploit radial symmetry so the surface integral becomes one one-variable integral.

Step 1: Surface element With $g(x, y) = x^2 + y^2$,

$$dS = \sqrt{1 + 4x^2 + 4y^2} dA.$$

In polar coordinates, $z = r^2$ and $dA = r dr d\theta$, so

$$\boxed{dS = \sqrt{1 + 4r^2} r dr d\theta}.$$



Step 2: Integrate

$$\iint_S z dS = 2\pi \int_0^1 r^3 \sqrt{1 + 4r^2} dr.$$

Let $t = 1 + 4r^2$. Then $r dr = dt/8$ and $r^2 = (t - 1)/4$, giving

$$\begin{aligned} \iint_S z dS &= \frac{\pi}{16} \int_1^5 (t^{3/2} - t^{1/2}) dt \\ &= \frac{\pi}{16} \left[\frac{2}{5} t^{5/2} - \frac{2}{3} t^{3/2} \right]_1^5 \\ &= \boxed{\frac{\pi(25\sqrt{5} + 1)}{60}}. \end{aligned}$$

Verification The integrand is nonnegative. Also $0 \leq z \leq 1$, so the answer must not exceed the surface area; numerically it is about 2.98, consistent with that bound.

Question 3

A thin surface occupies the lateral cylinder

$$S: x^2 + y^2 = 4, \quad 0 \leq z \leq 3,$$

with surface density $\delta(x, y, z) = 1 + z$.

Tasks

1. Parametrize S and compute its surface element.
2. Find the total mass.
3. Find the center of mass, using symmetry where appropriate.

Question 3 – Solution

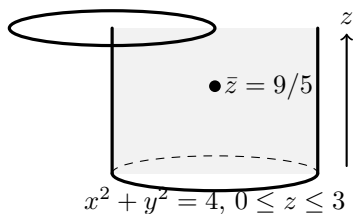
Strategy. Parametrize by angle and height. Rotational symmetry immediately locates two coordinates of the center of mass.

Step 1: Geometry

$$\mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3.$$

Since $\|\mathbf{r}_\theta \times \mathbf{r}_z\| = 2$,

$$dS = 2 \, d\theta \, dz.$$

**Step 2: Mass**

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^3 (1+z) 2 \, dz \, d\theta \\ &= 4\pi \left[z + \frac{z^2}{2} \right]_0^3 = 30\pi. \end{aligned}$$

Step 3: Center of mass Symmetry gives $\bar{x} = \bar{y} = 0$. The first moment about the xy -plane is

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^3 z(1+z) 2 \, dz \, d\theta \\ &= 4\pi \left[\frac{z^2}{2} + \frac{z^3}{3} \right]_0^3 = 54\pi. \end{aligned}$$

Thus

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{9}{5} \right).$$

Verification Because density increases with height, $\bar{z} = 1.8$ is correctly above the geometric midpoint 1.5 and remains inside $0 \leq z \leq 3$.

Question 4

Let S be the conical frustum

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 3.$$

Tasks

1. Parametrize S and derive dS .
2. Compute the surface area.
3. Evaluate $\iint_S \frac{1}{z} dS$ and its average value on S .

Question 4 – Solution

Strategy. Use height as the radius parameter. The factor of u in dS cancels against the weight $1/z$ in the second integral.

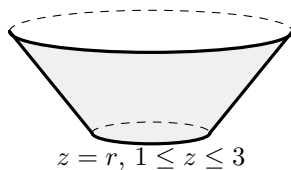
Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 1 \leq u \leq 3, \quad 0 \leq v \leq 2\pi.$$

The tangent vectors have cross-product magnitude

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{2},$$

so $\boxed{dS = u\sqrt{2} \, du \, dv}.$



Step 2: Area

$$\text{area}(S) = \int_0^{2\pi} \int_1^3 u\sqrt{2} \, du \, dv = \boxed{8\pi\sqrt{2}}.$$

Step 3: Weighted integral

Since $z = u$,

$$\iint_S \frac{1}{z} \, dS = \int_0^{2\pi} \int_1^3 \frac{1}{u} u\sqrt{2} \, du \, dv = \boxed{4\pi\sqrt{2}}.$$

Its average value is therefore

$$\boxed{\frac{4\pi\sqrt{2}}{8\pi\sqrt{2}} = \frac{1}{2}}.$$

Verification On the frustum, $1/3 \leq 1/z \leq 1$; the average $1/2$ lies in this interval.

Question 5

On the sphere $S : x^2 + y^2 + z^2 = 4$, evaluate the scalar surface integral

$$\iint_S z^2 dS.$$

Then find the average value of z^2 over the sphere.

Tasks

1. Choose spherical parameters and compute dS .
2. Evaluate the integral exactly.
3. Use symmetry to check the average value.

Question 5 – Solution

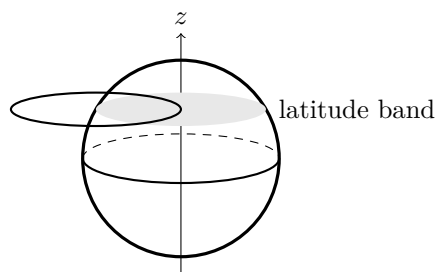
Strategy. In spherical parameters, z and dS separate into a polar factor and an azimuthal factor.

Step 1: Parametrize For radius 2, use

$$\mathbf{r}(\phi, \theta) = \langle 2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi \rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Then

$$z = 2 \cos \phi, \quad dS = 4 \sin \phi \, d\phi \, d\theta.$$



Step 2: Integrate

$$\begin{aligned} \iint_S z^2 \, dS &= \int_0^{2\pi} \int_0^\pi 16 \cos^2 \phi \sin \phi \, d\phi \, d\theta \\ &= 32\pi \int_0^\pi \cos^2 \phi \sin \phi \, d\phi \\ &= 32\pi \left[\frac{-\cos^3 \phi}{3} \right]_0^\pi = \boxed{\frac{64\pi}{3}}. \end{aligned}$$

Because $\text{area}(S) = 4\pi(2^2) = 16\pi$, the average is

$$\boxed{\text{Avg}_S(z^2) = \frac{4}{3}}.$$

Verification Rotational symmetry gives equal averages of x^2, y^2, z^2 . Their sum is always 4, so each average must be $4/3$.

Question 6

Let S be the upper hemisphere

$$z = \sqrt{9 - x^2 - y^2}, \quad z \geq 0.$$

Evaluate $\iint_S z \, dS$ by viewing S as a graph, and find the average value of z on S .

Tasks

1. Derive the graph surface element and address its boundary singularity.
2. Explain the cancellation that simplifies the integrand.
3. Evaluate the integral and average value.

Question 6 – Solution

Strategy. The graph factor becomes unbounded at the equator, but multiplication by z cancels it exactly in the interior.

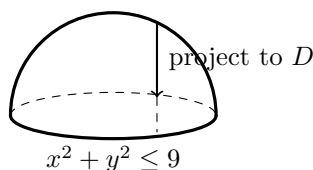
Step 1: Graph factor Let $g(x, y) = \sqrt{9 - x^2 - y^2}$ over the disk $D : x^2 + y^2 \leq 9$. Since

$$g_x = -\frac{x}{z}, \quad g_y = -\frac{y}{z},$$

we have, for $z > 0$,

$$\sqrt{1 + g_x^2 + g_y^2} = \sqrt{1 + \frac{x^2 + y^2}{z^2}} = \frac{3}{z}.$$

Thus $dS = (3/z)dA$ away from the boundary.



Step 2: Cancel and integrate The weighted element is simply

$$z dS = z \frac{3}{z} dA = 3 dA.$$

Therefore

$$\boxed{\iint_S z dS = 3 \text{ area}(D) = 27\pi}.$$

The hemisphere has area $2\pi(3^2) = 18\pi$, so

$$\boxed{\text{Avg}_S(z) = \frac{27\pi}{18\pi} = \frac{3}{2}}.$$

Verification Integrating over disks of radius $R < 3$ gives $3\pi R^2$; taking $R \rightarrow 3^-$ justifies the boundary limit and yields 27π .

Question 7

Let S be the entire boundary of the unit cube $0 \leq x, y, z \leq 1$.

Tasks

1. Decompose S into its six planar faces.
2. Evaluate $\iint_S (x + y + z) dS$.
3. Find the average value of $x + y + z$ on the cube's boundary and verify it by symmetry.

Question 7 – Solution

Strategy. Pair opposite faces. Each pair produces the same contribution, avoiding six unrelated computations.

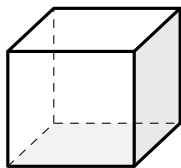
Step 1: Pair the x -faces On $x = 0$, $dS = dy dz$ and

$$\int_0^1 \int_0^1 (y + z) dy dz = 1.$$

On $x = 1$,

$$\int_0^1 \int_0^1 (1 + y + z) dy dz = 2.$$

Thus the pair perpendicular to the x -axis contributes 3.



six faces of the unit cube

Step 2: Use coordinate symmetry The cube and integrand are unchanged by permuting x, y, z . Consequently, the two y -faces and two z -faces also contribute 3 apiece. Hence

$$\boxed{\iint_S (x + y + z) dS = 3 + 3 + 3 = 9}.$$

Step 3: Average The six unit-square faces give $\text{area}(S) = 6$, so

$$\boxed{\text{Avg}_S(x + y + z) = \frac{9}{6} = \frac{3}{2}}.$$

Verification The map $(x, y, z) \mapsto (1 - x, 1 - y, 1 - z)$ preserves the boundary and pairs values whose sum is 3. Therefore their average is $3/2$, confirming the computation.

Question 8

Find the area of the helicoid patch

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.$$

Tasks

1. Compute the tangent vectors and surface element.
2. Evaluate the resulting nonpolynomial integral.
3. Check regularity and the scale of the area.

Question 8 – Solution

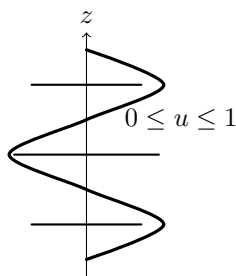
Strategy. Compute the cross product in the given parameters; its magnitude depends only on u , so the v -integration separates.

Step 1: Surface element

$$\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 1 \rangle.$$

Thus

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \rangle, \quad \boxed{dS = \sqrt{1 + u^2} \, du \, dv}.$$



Step 2: Integrate

$$\begin{aligned} \text{area}(S) &= 2\pi \int_0^1 \sqrt{1 + u^2} \, du \\ &= 2\pi \left[\frac{1}{2} \left(u\sqrt{1 + u^2} + \ln(u + \sqrt{1 + u^2}) \right) \right]_0^1 \\ &= \boxed{\pi \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)}. \end{aligned}$$

Step 3: Check The cross product never vanishes because its first two components have squared sum 1. Also $1 \leq \sqrt{1 + u^2} \leq \sqrt{2}$, so

$$2\pi \leq \text{area}(S) \leq 2\pi\sqrt{2},$$

and the exact result lies between these bounds.

Question 9

Let S be the entire unbounded paraboloid $z = x^2 + y^2$. Determine whether the improper scalar surface integral

$$\iint_S \frac{e^{-z}}{\sqrt{1+4z}} dS$$

converges, and evaluate it if it does.

Tasks

1. Truncate the surface above disks $x^2 + y^2 \leq R^2$.
2. Reduce the truncated surface integral to one variable.
3. Take the improper limit and justify convergence.

Question 9 – Solution

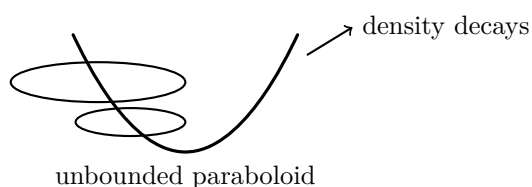
Strategy. On this paraboloid, the denominator in the density is exactly the graph-area factor, leaving a Gaussian radial integral.

Step 1: Truncate Let S_R be the part above $0 \leq r \leq R$. For $z = r^2$,

$$dS = \sqrt{1 + 4r^2} r dr d\theta.$$

Therefore the density times the area element simplifies to

$$\frac{e^{-r^2}}{\sqrt{1 + 4r^2}} \sqrt{1 + 4r^2} r dr d\theta = e^{-r^2} r dr d\theta.$$



Step 2: Evaluate the truncation

$$\begin{aligned} \iint_{S_R} \frac{e^{-z}}{\sqrt{1 + 4z}} dS &= 2\pi \int_0^R e^{-r^2} r dr \\ &= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^R \\ &= \pi(1 - e^{-R^2}). \end{aligned}$$

Step 3: Take the limit The integrand is nonnegative and the truncated values increase to a finite limit:

$$\boxed{\iint_S \frac{e^{-z}}{\sqrt{1 + 4z}} dS = \lim_{R \rightarrow \infty} \pi(1 - e^{-R^2}) = \pi.}$$

Verification The omitted tail equals πe^{-R^2} , which tends to zero and quantifies the convergence.

Question 10

The same planar patch is described over the unit square by

$$\mathbf{r}(u, v) = \langle u, v, u + 2v \rangle, \quad \mathbf{s}(p, q) = \langle 1 - p, q, 1 - p + 2q \rangle.$$

Tasks

1. Show that the parametrizations have the same image but opposite orientations.
2. Evaluate $\iint_S (x + y) \, dS$ using each parametrization.
3. Explain why reversing orientation cannot change a scalar surface integral.

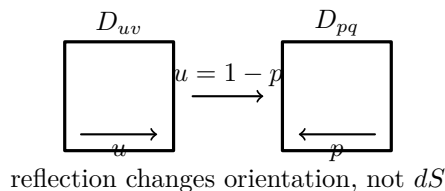
Question 10 – Solution

Strategy. Recognize $\mathbf{s}(p, q) = \mathbf{r}(1 - p, q)$, then compare cross products by magnitude rather than sign.

Step 1: Image and orientation The reflection $(u, v) = (1 - p, q)$ maps the unit square onto itself, so the images agree. Moreover,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1, -2, 1 \rangle, \quad \mathbf{s}_p \times \mathbf{s}_q = \langle 1, 2, -1 \rangle.$$

The normals are negatives, so the orientations are opposite; both magnitudes are $\sqrt{6}$.



Step 2: First parametrization

$$\iint_S (x + y) dS = \int_0^1 \int_0^1 (u + v) \sqrt{6} du dv = \boxed{\sqrt{6}}.$$

Step 3: Second parametrization

$$\iint_S (x + y) dS = \int_0^1 \int_0^1 (1 - p + q) \sqrt{6} dp dq = \boxed{\sqrt{6}}.$$

Verification A scalar surface integral uses $\|\mathbf{r}_u \times \mathbf{r}_v\|$, so reversing the cross product does not change dS . Here the reflection has Jacobian determinant -1 , whose absolute value is 1, confirming the equality under change of parameters.

Question 1

Let S be the part of the plane $z = 1 + x + y$ above the unit square $0 \leq x, y \leq 1$, oriented upward. Let

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Tasks

1. Find the upward-oriented vector surface element.
2. Compute the flux $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$.
3. State how the result changes under downward orientation.

Question 1 – Solution

Strategy. Parametrize the graph by (x, y) and retain the cross product itself, not merely its magnitude.

Step 1: Oriented element Use

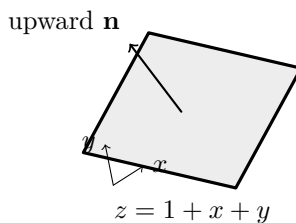
$$\mathbf{r}(x, y) = \langle x, y, 1 + x + y \rangle.$$

Then

$$\mathbf{r}_x = \langle 1, 0, 1 \rangle, \quad \mathbf{r}_y = \langle 0, 1, 1 \rangle,$$

so the upward vector area element is

$$\mathbf{n} dS = (\mathbf{r}_x \times \mathbf{r}_y) dx dy = \langle -1, -1, 1 \rangle dx dy.$$



Step 2: Dot product On the surface, $z = 1 + x + y$, and hence

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -x - y + z = 1.$$

Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \int_0^1 \int_0^1 1 dx dy = 1.$$

Step 3: Reverse orientation A downward normal is the negative of the chosen one, so the downward flux is $\boxed{-1}$.

Verification The cross product has positive third component, confirming the upward orientation; changing only the orientation must negate a flux integral.

Question 2

Let S be the paraboloid patch $z = x^2 + y^2$ above $x^2 + y^2 \leq 1$, oriented upward. For

$$\mathbf{F}(x, y, z) = \langle x, y, 1 \rangle,$$

compute the flux across S .

Tasks

1. Derive the upward vector surface element.
2. Evaluate the flux in polar coordinates.
3. Explain how positive and negative contributions cancel.

Question 2 – Solution

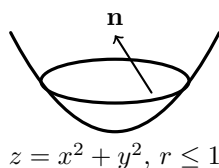
Strategy. Use the upward graph element $\langle -g_x, -g_y, 1 \rangle dA$ and then exploit radial symmetry.

Step 1: Vector area element For $g(x, y) = x^2 + y^2$,

$$\mathbf{n} dS = \langle -2x, -2y, 1 \rangle dA.$$

Thus

$$\mathbf{F} \cdot \mathbf{n} dS = (-2x^2 - 2y^2 + 1) dA = (1 - 2r^2) dA.$$



Step 2: Integrate

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^1 (1 - 2r^2) r dr d\theta \\ &= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{2} \right]_0^1 = \boxed{0}. \end{aligned}$$

Step 3: Locate the cancellation The normal component is positive for $r < 1/\sqrt{2}$ and negative for $r > 1/\sqrt{2}$. The inner contribution is

$$2\pi \left[\frac{r^2}{2} - \frac{r^4}{2} \right]_0^{1/\sqrt{2}} = \frac{\pi}{4},$$

while the outer annulus contributes $-\pi/4$.

Verification The exact cancellation explains why zero flux does not mean the field is tangent everywhere.

Question 3

Let S be only the curved side of the cylinder

$$x^2 + y^2 = 4, \quad 0 \leq z \leq 3,$$

oriented away from the z -axis. For $\mathbf{F} = \langle x, y, z^2 \rangle$, find the outward flux.

Tasks

1. Parametrize the curved side with the correct orientation.
2. Compute the oriented vector area element.
3. Evaluate and interpret the flux.

Question 3 – Solution

Strategy. Use angle and height parameters. The outward cross product is horizontal, so the vertical component of the field contributes nothing.

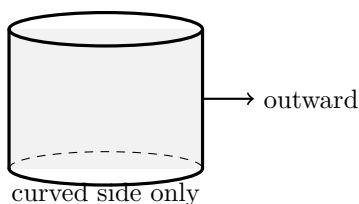
Step 1: Parametrize

$$\mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3.$$

The ordered cross product

$$\mathbf{r}_\theta \times \mathbf{r}_z = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle$$

points away from the axis.



Step 2: Form the integrand On S ,

$$\mathbf{F} = \langle 2 \cos \theta, 2 \sin \theta, z^2 \rangle,$$

and therefore

$$\mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) = 4 \cos^2 \theta + 4 \sin^2 \theta = 4.$$

Step 3: Integrate

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^3 4 \, dz \, d\theta = 24\pi.$$

Verification The flux density is constantly positive, so the field points outward through every point of the curved side. The z^2 component is tangent to that side and correctly disappears.

Question 4

For $a > 0$, let S_a be the sphere $x^2 + y^2 + z^2 = a^2$, oriented outward. Consider the inverse-square radial field

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Find $\mathbf{F} \cdot \mathbf{n}$ on S_a .
2. Compute the outward flux directly.
3. Explain why the result is independent of a .

Question 4 – Solution

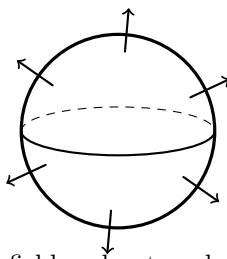
Strategy. On a centered sphere, the field and unit normal are parallel; only their magnitudes need to be compared.

Step 1: Normal component Write $\mathbf{x} = \langle x, y, z \rangle$. On S_a ,

$$\mathbf{n} = \frac{\mathbf{x}}{a}, \quad \mathbf{F} = \frac{\mathbf{x}}{a^3}.$$

Consequently,

$$\mathbf{F} \cdot \mathbf{n} = \frac{\mathbf{x} \cdot \mathbf{x}}{a^4} = \frac{1}{a^2}.$$



radial field and outward normal

Step 2: Integrate Since the sphere has area $4\pi a^2$,

$$\iint_{S_a} \mathbf{F} \cdot \mathbf{n} \, dS = \frac{1}{a^2} (4\pi a^2) = 4\pi.$$

Step 3: Interpret the scale Increasing the radius weakens the normal component by the factor a^{-2} while increasing spherical area by a^2 . These factors cancel exactly.

Verification The field is everywhere outward on S_a , so the positive sign is correct. The calculation is direct and uses no theorem about closed surfaces.

Question 5

Let S be the lateral cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \leq z \leq 2,$$

oriented away from the z -axis. For $\mathbf{F} = \langle 0, 0, z \rangle$, compute the flux across S .

Tasks

1. Parametrize the cone and choose the requested orientation.
2. Evaluate the oriented surface integral.
3. Address the singular parameter point at the cone's vertex.

Question 5 – Solution

Strategy. The normal pointing horizontally away from the axis also has a downward vertical component, which determines the sign of the flux.

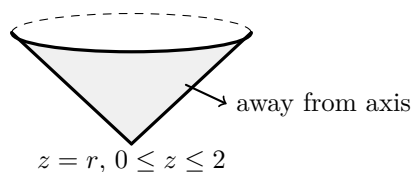
Step 1: Oriented cross product Use

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

The ordered product

$$\mathbf{r}_v \times \mathbf{r}_u = \langle u \cos v, u \sin v, -u \rangle$$

has a radial component pointing away from the axis.



Step 2: Integrate On the cone, $\mathbf{F} = \langle 0, 0, u \rangle$, so

$$\mathbf{F} \cdot (\mathbf{r}_v \times \mathbf{r}_u) = -u^2.$$

Therefore

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^2 -u^2 \, du \, dv \\ &= \boxed{-\frac{16\pi}{3}}. \end{aligned}$$

Step 3: Vertex The parametrization is singular at $u = 0$, but that single surface point has zero area. Equivalently, integrate over $\varepsilon \leq u \leq 2$ and let $\varepsilon \rightarrow 0^+$.

Verification The chosen normal has negative z -component while the field points upward, so a negative flux is required.

Question 6

Let E be the tetrahedron in the first octant bounded by the coordinate planes and $x + y + z = 1$. Let $S = \partial E$ carry the outward orientation, and let

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Compute the total outward flux directly by summing the four faces.

Tasks

1. Determine the flux through the three coordinate-plane faces.
2. Parametrize the slanted face with its outward orientation.
3. Sum the face contributions and verify the sign.

Question 6 – Solution

Strategy. The field has zero normal component on each coordinate face, leaving only the slanted triangular face.

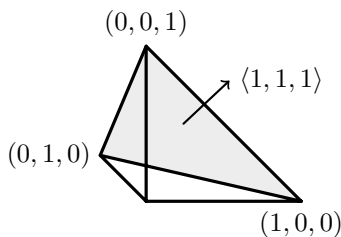
Step 1: Coordinate faces On $x = 0$, the outward normal is $-\mathbf{i}$ and $\mathbf{F} \cdot (-\mathbf{i}) = -x = 0$. Similarly, the flux densities on $y = 0$ and $z = 0$ vanish. These three fluxes are all zero.

Step 2: Slanted face Write $z = 1 - x - y$ over

$$D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}.$$

The outward side points away from the origin, and its vector element is

$$\mathbf{n} dS = \langle 1, 1, 1 \rangle dx dy.$$



On this face,

$$\mathbf{F} \cdot \langle 1, 1, 1 \rangle = x + y + z = 1.$$

Step 3: Integrate and sum

$$\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} dS = 0 + \iint_D 1 dA = \boxed{\frac{1}{2}}.$$

Verification The field points away from the origin and is tangent to the coordinate faces, so all nonzero flux must leave through the slanted face and must be positive.

Question 7

Let S be the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented outward. Compute the flux of the constant field $\mathbf{F} = \langle 0, 0, 1 \rangle$ across S .

Tasks

1. Express the hemisphere as a graph and find its upward vector element.
2. Evaluate the flux using projection onto the xy -plane.
3. State the result for the opposite orientation.

Question 7 – Solution

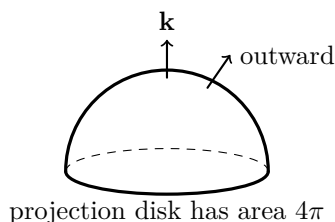
Strategy. A constant vertical field dotted with the upward graph element leaves exactly the projected area element.

Step 1: Graph orientation Write

$$z = g(x, y) = \sqrt{4 - x^2 - y^2}$$

over the disk $D : x^2 + y^2 \leq 4$. The outward orientation on the upper hemisphere is upward, so

$$\mathbf{n} dS = \langle -g_x, -g_y, 1 \rangle dA.$$



Step 2: Project Since $\mathbf{F} = \mathbf{k}$,

$$\mathbf{F} \cdot \mathbf{n} dS = \mathbf{k} \cdot \langle -g_x, -g_y, 1 \rangle dA = dA.$$

Thus

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D dA = \pi(2^2) = 4\pi.}$$

Step 3: Reverse orientation With the inward orientation the vector element changes sign, so the flux is $\boxed{-4\pi}$.

Verification The graph derivatives become unbounded at the equator, but their components are annihilated by the dot product with $\mathbf{k}$. Truncating to radius $R < 2$ and taking $R \rightarrow 2^-$ yields the same result.

Question 8

The Möbius band can be parametrized by

$$\mathbf{r}(u, v) = \left\langle \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \cos u, \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \sin u, \frac{v}{2} \sin \frac{u}{2} \right\rangle,$$

where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.

Tasks

1. Verify the seam identification $\mathbf{r}(0, v) = \mathbf{r}(2\pi, -v)$.
2. Compare the transverse tangent direction at the two identified edges.
3. Explain why a globally oriented flux integral is not defined on the band.

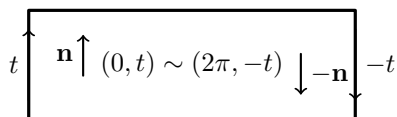
Question 8 – Solution

Strategy. Follow a normal direction once around the band and test whether it returns consistently at the identified seam.

Step 1: Identify the seam Direct substitution gives

$$\mathbf{r}(0, v) = \left\langle 1 + \frac{v}{2}, 0, 0 \right\rangle = \mathbf{r}(2\pi, -v).$$

Thus the left edge at height v is glued to the right edge at height $-v$.



the seam reverses the transverse direction

Step 2: Compare transverse tangents Along the centerline $v = 0$,

$$\mathbf{r}_v(0, 0) = \left\langle \frac{1}{2}, 0, 0 \right\rangle, \quad \mathbf{r}_v(2\pi, 0) = \left\langle -\frac{1}{2}, 0, 0 \right\rangle.$$

Also $\mathbf{r}_u(0, 0) = \mathbf{r}_u(2\pi, 0) = \langle 0, 1, 0 \rangle$. Hence

$$(\mathbf{r}_u \times \mathbf{r}_v)(0, 0) = \left\langle 0, 0, -\frac{1}{2} \right\rangle, \quad (\mathbf{r}_u \times \mathbf{r}_v)(2\pi, 0) = \left\langle 0, 0, \frac{1}{2} \right\rangle.$$

Step 3: Conclude The two parameter points represent the same surface point, yet continuation around the strip reverses the normal. Therefore no continuous global unit normal exists, and

a global oriented flux integral over the Möbius band is not defined.

Verification Cutting the band at the seam permits a local choice of normal, but gluing the seam back forces opposite choices, exposing the obstruction.

Question 9

A plane patch over the unit square has two parametrizations

$$\mathbf{r}(u, v) = \langle u, v, u + 2v \rangle, \quad \mathbf{s}(p, q) = \langle q, p, q + 2p \rangle.$$

For $\mathbf{F} = \langle 1, 0, 0 \rangle$, compare the flux values produced by the ordered cross products of the two parametrizations.

Tasks

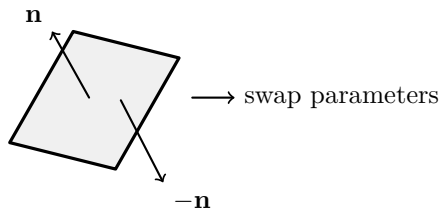
1. Show that $\mathbf{s}$ swaps the parameters of $\mathbf{r}$.
2. Compute both signed flux integrals.
3. Explain how to make the two computations represent the same geometric orientation.

Question 9 – Solution

Strategy. Swapping parameters reverses the cross product, so a vector surface integral detects a sign change that a scalar surface integral would not.

Step 1: Compare cross products Because $\mathbf{s}(p, q) = \mathbf{r}(q, p)$, the images are the same. However,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1, -2, 1 \rangle, \quad \mathbf{s}_p \times \mathbf{s}_q = \langle 1, 2, -1 \rangle.$$



Step 2: Compute signed fluxes Since $\mathbf{F} = \langle 1, 0, 0 \rangle$ and both parameter domains have area 1,

$$\iint \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv = \boxed{-1},$$

whereas

$$\iint \mathbf{F} \cdot (\mathbf{s}_p \times \mathbf{s}_q) \, dp \, dq = \boxed{1}.$$

Step 3: Match orientation If the orientation from $\mathbf{r}_u \times \mathbf{r}_v$ is prescribed geometrically, then the $\mathbf{s}$ calculation must use $-(\mathbf{s}_p \times \mathbf{s}_q)$. It then also gives -1 .

Verification The two raw answers are negatives, exactly as required because the ordered normals are negatives. Flux is invariant under reparametrization only after the geometric orientation is held fixed.

Question 10

Let S be the closed cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 2$, including its top and bottom disks, with outward orientation. For

$$\mathbf{F}(x, y, z) = \langle x, y, -2z \rangle,$$

compute the net outward flux directly, without using a general flux theorem.

Tasks

1. Compute the flux through the curved side.
2. Compute the flux through each cap with the correct outward normal.
3. Sum the contributions and interpret the cancellation.

Question 10 – Solution

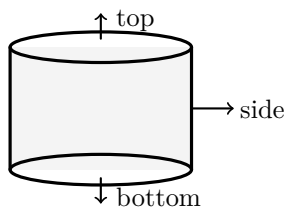
Strategy. Decompose the closed surface into three oriented pieces and track the outward normal on each one.

Step 1: Curved side Parametrize by

$$\mathbf{r}(\theta, z) = \langle \cos \theta, \sin \theta, z \rangle.$$

The outward vector element is $\langle \cos \theta, \sin \theta, 0 \rangle d\theta dz$. Its dot product with $\mathbf{F}$ is 1, so

$$\Phi_{\text{side}} = \int_0^{2\pi} \int_0^2 1 dz d\theta = 4\pi.$$



Step 2: Caps On the top disk $z = 2$, the outward normal is $\mathbf{k}$, and

$$\mathbf{F} \cdot \mathbf{k} = -2z = -4.$$

Thus $\Phi_{\text{top}} = -4\pi$. On the bottom disk $z = 0$, the outward normal is $-\mathbf{k}$, but the vertical field component is zero, so $\Phi_{\text{bottom}} = 0$.

Step 3: Net flux

$$\Phi_{\text{net}} = 4\pi - 4\pi + 0 = 0.$$

Verification The side carries 4π units outward while the top carries the same amount inward; the signed balance is zero even though neither contribution vanishes.

Question 1

Let C be the unit circle $x^2 + y^2 = 1$ in the plane $z = 0$, oriented counterclockwise when viewed from above. For

$$\mathbf{F}(x, y, z) = \langle -y, x, z \rangle,$$

evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem.

Tasks

1. Choose an oriented spanning surface and compute $\nabla \times \mathbf{F}$.
2. Apply Stokes' Theorem.
3. Verify the answer by direct parametrization of C .

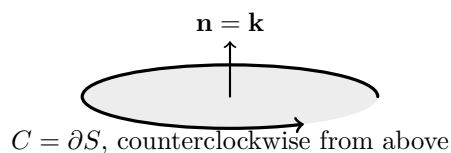
Question 1 – Solution

Strategy. Span the circle by its flat disk; the prescribed counterclockwise direction selects the upward normal.

Step 1: Curl and orientation

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & z \end{vmatrix} = \langle 0, 0, 2 \rangle.$$

Let S be the unit disk with $\mathbf{n} = \mathbf{k}$.



Step 2: Apply Stokes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_S 2 \, dA = 2\pi.$$

Step 3: Direct check With $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$,

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t, 0 \rangle = \mathbf{r}'(t).$$

Therefore $\int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}' \, dt = \int_0^{2\pi} 1 \, dt = 2\pi$.

Verification The boundary is counterclockwise from the tip of $\mathbf{k}$, so the orientation pairing is correct.

Question 2

Let C be the triangular boundary of the first-octant portion of the plane $x + y + z = 1$. Orient C consistently with the normal having positive components. For

$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Compute the curl and an oriented vector area element.
2. Evaluate the surface integral over the projected triangle.
3. Verify the result by integrating along the three edges.

Question 2 – Solution

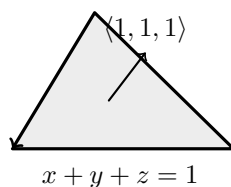
Strategy. Use the triangular plane itself; its graph vector element is constant and points in the prescribed direction.

Step 1: Curl and surface element

$$\nabla \times \mathbf{F} = \langle 1, 1, 1 \rangle.$$

Writing $z = 1 - x - y$ over $D = \{x, y \geq 0 : x + y \leq 1\}$ gives

$$\mathbf{n} dS = \langle 1, 1, 1 \rangle dx dy.$$



Step 2: Apply Stokes

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle dA \\ &= 3 \text{ area}(D) = \boxed{\frac{3}{2}}. \end{aligned}$$

Step 3: Edge check The orientation follows

$$(1, 0, 0) \rightarrow (0, 1, 0) \rightarrow (0, 0, 1) \rightarrow (1, 0, 0).$$

Parametrizing each edge by $0 \leq t \leq 1$ gives integrands $1 - t$, $1 - t$, and $1 - t$, respectively. Hence the three line integrals are each $1/2$, totaling $3/2$.

Verification Two successive directed edge vectors have cross product parallel to $\langle 1, 1, 1 \rangle$, confirming the boundary orientation.

Question 3

Let C be the equator of the upper hemisphere $x^2 + y^2 + z^2 = 9$, oriented as the boundary of the outward-oriented hemisphere. For

$$\mathbf{F} = \langle -y, x, z \rangle,$$

find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ without parametrizing the hemisphere.

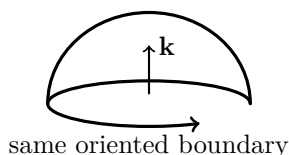
Tasks

1. Determine the induced direction of C as viewed from above.
2. Replace the hemisphere by a simpler spanning surface.
3. Evaluate the circulation and explain why the replacement is valid.

Question 3 – Solution

Strategy. Stokes' Theorem permits any smooth oriented spanning surface with the same oriented boundary; use the equatorial disk.

Step 1: Boundary orientation The outward normal on the upper hemisphere induces counterclockwise motion around the equator when viewed from $+z$.



Step 2: Choose the disk Let D be $x^2 + y^2 \leq 9$, $z = 0$, with upward normal. It has the same oriented boundary as the hemisphere. Since

$$\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle,$$

Stokes' Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 2 \, dA = 2(9\pi) = 18\pi.$$

Step 3: Explain the replacement Both the hemisphere and disk are orientable, share C , and lie where $\mathbf{F}$ has continuous derivatives. Each curl flux therefore equals the same boundary circulation.

Verification On the equator, $\mathbf{F} = \langle -y, x, 0 \rangle$ is tangent in the counterclockwise direction and has magnitude 3; multiplying by the circumference 6π also gives 18π .

Question 4

Let C be the intersection of the cylinder $x^2 + y^2 = 4$ with the plane $z = x + y$. Orient C consistently with the upward normal to the planar region it bounds. For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Identify a convenient spanning surface and its vector element.
2. Compute the curl flux.
3. Explain why the tilt of the curve does not alter the result.

Question 4 – Solution

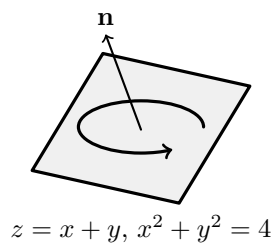
Strategy. Use the planar patch $z = x + y$ over the radius-2 disk; its vertical vector-area component is especially simple.

Step 1: Curl and vector area We have

$$\nabla \times \mathbf{F} = \langle 0, 0, 1 \rangle.$$

For $z = x + y$, the upward vector element is

$$\mathbf{n} dS = \langle -1, -1, 1 \rangle dA.$$



Step 2: Apply Stokes

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_{x^2+y^2 \leq 4} \langle 0, 0, 1 \rangle \cdot \langle -1, -1, 1 \rangle dA \\ &= \iint_{x^2+y^2 \leq 4} 1 dA = \boxed{4\pi}. \end{aligned}$$

Step 3: Interpret The curl is vertical, so only the vertical component of the oriented area vector matters. That component equals the area element of the horizontal projection, whose area is 4π .

Verification Reversing the curve would force the downward normal and change the answer to -4π .

Question 5

Let S be the annulus $1 \leq x^2 + y^2 \leq 4$ in the plane $z = 0$, oriented upward. Its boundary consists of an outer circle C_o and inner circle C_i . For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

evaluate the circulation around the fully oriented boundary ∂S .

Tasks

1. Determine the induced direction on each boundary component.
2. Apply Stokes' Theorem to the annulus.
3. Verify the result from the two separate circle integrals.

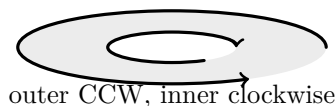
Question 5 – Solution

Strategy. For an upward annulus, the outer boundary is counterclockwise and the inner boundary is clockwise.

Step 1: Orient the boundary The right-hand rule gives

$$\partial S = C_o - C_i^{\text{CCW}}.$$

Equivalently, traverse the outer circle counterclockwise and the inner circle clockwise.



Step 2: Apply Stokes Since $\nabla \times \mathbf{F} = \mathbf{k}$,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S 1 \, dA = \pi(2^2 - 1^2) = 3\pi.$$

Step 3: Check components For a counterclockwise circle of radius R ,

$$\oint \mathbf{F} \cdot d\mathbf{r} = \pi R^2.$$

Thus the outer circle contributes 4π , while the clockwise inner circle contributes $-\pi$. Their sum is 3π .

Verification Assigning both circles counterclockwise would give 5π and violate the induced-boundary convention.

Question 6

Use Stokes' Theorem to derive an area formula for a positively oriented simple closed curve C in the xy -plane. Then apply it to the ellipse

$$C : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0.$$

Tasks

1. Choose a vector field with constant vertical curl.
2. Derive $A = \frac{1}{2} \oint_C (x \, dy - y \, dx)$.
3. Evaluate the formula for the ellipse.

Question 6 – Solution

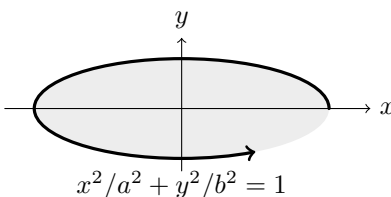
Strategy. Encode planar area as the curl flux of a field whose vertical curl is 1.

Step 1: Choose the field Let

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle.$$

Then $\nabla \times \mathbf{F} = \mathbf{k}$. If D is the region enclosed by positively oriented C , Stokes gives

$$\text{area}(D) = \iint_D 1 \, dA = \oint_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\frac{1}{2} \oint_C (x \, dy - y \, dx)}.$$



Step 2: Parametrize the ellipse Use

$$x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi.$$

Then

$$x \, dy - y \, dx = ab(\cos^2 t + \sin^2 t) \, dt = ab \, dt.$$

Therefore

$$\boxed{\text{area}(D) = \frac{1}{2} \int_0^{2\pi} ab \, dt = \pi ab}.$$

Verification Setting $a = b = R$ recovers the circle area πR^2 , including the correct scale and orientation sign.

Question 7

Let C be the unit circle in $z = 0$, counterclockwise from above. Two surfaces span it: the upper hemisphere S_1 and the paraboloid $S_2 : z = x^2 + y^2 - 1, x^2 + y^2 \leq 1$, both oriented to induce the given direction. For

$$\mathbf{F} = \langle -y, x, z \rangle,$$

compare the curl fluxes through S_1 and S_2 .

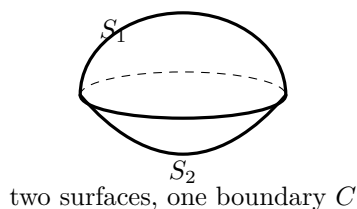
Tasks

1. Determine compatible orientations on both surfaces.
2. Evaluate one common boundary circulation or curl flux.
3. Explain why the two fluxes must agree.

Question 7 – Solution

Strategy. Use Stokes' Theorem as a surface-independence statement: each compatible curl flux equals the same circulation around C .

Step 1: Orient the surfaces Use the outward orientation on the upper hemisphere and the upward orientation on the paraboloid. Both induce counterclockwise motion on C viewed from above.



Step 2: Compute the common value Since

$$\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle,$$

we may use the upward unit disk D spanning C :

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 2 \, dA = 2\pi.$$

Consequently,

$$\boxed{\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 \, dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, dS = 2\pi.}$$

Step 3: Explain Stokes' Theorem ties curl flux to the oriented boundary, not to the particular spanning surface, provided the field is smooth throughout the relevant surfaces.

Verification Both surfaces project once onto the unit disk with positive vertical orientation, so integrating the vertical curl 2 directly would also give 2π on each.

Question 8

Let C be the boundary of the part of the plane $z = 2x - y$ above

$$0 \leq x \leq 1, \quad 0 \leq y \leq 2,$$

oriented consistently with the normal having positive z -component. Let

$$\mathbf{F} = \frac{1}{2} \langle 2z - 3y, 3x - z, y - 2x \rangle.$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Compute the curl of $\mathbf{F}$.
2. Find the oriented vector surface element for the plane.
3. Apply Stokes' Theorem and verify the orientation.

Question 8 – Solution

Strategy. The field was chosen to have constant curl, while the plane has a constant vector area element.

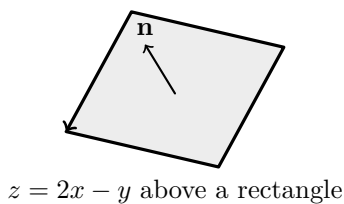
Step 1: Curl Direct differentiation gives

$$\nabla \times \mathbf{F} = \langle 1, 2, 3 \rangle.$$

Step 2: Surface element With $z = g(x, y) = 2x - y$,

$$\mathbf{n} dS = \langle -g_x, -g_y, 1 \rangle dA = \langle -2, 1, 1 \rangle dA.$$

Its third component is positive, as required.



Step 3: Apply Stokes The curl flux density is constant:

$$\langle 1, 2, 3 \rangle \cdot \langle -2, 1, 1 \rangle = 3.$$

The projected rectangle has area 2, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R 3 dA = 6.$$

Verification Reversing the traversal of all four edges would correspond to the negative normal and would change the value to -6 .

Question 9

Let S consist of the top and four side faces of the unit cube $0 \leq x, y, z \leq 1$, but not the bottom face. Give S the outward orientation. Its boundary C is the rim of the missing bottom square. For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem on the composite surface.

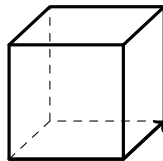
Tasks

1. Determine the induced direction of C as viewed from above.
2. Compute the curl flux through the five faces.
3. Verify the result using the missing bottom square.

Question 9 – Solution

Strategy. The curl is vertical, so the four vertical faces contribute zero; only the top face remains.

Step 1: Boundary direction The outward orientation on the four sides induces counterclockwise traversal of the bottom rim when viewed from above.



C : rim of the missing bottom

Step 2: Flux through S Since

$$\nabla \times \mathbf{F} = \mathbf{k},$$

its dot product with every horizontal side-face normal is zero. On the top, the outward normal is $\mathbf{k}$, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0 + \iint_{[0,1]^2} 1 \, dA = 1.$$

Step 3: Replace the surface The upward-oriented bottom square has the same counterclockwise boundary C . Applying Stokes to that single square also gives $\iint 1 \, dA = 1$.

Verification Using the outward normal $-\mathbf{k}$ on the missing bottom face would induce the opposite boundary direction; it is not the compatible replacement for the stated C .

Question 10

On $\mathbb{R}^3$ minus the z -axis, define

$$\mathbf{F}(x, y, z) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right\rangle.$$

Let C be the unit circle in $z = 0$, counterclockwise from above.

Tasks

1. Compute $\nabla \times \mathbf{F}$ where the field is defined.
2. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly.
3. Explain why applying Stokes to the full disk is invalid, and reconcile the result using a punctured disk.

Question 10 – Solution

Strategy. The apparent contradiction “zero curl but nonzero circulation” signals a failed hypothesis: the field is singular on every full disk spanning C .

Step 1: Curl and direct integral Differentiation gives

$$\nabla \times \mathbf{F} = \mathbf{0} \quad (x^2 + y^2 > 0).$$

On C , let $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t, 0 \rangle = \mathbf{r}'(t),$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 \, dt = 2\pi.$$



Step 2: Locate the failed hypothesis The field is undefined at the origin, which lies on the full spanning disk. Thus $\mathbf{F}$ does not have continuous derivatives on an open set containing that disk, and Stokes' Theorem cannot be applied there.

Step 3: Use an annulus Remove a radius- ε disk. On the resulting annulus the curl is zero, so Stokes gives

$$0 = \oint_{C_{\text{outer}}} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_{\text{inner, clockwise}}} \mathbf{F} \cdot d\mathbf{r} = 2\pi - 2\pi.$$

Verification The inner boundary carries exactly the circulation needed to account for the puncture; no contradiction remains.

Question 1

Let S be the sphere $x^2 + y^2 + z^2 = 4$, oriented outward, and let

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Tasks

1. Compute $\nabla \cdot \mathbf{F}$.
2. Use the Divergence Theorem to find the outward flux through S .
3. Verify the result directly from the geometry of the sphere.

Question 1 – Solution

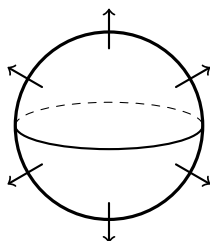
Strategy. Replace the closed-surface flux by a volume integral over the radius-2 ball.

Step 1: Divergence

$$\boxed{\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3}.$$

If E is the ball bounded by S , then

$$\text{vol}(E) = \frac{4}{3}\pi(2^3) = \frac{32\pi}{3}.$$



outward flux through $r = 2$

Step 2: Apply the theorem

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E 3 \, dV = 3 \left(\frac{32\pi}{3} \right) = 32\pi}.$$

Step 3: Direct check On the sphere, the outward unit normal is $\mathbf{n} = \langle x, y, z \rangle / 2$. Therefore

$$\mathbf{F} \cdot \mathbf{n} = \frac{x^2 + y^2 + z^2}{2} = 2.$$

Multiplying by the surface area $4\pi(2^2) = 16\pi$ gives $2(16\pi) = 32\pi$.

Verification The radial field points outward everywhere, so the positive sign is required.

Question 2

Let $E = [0, 1] \times [0, 2] \times [0, 3]$ and let $S = \partial E$ have outward orientation. For

$$\mathbf{F}(x, y, z) = \langle x^2, yz, z \rangle,$$

compute the total outward flux through the six faces.

Tasks

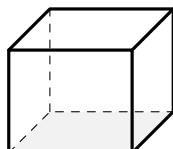
1. Find the divergence.
2. Evaluate the resulting triple integral by separating its terms.
3. Explain why this is preferable to six direct surface integrals.

Question 2 – Solution

Strategy. The box is closed and the field is smooth, so one volume integral replaces six oriented face calculations.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = 2x + z + 1.$$



$1 \times 2 \times 3$ box

Step 2: Integrate term by term The three contributions over E are

$$\begin{aligned}\iiint_E 2x \, dV &= \left(\int_0^1 2x \, dx \right) (2)(3) = 6, \\ \iiint_E z \, dV &= (1)(2) \int_0^3 z \, dz = 9, \\ \iiint_E 1 \, dV &= (1)(2)(3) = 6.\end{aligned}$$

Hence

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E (2x + z + 1) \, dV = 21.$$

Step 3: Interpret A direct method would require a different outward normal and parameter domain on every face. The divergence automatically combines all six signed contributions.

Verification The divergence is positive throughout E , so a positive net outward flux is consistent.

Question 3

Let E be the solid cylinder $x^2 + y^2 \leq 4$, $0 \leq z \leq 3$, and let $S = \partial E$ be oriented outward. For

$$\mathbf{F}(x, y, z) = \langle xz, yz, z^2 \rangle,$$

find the net outward flux and verify it by splitting S into its side and caps.

Tasks

1. Apply the Divergence Theorem in cylindrical coordinates.
2. Compute the flux through the curved side and both caps directly.
3. Compare the totals.

Question 3 – Solution

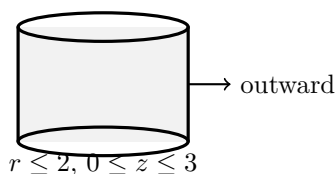
Strategy. First obtain the net flux from the divergence, then use the simple normals of a cylinder as an independent check.

Step 1: Volume integral Since

$$\nabla \cdot \mathbf{F} = z + z + 2z = 4z,$$

we obtain

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^2 \int_0^3 4z \, r \, dz \, dr \, d\theta \\ &= \boxed{72\pi}. \end{aligned}$$



Step 2: Curved side On $r = 2$, the outward vector element is $\langle 2 \cos \theta, 2 \sin \theta, 0 \rangle \, d\theta \, dz$. Thus the dot product is $4z$, and

$$\Phi_{\text{side}} = \int_0^{2\pi} \int_0^3 4z \, dz \, d\theta = 36\pi.$$

Step 3: Caps On the top $z = 3$, $\mathbf{F} \cdot \mathbf{k} = 9$, so $\Phi_{\text{top}} = 9(4\pi) = 36\pi$. On the bottom, $z = 0$, so $\Phi_{\text{bottom}} = 0$.

$$\boxed{\Phi_{\text{side}} + \Phi_{\text{top}} + \Phi_{\text{bottom}} = 72\pi}.$$

Verification The direct total matches the volume-integral result exactly.

Question 4

Let E be the spherical shell $a \leq \sqrt{x^2 + y^2 + z^2} \leq b$, where $0 < a < b$, with outward orientation on its entire boundary. For

$$\mathbf{F} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}},$$

find the net flux through the shell boundary.

Tasks

1. Compute the divergence in the shell.
2. Track the outward normal on both spherical boundary components.
3. Reconcile the separate nonzero fluxes with the net result.

Question 4 – Solution

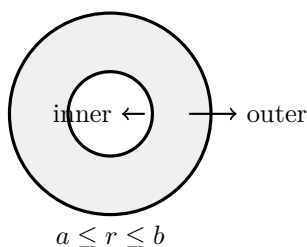
Strategy. The origin is excluded from the shell, so the field is smooth there and has zero divergence; the inner normal points toward the origin.

Step 1: Apply the theorem For $r > 0$,

$$\nabla \cdot \left(\frac{\langle x, y, z \rangle}{r^3} \right) = 0.$$

Therefore

$$\boxed{\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E 0 \, dV = 0}.$$



Step 2: Outer sphere At $r = b$, $\mathbf{F}$ points in the outward-normal direction with normal component $1/b^2$. Hence

$$\Phi_{r=b} = \frac{1}{b^2}(4\pi b^2) = 4\pi.$$

Step 3: Inner sphere The outward normal for the shell at $r = a$ points into the cavity, namely $-\mathbf{e}_r$. Thus

$$\Phi_{r=a} = -\frac{1}{a^2}(4\pi a^2) = -4\pi.$$

The two boundary components sum to zero.

Verification The cancellation depends on reversing the normal at the inner boundary; using the radial normal on both spheres would not describe the shell's outward orientation.

Question 5

Let S be the curved upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented outward. For

$$\mathbf{F}(x, y, z) = \langle x, y, z + 1 \rangle,$$

find the flux through S by closing it with its base disk.

Tasks

1. Apply the Divergence Theorem to the closed upper half-ball.
2. Compute the flux through the base disk with its outward normal.
3. Isolate and verify the curved-surface flux.

Question 5 – Solution

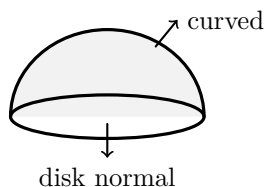
Strategy. Add the base disk, compute the total closed flux, and subtract the disk contribution with careful attention to its downward normal.

Step 1: Closed flux The divergence is

$$\nabla \cdot \mathbf{F} = 3.$$

The upper half-ball has volume $\frac{1}{2} \cdot \frac{4}{3}\pi(2^3) = 16\pi/3$. Therefore

$$\Phi_{\text{closed}} = 3 \left(\frac{16\pi}{3} \right) = 16\pi.$$



Step 2: Base disk On $z = 0$, the outward normal for the half-ball is $-\mathbf{k}$. Since $\mathbf{F} = \langle x, y, 1 \rangle$ there,

$$\Phi_{\text{disk}} = \iint_D (-1) dA = -4\pi.$$

Step 3: Curved part Because $\Phi_{\text{closed}} = \Phi_S + \Phi_{\text{disk}}$,

$$\boxed{\Phi_S = 16\pi - (-4\pi) = 20\pi}.$$

Verification On the sphere, $\mathbf{F} = \langle x, y, z \rangle + \mathbf{k}$. The radial part contributes $2(8\pi) = 16\pi$ over the hemisphere, and the constant vertical part contributes its projected area 4π , totaling 20π .

Question 6

Let E be the first-octant tetrahedron $x + y + z \leq 1$, and let $S = \partial E$ be outward oriented. For

$$\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle,$$

compute the total outward flux.

Tasks

1. Find the divergence.
2. Evaluate the needed coordinate moments over the tetrahedron.
3. Check the answer directly on the slanted face.

Question 6 – Solution

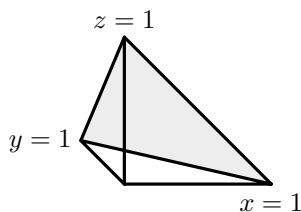
Strategy. Symmetry makes the three first moments equal; the coordinate-plane faces also provide a short direct check.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z.$$

The tetrahedron has volume $1/6$ and centroid $(1/4, 1/4, 1/4)$, so

$$\iiint_E x \, dV = \iiint_E y \, dV = \iiint_E z \, dV = \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}.$$

**Step 2: Apply the theorem**

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = 2 \left(\frac{1}{24} + \frac{1}{24} + \frac{1}{24} \right) = \frac{1}{4}.}$$

Step 3: Direct check On each coordinate face the corresponding field component is zero, so those fluxes vanish. On $z = 1 - x - y$, the outward vector element is $\langle 1, 1, 1 \rangle dA$, giving

$$\iint_D (x^2 + y^2 + (1 - x - y)^2) \, dA = \frac{1}{4}.$$

Verification The direct slanted-face integral agrees with the divergence result and is positive, as expected in the first octant.

Question 7

Let E be the solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Its closed boundary is oriented outward. For

$$\mathbf{F}(x, y, z) = \langle 0, 0, z \rangle,$$

compute the net outward flux and verify the contributions from the two boundary pieces.

Tasks

1. Describe the cylindrical-coordinate bounds for E .
2. Apply the Divergence Theorem.
3. Compute the plane and paraboloid fluxes separately.

Question 7 – Solution

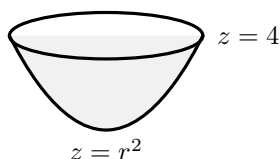
Strategy. The divergence equals 1, so the net flux is the volume; the direct check tests the downward orientation on the paraboloid.

Step 1: Bounds and volume The surfaces meet at $r = 2$, and

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad r^2 \leq z \leq 4.$$

Since $\nabla \cdot \mathbf{F} = 1$,

$$\Phi_{\text{net}} = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dz \, dr \, d\theta = \boxed{8\pi}.$$



Step 2: Top plane On $z = 4$, the outward normal is $\mathbf{k}$, so

$$\Phi_{\text{top}} = \iint_{r \leq 2} 4 \, dA = 16\pi.$$

Step 3: Paraboloid The solid lies above the paraboloid, so its outward vector element is downward:

$$\mathbf{n} \, dS = \langle 2x, 2y, -1 \rangle \, dA.$$

Thus

$$\Phi_{\text{paraboloid}} = \iint_{r \leq 2} -r^2 \, dA = -2\pi \int_0^2 r^3 \, dr = -8\pi.$$

The direct total is $16\pi - 8\pi = 8\pi$.

Verification The sign of the lower contribution is negative because the field points upward while the lower outward normal points downward.

Question 8

For a constant a , let

$$\mathbf{F}_a(x, y, z) = \langle ax, 2y, 3z \rangle.$$

The measured outward flux through the unit sphere is 24π . Determine a .

Tasks

1. Express the divergence in terms of a .
2. Use the flux measurement and the unit-ball volume.
3. Verify the recovered value directly on the sphere.

Question 8 – Solution

Strategy. The divergence is constant, so the measured flux gives a linear equation for the unknown coefficient.

Step 1: Apply the theorem

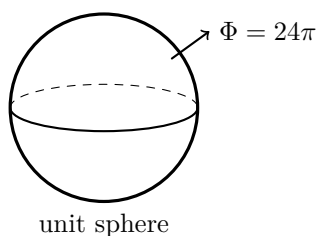
$$\nabla \cdot \mathbf{F}_a = a + 2 + 3 = a + 5.$$

The unit ball has volume $4\pi/3$, so

$$24\pi = (a + 5) \frac{4\pi}{3}.$$

Therefore $a + 5 = 18$, and

$$\boxed{a = 13}.$$



Step 2: Direct verification On the unit sphere, $\mathbf{n} = \langle x, y, z \rangle$, so with $a = 13$,

$$\mathbf{F}_{13} \cdot \mathbf{n} = 13x^2 + 2y^2 + 3z^2.$$

By spherical symmetry,

$$\iint_S x^2 dS = \iint_S y^2 dS = \iint_S z^2 dS = \frac{4\pi}{3}.$$

Hence the direct flux is

$$(13 + 2 + 3) \frac{4\pi}{3} = 24\pi.$$

Verification The recovered parameter reproduces the stated measurement by both methods.

Question 9

On $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, consider

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Let S be the outward-oriented unit sphere.

Tasks

1. Compute the flux through S directly.
2. Explain why zero divergence away from the origin does not permit applying the Divergence Theorem to the full unit ball.
3. Reconcile the result using a punctured ball.

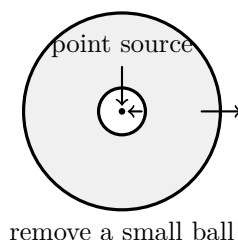
Question 9 – Solution

Strategy. The nonzero flux is caused by a singularity inside the sphere; remove a small ball so the theorem's hypotheses hold.

Step 1: Direct flux On the unit sphere, $\mathbf{F} = \mathbf{n}$, so

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S 1 \, dS = 4\pi}.$$

Although $\nabla \cdot \mathbf{F} = 0$ for $r > 0$, the field is undefined at the origin and is not smooth throughout the full ball.



Step 2: Puncture the ball Remove the ball $r < \varepsilon$. On the shell $\varepsilon \leq r \leq 1$, the divergence is zero, so the total outward flux is zero.

Step 3: Track both boundaries The outer flux is 4π . On $r = \varepsilon$, the shell's outward normal is $-\mathbf{e}_r$, while $\mathbf{F} = \mathbf{e}_r/\varepsilon^2$. Thus

$$\Phi_{r=\varepsilon} = -\frac{1}{\varepsilon^2}(4\pi\varepsilon^2) = -4\pi.$$

Therefore $4\pi - 4\pi = 0$, consistent with the theorem on the punctured region.

Verification The inner flux remains -4π as $\varepsilon \rightarrow 0^+$, recording the enclosed point source rather than disappearing.

Question 10

Let E be a bounded solid with piecewise smooth closed boundary S and outward unit normal $\mathbf{n}$.

Tasks

1. Use the Divergence Theorem with a suitable vector field to prove $\text{vol}(E) = \frac{1}{3} \iint_S \langle x, y, z \rangle \cdot \mathbf{n} \, dS$.
2. Apply the identity to a right circular cone of base radius R and height h , with vertex at the origin and base in the plane $z = h$.
3. Explain why the lateral surface contributes zero.

Question 10 – Solution

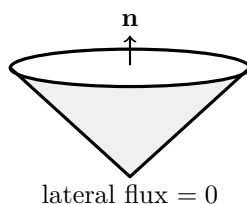
Strategy. Choose a field with divergence 1, then exploit the fact that the cone's radial position vector is tangent to its lateral generators.

Step 1: Derive the identity Let

$$\mathbf{G} = \frac{1}{3}\langle x, y, z \rangle.$$

Then $\nabla \cdot \mathbf{G} = 1$, so

$$\text{vol}(E) = \iiint_E 1 \, dV = \iint_S \mathbf{G} \cdot \mathbf{n} \, dS = \boxed{\frac{1}{3} \iint_S \langle x, y, z \rangle \cdot \mathbf{n} \, dS}.$$



Step 2: Lateral surface Every ray from the origin along the cone is tangent to the lateral surface. Hence $\langle x, y, z \rangle$ is tangent there and

$$\langle x, y, z \rangle \cdot \mathbf{n} = 0.$$

Step 3: Base On the base, $z = h$ and $\mathbf{n} = \mathbf{k}$, so the dot product equals h . Therefore

$$\boxed{\text{vol}(E) = \frac{1}{3} \iint_{\text{base}} h \, dA = \frac{1}{3} h (\pi R^2) = \frac{1}{3} \pi R^2 h}.$$

Verification The formula has the expected “one-third base times height” form and correct cubic units.