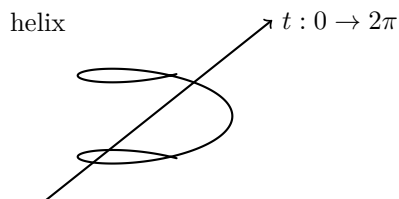


Question 1

A particle follows the helix $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$ for $0 \leq t \leq 2\pi$. Find the exact distance traveled. Then explain why the vertical rise alone is not the distance.

Question 1 – Solution

Strategy Distance traveled is the integral of speed, not the net change in any one coordinate.



Speed

$$\vec{r}'(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \|\vec{r}'(t)\| = \sqrt{9 \sin^2 t + 9 \cos^2 t + 16} = 5.$$

Arc length Therefore

$$L = \int_0^{2\pi} 5 \, dt = 10\pi.$$

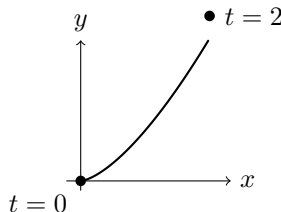
Interpretation The vertical rise is $4(2\pi) = 8\pi$, but the particle simultaneously travels around the cylinder. The velocity has perpendicular horizontal and vertical components of magnitudes 3 and 4, producing total speed 5.

Question 2

Find the exact arc length of $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3 \rangle$ on $0 \leq t \leq 2$. Show the algebraic step that makes the integral elementary.

Question 2 – Solution

Strategy Differentiate first, factor the speed, and use $u = 1 + t^2$.



Speed

$$\vec{r}'(t) = \langle 2t, 2t^2 \rangle, \quad \|\vec{r}'(t)\| = 2t\sqrt{1+t^2},$$

where $t \geq 0$ allows $\sqrt{t^2} = t$.

Integration

$$L = \int_0^2 2t\sqrt{1+t^2} dt = \frac{2}{3} \left[(1+t^2)^{3/2} \right]_0^2 = \boxed{\frac{2}{3}(5\sqrt{5} - 1)}.$$

Check Both coordinate functions increase. The straight-line distance between the endpoints is

$$\sqrt{4^2 + (16/3)^2} = \frac{20}{3}.$$

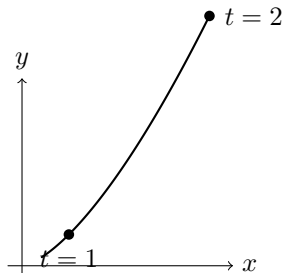
The computed curve length is larger, as it must be.

Question 3

For $\vec{r}(t) = \langle t^2, \frac{2}{3}t^3, 0 \rangle$, find the arc length from the point corresponding to $t = 1$ to the point corresponding to $t = 2$. Also find the chord length and compare the two quantities.

Question 3 – Solution

Strategy Use the same speed structure as in Question 2, but retain the nonzero lower limit.



Arc length

$$L = \int_1^2 2t\sqrt{1+t^2} dt = \boxed{\frac{2}{3}(5\sqrt{5} - 2\sqrt{2})}.$$

Chord The endpoints are $P = (1, 2/3, 0)$ and $Q = (4, 16/3, 0)$, so

$$\|Q - P\| = \sqrt{3^2 + \left(\frac{14}{3}\right)^2} = \boxed{\frac{\sqrt{277}}{3}}.$$

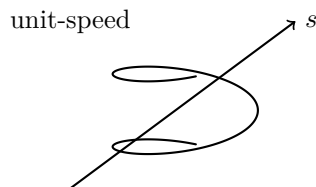
Comparison Numerically, $L \approx 5.568$ while the chord is about 5.548. The curve length is slightly larger, as required; equality would occur only for a straight segment traversed without reversal.

Question 4

Let $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$ with starting point at $t = 0$. Find its arc-length function $s(t)$ and reparameterize the helix by arc length. Verify directly that the new parameterization has unit speed.

Question 4 – Solution

Strategy Integrate speed from the stated base point, solve for t , and substitute.



Arc-length coordinate From Question 1, $\|\vec{r}'(t)\| = 5$. Hence $s(t) = \int_0^t 5 \, du = 5t$, so $t = s/5$.

Reparameterization

$$\boxed{\vec{R}(s) = \langle 3 \cos(s/5), 3 \sin(s/5), 4s/5 \rangle}, \quad s \geq 0.$$

Verification

$$\vec{R}'(s) = \left\langle -\frac{3}{5} \sin(s/5), \frac{3}{5} \cos(s/5), \frac{4}{5} \right\rangle, \quad \|\vec{R}'(s)\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1.$$

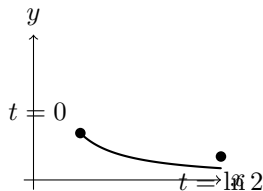
Thus an increase of one unit in s represents exactly one unit of distance along the helix.

Question 5

Evaluate exactly the length of $\vec{r}(t) = \langle e^t, e^{-t}, \sqrt{2}t \rangle$ for $0 \leq t \leq \ln 2$. The square root in the speed appears complicated; identify its hidden perfect square.

Question 5 – Solution

Strategy Expand the squared speed and recognize $(e^t + e^{-t})^2$.



Speed

$$\|\vec{r}'(t)\|^2 = e^{2t} + e^{-2t} + 2 = (e^t + e^{-t})^2.$$

Both terms are positive, so $\|\vec{r}'(t)\| = e^t + e^{-t}$.

Length

$$L = \int_0^{\ln 2} (e^t + e^{-t}) dt = [e^t - e^{-t}]_0^{\ln 2} = \boxed{\frac{3}{2}}.$$

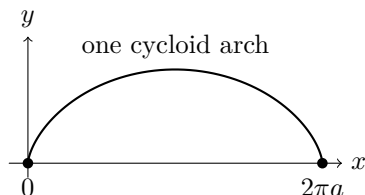
Insight Simplifying the norm before choosing a substitution is decisive here; treating the radical as generic would hide the elementary antiderivative.

Question 6

One arch of a cycloid is parameterized by $\vec{r}(t) = \langle a(t - \sin t), a(1 - \cos t) \rangle$, $0 \leq t \leq 2\pi$, where $a > 0$. Prove that its length is $8a$. Explain the absolute-value issue in simplifying the speed.

Question 6 – Solution

Strategy Use half-angle identities and determine the sign of $\sin(t/2)$ on the interval.



Speed

$$\vec{r}' = a \langle 1 - \cos t, \sin t \rangle, \quad \|\vec{r}'\| = a\sqrt{2 - 2\cos t} = 2a \left| \sin \frac{t}{2} \right|.$$

For $0 \leq t \leq 2\pi$, $\sin(t/2) \geq 0$, so the absolute value may be removed.

Length

$$L = \int_0^{2\pi} 2a \sin \frac{t}{2} dt = \left[-4a \cos \frac{t}{2} \right]_0^{2\pi} = \boxed{8a}.$$

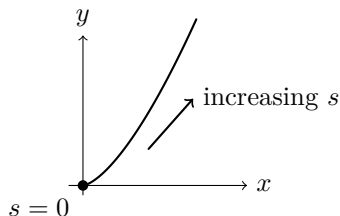
Endpoint note The velocity vanishes at the two cusps, but the speed is continuous and integrable, so the arch still has a well-defined finite length.

Question 7

For $t \geq 0$, let $\vec{r}(t) = \langle \frac{1}{2}t^2, \frac{1}{3}t^3 \rangle$. Construct the arc-length coordinate measured from $t = 0$, invert it explicitly, and give a unit-speed parameterization $\vec{R}(s)$.

Question 7 – Solution

Strategy The factor $t \geq 0$ makes the speed easy to integrate; then solve algebraically for t^2 .



Arc-length function

$$\|\vec{r}'(t)\| = \|\langle t, t^2 \rangle\| = t\sqrt{1+t^2}, \quad s(t) = \frac{1}{3}((1+t^2)^{3/2} - 1).$$

Inverse From $(1+t^2)^{3/2} = 3s+1$,

$$t(s) = \sqrt{(3s+1)^{2/3} - 1}.$$

Unit-speed form With $u = \sqrt{(3s+1)^{2/3} - 1}$,

$$\boxed{\vec{R}(s) = \left\langle \frac{1}{2}u^2, \frac{1}{3}u^3 \right\rangle}, \quad s \geq 0.$$

Because s was defined by integrating speed and is strictly increasing for $t > 0$, the chain rule gives $\|d\vec{R}/ds\| = 1$ there; the right-hand unit-speed limit also exists at $s = 0$.

Question 8

Suppose $\vec{q}(u) = \vec{r}(\phi(u))$ traces the same regular curve, where ϕ is continuously differentiable and strictly monotone. Prove that the arc length is unchanged by this reparameterization, including the case in which orientation is reversed.

Question 8 – Solution

Strategy Apply the chain rule to speed, then use substitution while carefully retaining the absolute value.

same trace, opposite orientations



Transformed speed

$$\vec{q}'(u) = \vec{r}'(\phi(u))\phi'(u), \quad \|\vec{q}'(u)\| = \|\vec{r}'(\phi(u))\| |\phi'(u)|.$$

Increasing case If $\phi' \geq 0$ and $\phi(\alpha) = a, \phi(\beta) = b$, substitution gives

$$\int_{\alpha}^{\beta} \|\vec{q}'(u)\| du = \int_a^b \|\vec{r}'(t)\| dt.$$

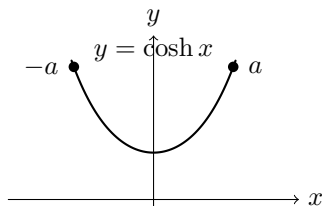
Decreasing case If $\phi' \leq 0$, then $\phi(\alpha) = b$ and $\phi(\beta) = a$. Since $|\phi'| = -\phi'$, reversing the substituted limits again produces $\int_a^b \|\vec{r}'(t)\| dt$. Thus length depends on the traced curve, not its orientation or timing.

Question 9

The catenary $\vec{r}(t) = \langle t, \cosh t \rangle$ is restricted to $-a \leq t \leq a$, where $a > 0$. Find its exact length in terms of a . Then determine the value of a that makes the length equal to 6.

Question 9 – Solution

Strategy Use the identity $1 + \sinh^2 t = \cosh^2 t$ and the positivity of $\cosh t$.



Length

$$\|\vec{r}'(t)\| = \sqrt{1 + \sinh^2 t} = \cosh t, \quad L = \int_{-a}^a \cosh t \, dt = 2 \sinh a.$$

Therefore $\boxed{L = 2 \sinh a}$.

Prescribed length Setting $2 \sinh a = 6$ gives $\sinh a = 3$, hence

$$\boxed{a = \operatorname{arsinh}(3) = \ln(3 + \sqrt{10})}.$$

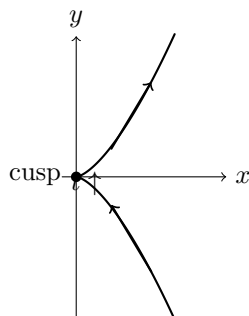
Symmetry The even speed makes the two halves have equal length; equivalently, $L = 2 \int_0^a \cosh t \, dt$.

Question 10

Consider $\vec{r}(t) = \langle t^2, t^3 \rangle$ on $-1 \leq t \leq 1$. The parameterization is not regular at $t = 0$. Determine the total arc length exactly and explain why one must split the integral or preserve an absolute value.

Question 10 – Solution

Strategy The speed contains $\sqrt{t^2} = |t|$; use symmetry across the cusp.



Speed

$$\vec{r}'(t) = \langle 2t, 3t^2 \rangle, \quad \|\vec{r}'(t)\| = |t|\sqrt{4 + 9t^2}.$$

Length

$$L = 2 \int_0^1 t\sqrt{4 + 9t^2} dt = \frac{2}{27} \left[(4 + 9t^2)^{3/2} \right]_0^1 = \boxed{\frac{2}{27}(13\sqrt{13} - 8)}.$$

Why care about $|t|$? Replacing $|t|$ by t on the full symmetric interval would make the integrand odd and falsely give zero. Although velocity vanishes at the cusp, each half is smooth and their nonnegative lengths add.