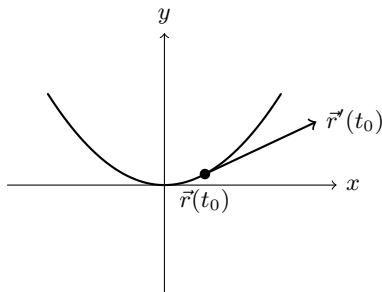


Question 1

For $\vec{r}(t) = \langle t^2 - 1, e^t, \sin(2t) \rangle$, find $\vec{r}'(t)$ and $\vec{r}''(t)$. Find the tangent line at $t = 0$ and a second-order approximation to $\vec{r}(0.1)$.

Question 1 – Solution

Strategy Differentiate componentwise; use the vector Taylor polynomial through second order.



Derivatives $\vec{r}'(t) = \langle 2t, e^t, 2\cos(2t) \rangle$ and $\vec{r}''(t) = \langle 2, e^t, -4\sin(2t) \rangle$. At $t = 0$, $\vec{r}(0) = \langle -1, 1, 0 \rangle$ and $\vec{r}'(0) = \langle 0, 1, 2 \rangle$.

Tangent line $\vec{L}(s) = \langle -1, 1, 0 \rangle + s \langle 0, 1, 2 \rangle$.

Approximation Since $\vec{r}''(0) = \langle 2, 1, 0 \rangle$,

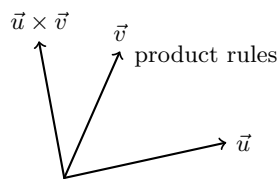
$$\vec{r}(0.1) \approx \vec{r}(0) + 0.1\vec{r}'(0) + \frac{0.1^2}{2}\vec{r}''(0) = \langle -0.99, 1.105, 0.2 \rangle.$$

Question 2

Let $\vec{u}(t) = \langle t, t^2, 1 \rangle$ and $\vec{v}(t) = \langle e^t, \cos t, t \rangle$. Compute $\frac{d}{dt}(\vec{u} \cdot \vec{v})$ and $\frac{d}{dt}(\vec{u} \times \vec{v})$ at $t = 0$ using product rules, then verify the dot-product result by expanding first.

Question 2 – Solution

Strategy Use $(\vec{u} \cdot \vec{v})' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$ and $(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$.



Values at zero $\vec{u}(0) = \langle 0, 0, 1 \rangle$, $\vec{u}'(0) = \langle 1, 0, 0 \rangle$, $\vec{v}(0) = \langle 1, 1, 0 \rangle$, and $\vec{v}'(0) = \langle 1, 0, 1 \rangle$.

Results Thus

$$(\vec{u} \cdot \vec{v})'(0) = 1 + 1 = \boxed{2},$$

and

$$(\vec{u} \times \vec{v})'(0) = \langle 0, 0, 1 \rangle + \langle 0, 1, 0 \rangle = \boxed{\langle 0, 1, 1 \rangle}.$$

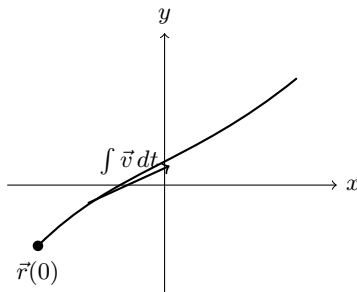
Expanding $\vec{u} \cdot \vec{v} = te^t + t^2 \cos t + t$ and differentiating also gives 2 at zero.

Question 3

A curve has velocity $\vec{r}'(t) = \langle 2t, 3t^2, e^{-t} \rangle$ and passes through $P = (1, -2, 4)$ at $t = 0$. Find $\vec{r}(t)$ and the net displacement from $t = 0$ to $t = 2$.

Question 3 – Solution

Strategy Integrate each velocity component and use the initial condition to determine the constant vector.



Antiderivative $\vec{r}(t) = \langle t^2, t^3, -e^{-t} \rangle + \vec{C}$. Since $\vec{r}(0) = \langle 0, 0, -1 \rangle + \vec{C} = P$, $\vec{C} = \langle 1, -2, 5 \rangle$. Therefore

$$\boxed{\vec{r}(t) = \langle t^2 + 1, t^3 - 2, 5 - e^{-t} \rangle}.$$

Displacement $\vec{r}(2) - \vec{r}(0) = \int_0^2 \vec{r}'(t) dt = \boxed{\langle 4, 8, 1 - e^{-2} \rangle}$.

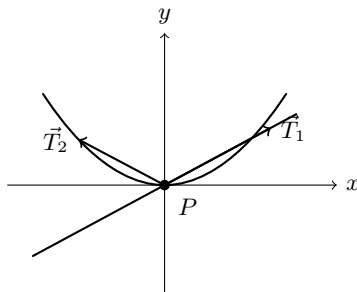
Verification Differentiation recovers the given velocity and substitution gives P .

Question 4

The curves $\vec{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\vec{r}_2(s) = \langle s^2, s, s^3 \rangle$ meet at the origin and at $(1, 1, 1)$. Find the smaller angle between their tangent lines at each intersection.

Question 4 – Solution

Strategy Differentiate each parametrization and evaluate using its own parameter at the shared point.



At the origin Both use parameter 0. The tangent vectors are $\vec{r}_1'(0) = \langle 1, 0, 0 \rangle$ and $\vec{r}_2'(0) = \langle 0, 1, 0 \rangle$, so the angle is $\boxed{90^\circ}$.

At $(1, 1, 1)$ Both use parameter 1. The tangents are $\langle 1, 2, 3 \rangle$ and $\langle 2, 1, 3 \rangle$. Their dot product is 13 and both magnitudes are $\sqrt{14}$, so

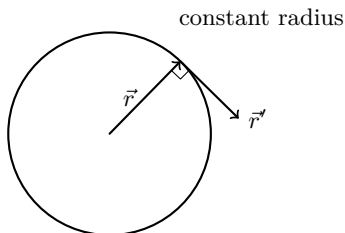
$$\boxed{\theta = \cos^{-1}(13/14) \approx 21.8^\circ}.$$

Question 5

Suppose a differentiable vector function satisfies $\|\vec{r}(t)\| = 5$ for every t . Prove that $\vec{r}(t) \perp \vec{r}'(t)$. Apply the result to $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4 \rangle$ and verify directly.

Question 5 – Solution

Strategy Differentiate the squared magnitude rather than the square root.



Proof Since $\vec{r} \cdot \vec{r} = 25$, differentiation gives $2\vec{r} \cdot \vec{r}' = 0$. Thus $\boxed{\vec{r} \cdot \vec{r}' = 0}$, proving orthogonality wherever $\vec{r}'$ is nonzero.

Application Here $\vec{r}'(t) = \langle -3 \sin t, 3 \cos t, 0 \rangle$ and

$$\vec{r} \cdot \vec{r}' = -9 \cos t \sin t + 9 \sin t \cos t = 0.$$

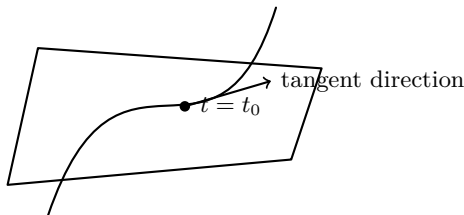
Also $\|\vec{r}(t)\| = \sqrt{9 + 16} = 5$, as required.

Question 6

For $\vec{r}(t) = \langle t, t^2, t^3 - 3t \rangle$, find all points where the tangent line is parallel to the plane $2x - y + z = 4$. Determine whether any tangent line is perpendicular to that plane.

Question 6 – Solution

Strategy A tangent is parallel to a plane when its direction is orthogonal to the plane normal; it is perpendicular when its direction is parallel to the normal.



Parallel condition $\vec{r}'(t) = \langle 1, 2t, 3t^2 - 3 \rangle$ and $\vec{n} = \langle 2, -1, 1 \rangle$. Thus

$$\vec{r}'(t) \cdot \vec{n} = 3t^2 - 2t - 1 = (3t + 1)(t - 1).$$

So $t = 1$ or $t = -1/3$, giving points $\boxed{(1, 1, -2)}$ and $\boxed{(-1/3, 1/9, 26/27)}$.

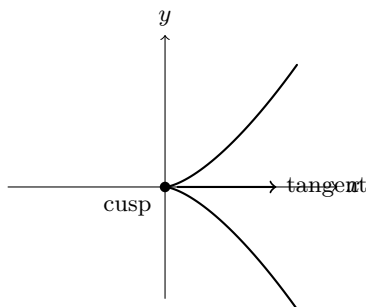
Perpendicular condition Parallelism of $\vec{r}'(t)$ with $\vec{n}$ would force scale $1/2$ from the first component, then $2t = -1/2$, but the third component at $t = -1/4$ is not $1/2$. Hence none exist.

Question 7

Analyze the smoothness and regularity of $\vec{r}(t) = \langle t^2, t^3 \rangle$ near $t = 0$. Does the curve have a tangent line at the origin even though $\vec{r}'(0) = \vec{0}$? Justify using a reparameterization or limiting secant directions.

Question 7 – Solution

Strategy A zero derivative makes the given parametrization nonregular, but the geometric curve may still have a tangent.



Regularity $\vec{r}'(t) = \langle 2t, 3t^2 \rangle$, so $\vec{r}'(0) = \vec{0}$ and this polynomial parametrization is smooth but not regular at 0. Eliminating t gives $y^2 = x^3$, a semicubical cusp.

Tangent direction For $t \neq 0$, the secant direction from the origin is proportional to $\langle t^2, t^3 \rangle = t^2 \langle 1, t \rangle$, which approaches $\langle 1, 0 \rangle$ from both sides. Thus the geometric tangent line exists and is $y = 0$.

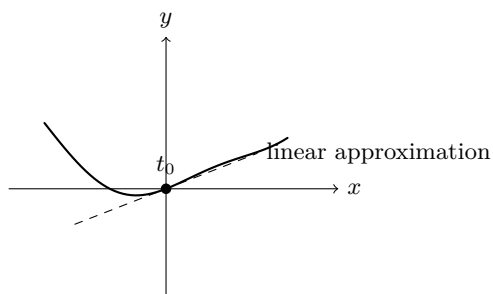
Distinction The curve has a tangent at the cusp, but no nonzero velocity there.

Question 8

For $\vec{r}(t) = \langle \cos t, \sin t, e^t \rangle$, form the second-order Taylor approximation about $t = 0$. Use it to approximate $\vec{r}(0.05)$ and state the order of the neglected error.

Question 8 – Solution

Strategy Use $\vec{r}(0) + t\vec{r}'(0) + \frac{1}{2}t^2\vec{r}''(0)$.



Data $\vec{r}(0) = \langle 1, 0, 1 \rangle$, $\vec{r}'(0) = \langle 0, 1, 1 \rangle$, and $\vec{r}''(0) = \langle -1, 0, 1 \rangle$. Hence

$$\vec{P}_2(t) = \langle 1 - t^2/2, t, 1 + t + t^2/2 \rangle.$$

At $t = 0.05$, $\vec{r}(0.05) \approx \langle 0.99875, 0.05, 1.05125 \rangle$.

Error Because the components have bounded third derivatives near zero, the remainder is $O(t^3)$, here on the scale of 0.05^3 .

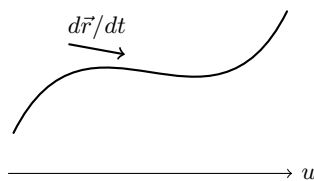
Question 9

Let $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ and reparameterize by $t = u^3 - u$. Find $d\vec{r}/du$. Identify every u where the reparameterized curve momentarily stops, and explain why the geometric helix itself remains smooth there.

Question 9 – Solution

Strategy Apply the vector chain rule and distinguish the image curve from the chosen parameter clock.

speed changes, image fixed



Chain rule With $t = g(u) = u^3 - u$,

$$\frac{d\vec{r}}{du} = \vec{r}'(g(u))g'(u) = (3u^2 - 1) \langle -\sin(u^3 - u), \cos(u^3 - u), 1 \rangle.$$

It vanishes when $3u^2 - 1 = 0$, so $\boxed{u = \pm 1/\sqrt{3}}$.

Interpretation The original derivative $\langle -\sin t, \cos t, 1 \rangle$ never vanishes. The stops occur because $dt/du = 0$; they reflect a nonregular reparameterization, not a cusp in the helix.

Question 10

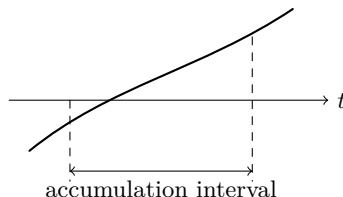
Evaluate

$$\int_0^{\pi} \langle t \sin t, t \cos t, \sin^2 t \rangle dt.$$

Then interpret the result as displacement if the integrand is velocity, and find the average velocity on $[0, \pi]$.

Question 10 – Solution

Strategy Integrate componentwise, using integration by parts for the first two components.



Integral We have

$$\int_0^\pi t \sin t \, dt = \pi, \quad \int_0^\pi t \cos t \, dt = -2, \quad \int_0^\pi \sin^2 t \, dt = \frac{\pi}{2}.$$

Therefore the displacement is $\boxed{\langle \pi, -2, \pi/2 \rangle}$.

Average velocity Divide displacement by the elapsed time π :

$$\boxed{\vec{v}_{\text{avg}} = \langle 1, -2/\pi, 1/2 \rangle}.$$

Verification Multiplying the average velocity by π recovers the net displacement.