

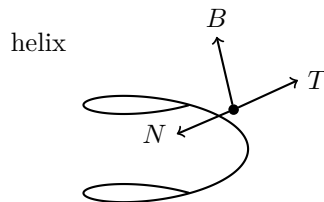
Question 1

For the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, find the unit tangent $\mathbf{T}$, principal unit normal $\mathbf{N}$, and binormal $\mathbf{B}$ for all t . Evaluate the frame at $t = 0$.

Calculus II Section 6-8: Tangent, Normal and Binormal Vectors

Question 1 – Solution

Strategy Normalize $\vec{r}'$, normalize $\mathbf{T}'$, then use $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.



Frame Since $\|\vec{r}'\| = \sqrt{2}$,

$$\mathbf{T} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle, \quad \mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle,$$

and

$$\mathbf{B} = \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle.$$

At $t = 0$ $\mathbf{T} = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle, \mathbf{N} = \langle -1, 0, 0 \rangle, \mathbf{B} = \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle.$

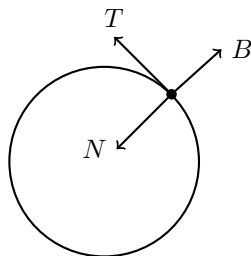
Verification The three vectors are unit, mutually orthogonal, and $\mathbf{T} \times \mathbf{N} = \mathbf{B}$.

Question 2

For the circle $\vec{r}(t) = \langle a \cos t, a \sin t, h \rangle$ with $a > 0$, compute $\mathbf{T}, \mathbf{N}, \mathbf{B}$. Explain why $\mathbf{N}$ is independent of a and why $\mathbf{B}$ is constant.

Question 2 – Solution

Strategy The parameter changes angular position while a changes only the circle's scale.



Computation $\vec{r}' = a \langle -\sin t, \cos t, 0 \rangle$, so

$$\mathbf{T} = \langle -\sin t, \cos t, 0 \rangle, \quad \mathbf{N} = \langle -\cos t, -\sin t, 0 \rangle, \quad \mathbf{B} = \langle 0, 0, 1 \rangle.$$

Interpretation The principal normal points radially inward regardless of radius. The curve lies in the fixed horizontal plane $z = h$, so the right-handed binormal is its constant upward normal.

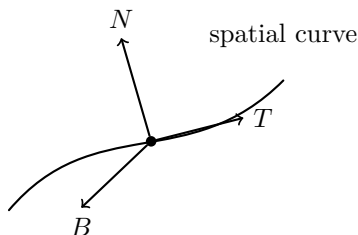
Question 3

Find the Frenet frame for $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$. Use the cross-product method for $\mathbf{B}$ and verify the orientation.

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Question 3 – Solution

Strategy Use $\mathbf{T} = \vec{r}' / \|\vec{r}'\|$, $\mathbf{B} = (\vec{r}' \times \vec{r}'') / \|\vec{r}' \times \vec{r}''\|$, and $\mathbf{N} = \mathbf{B} \times \mathbf{T}$.



Derivatives At $t = 1$, $\vec{r}' = \langle 1, 2, 3 \rangle$ and $\vec{r}'' = \langle 0, 2, 6 \rangle$. Thus

$$\mathbf{T} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle, \quad \mathbf{B} = \frac{1}{\sqrt{19}} \langle 3, -3, 1 \rangle.$$

Normal Therefore

$$\mathbf{N} = \mathbf{B} \times \mathbf{T} = \frac{1}{\sqrt{266}} \langle -11, -8, 9 \rangle.$$

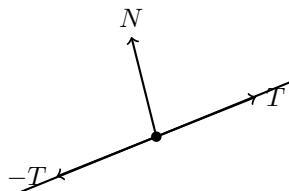
Verification Direct cross multiplication gives $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ and all norms equal 1.

Question 4

A regular curve $\vec{r}(t)$ with nonzero curvature is reparameterized with the opposite orientation by $\vec{q}(u) = \vec{r}(\phi(u))$, where $\phi'(u) < 0$. Determine how $\mathbf{T}, \mathbf{N}, \mathbf{B}$ change. Give a geometric explanation.

Question 4 – Solution

Strategy Apply the chain rule and track the sign introduced by the decreasing parameter.



Tangent Since $\vec{q}' = \vec{r}'(\phi)\phi'$, normalization gives $\mathbf{T}_q = -\mathbf{T}_r$.

Normal Differentiating $\mathbf{T}_q = -\mathbf{T}_r(\phi)$ with respect to u introduces another negative factor ϕ' , so its direction agrees with $\mathbf{N}_r$. Hence $\mathbf{N}_q = \mathbf{N}_r$.

Binormal $\mathbf{B}_q = \mathbf{T}_q \times \mathbf{N}_q = -\mathbf{B}_r$. Thus

$$\boxed{(\mathbf{T}, \mathbf{N}, \mathbf{B}) \mapsto (-\mathbf{T}, \mathbf{N}, -\mathbf{B})}.$$

The bending side is geometric, while forward and right-handed directions reverse.

Question 5

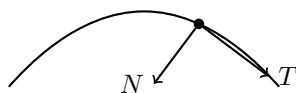
Let $\vec{r}(t) = \langle t, t^2, 0 \rangle$. Find $\mathbf{T}, \mathbf{N}, \mathbf{B}$ for all t . Explain why the binormal is constant and identify the direction in which the curve bends.

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Question 5 – Solution

Strategy Normalize $\vec{r}'$, differentiate $\mathbf{T}$, and use the planar geometry as a check.

B perpendicular to curve's plane



Tangent and normal With $w = \sqrt{1 + 4t^2}$,

$$\mathbf{T} = \frac{1}{w} \langle 1, 2t, 0 \rangle, \quad \mathbf{N} = \frac{1}{w} \langle -2t, 1, 0 \rangle.$$

Consequently $\boxed{\mathbf{B} = \langle 0, 0, 1 \rangle}$.

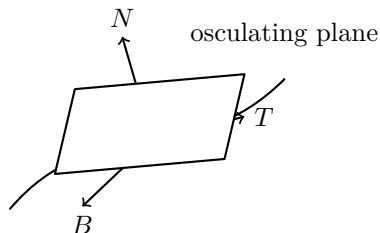
Interpretation The curve is contained in the xy -plane, whose oriented unit normal is constant. The normal has positive y -component for all t , so the upward-opening parabola bends toward its concave side.

Question 6

For $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at $t = 0$, find equations of the normal, osculating, and rectifying planes. State which Frenet vectors span each plane.

Question 6 – Solution

Strategy Use the frame from the helix and choose the Frenet vector perpendicular to each named plane.



Frame and point At $t = 0$, $P = (1, 0, 0)$, $\mathbf{T} \parallel \langle 0, 1, 1 \rangle$, $\mathbf{N} = \langle -1, 0, 0 \rangle$, and $\mathbf{B} \parallel \langle 0, -1, 1 \rangle$.

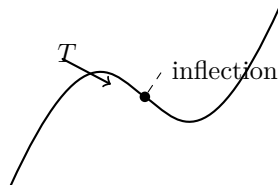
Planes The normal plane is perpendicular to $\mathbf{T}$: $y + z = 0$, and is spanned by $\mathbf{N}, \mathbf{B}$. The osculating plane is perpendicular to $\mathbf{B}$: $-y + z = 0$, spanned by $\mathbf{T}, \mathbf{N}$. The rectifying plane is perpendicular to $\mathbf{N}$: $x = 1$, spanned by $\mathbf{T}, \mathbf{B}$.

Question 7

For $\vec{r}(t) = \langle t, t^3, 0 \rangle$, determine where the Frenet principal normal is undefined. Does the curve still have a tangent there? Analyze the limiting normals on the two sides.

Question 7 – Solution

Strategy Compute $\mathbf{T}'$ and distinguish loss of curvature from loss of tangent direction.



Tangent $\vec{r}'(t) = \langle 1, 3t^2, 0 \rangle$ never vanishes, so the curve is regular and has tangent $\mathbf{T}(0) = \langle 1, 0, 0 \rangle$.

Normal failure Differentiating the normalized tangent shows $\mathbf{T}'(0) = \vec{0}$, so $\mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\|$ is undefined at 0. For $t > 0$, the normal tends to $\langle 0, 1, 0 \rangle$; for $t < 0$, it tends to $\langle 0, -1, 0 \rangle$.

Interpretation The origin is an inflection point: curvature vanishes and the bending side changes.

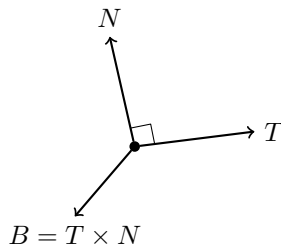
Question 8

Assume $\mathbf{T}$ is a differentiable unit tangent and $\mathbf{T}' \neq \vec{0}$. Prove that the Frenet vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form a right-handed orthonormal basis.

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Question 8 – Solution

Strategy Differentiate $\mathbf{T} \cdot \mathbf{T} = 1$, normalize $\mathbf{T}'$, and use cross-product identities.



Orthogonality Differentiation gives $2\mathbf{T} \cdot \mathbf{T}' = 0$. Since $\mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\|$, $\mathbf{T} \perp \mathbf{N}$, and both are unit.

Binormal Define $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. The cross product of perpendicular unit vectors is unit and perpendicular to both.

Orientation By definition $\mathbf{T} \times \mathbf{N} = \mathbf{B}$, so the ordered frame is right-handed. Hence its Gram matrix is the identity and its orientation determinant is $+1$.

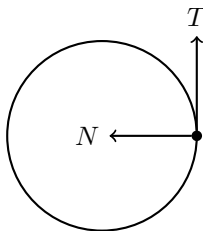
Question 9

The circle $x^2 + y^2 = 4$ is parametrized nonuniformly by $\vec{r}(t) = \langle 2 \cos(t^3), 2 \sin(t^3), 0 \rangle$ for $t > 0$. Find its Frenet frame and explain which parts depend on the varying speed.

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Question 9 – Solution

Strategy Let $\theta = t^3$ and separate angular position from the positive rate $d\theta/dt$.



same frame, nonuniform speed

Frame Because $3t^2 > 0$,

$$\mathbf{T} = \langle -\sin(t^3), \cos(t^3), 0 \rangle, \quad \mathbf{N} = \langle -\cos(t^3), -\sin(t^3), 0 \rangle, \quad \mathbf{B} = \langle 0, 0, 1 \rangle.$$

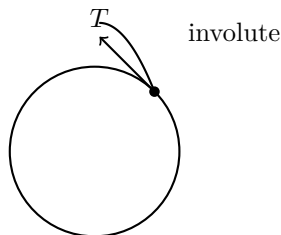
Dependence The frame at a geometric point depends on position and orientation, not on how fast that point is reached. The speed $\|\dot{\mathbf{r}}\| = 6t^2$ varies, but normalization removes it. At $t = 0$ the parametrization stops, so the usual Frenet construction from velocity is not regular there.

Question 10

For the unit circle $\vec{r}(t) = \langle \cos t, \sin t \rangle$, define its involute by $\vec{q}(t) = \vec{r}(t) - t\mathbf{T}(t)$ for $t > 0$. Find $\mathbf{T}_q(t)$ and show that it is parallel to the circle's principal normal at the corresponding point.

Question 10 – Solution

Strategy Differentiate the involute and use the circle's simple Frenet derivatives.



Circle frame $\mathbf{T}(t) = \langle -\sin t, \cos t \rangle$ and $\mathbf{T}'(t) = \langle -\cos t, -\sin t \rangle = \mathbf{N}(t)$.

Involute derivative Differentiating,

$$\vec{q}' = \vec{r}' - \mathbf{T} - t\mathbf{T}' = \mathbf{T} - \mathbf{T} - t\mathbf{N} = -t\mathbf{N}.$$

Since $t > 0$, $\|\vec{q}'\| = t$, so $\boxed{\mathbf{T}_q = -\mathbf{N} = \langle \cos t, \sin t \rangle}$.

Interpretation The taut string leaves the circle tangentially, while the involute's tangent points along the outward radial direction.