

Question 1

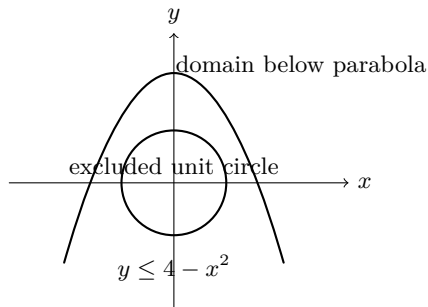
Find and sketch the domain of

$$f(x, y) = \frac{\sqrt{4 - x^2 - y^2}}{\ln(x^2 + y^2)}.$$

State which boundary pieces are included and whether the domain is bounded, open, closed, or neither.

Question 1 – Solution

Strategy Enforce the square-root, logarithm, and denominator restrictions simultaneously.



Restrictions We need $4 - x^2 - y \geq 0$, $x^2 + y^2 > 0$, and $\ln(x^2 + y^2) \neq 0$. Thus

$$D = \{(x, y) : y \leq 4 - x^2, x^2 + y^2 \neq 0, 1\}.$$

Boundary and topology The entire parabola $y = 4 - x^2$ is included; it does not meet the unit circle or the origin. The origin and entire unit circle are excluded. The region is unbounded downward. It is not open because the parabolic boundary is included, and not closed because excluded circle points are limit points.

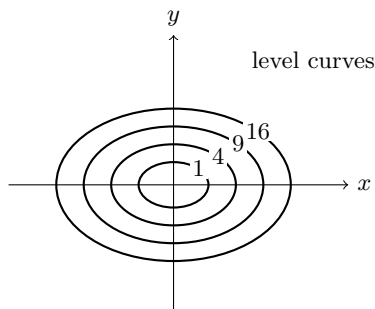
Check Every remaining point makes the radicand nonnegative and the denominator defined and nonzero.

Question 2

For $f(x, y) = 9 - 4(x - 1)^2 - (y + 2)^2$, find the range, global extrema, and level curves $f = c$. Describe how the curves change as c varies.

Question 2 – Solution

Strategy The subtracted squares are nonnegative; solve $f = c$ for the level sets.



Range and extrema The maximum is $\boxed{9}$ at $(1, -2)$. There is no minimum because $f \rightarrow -\infty$ as $\|(x, y)\| \rightarrow \infty$, so the range is $\boxed{(-\infty, 9]}$.

Level curves For $c < 9$,

$$4(x - 1)^2 + (y + 2)^2 = 9 - c,$$

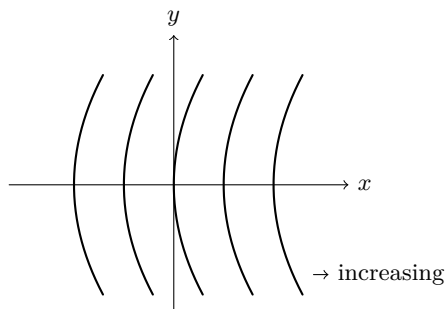
an ellipse centered at $(1, -2)$ with semiaxes $\sqrt{9 - c}/2$ and $\sqrt{9 - c}$. For $c = 9$ the level is one point; for $c > 9$ it is empty.

Question 3

A contour map consists of the curves $x - y^2 = c$ for all real c , labeled so values increase to the right. Reconstruct a simplest function producing the map. Find the level curve through $(3, -1)$ and decide where the function is positive.

Question 3 – Solution

Strategy Read the constant expression directly from the family of contours.



Function A simplest choice is $f(x, y) = x - y^2$. Any strictly increasing relabeling $g(f)$ would preserve the geometric contours but change their numerical labels.

Requested contour At $(3, -1)$, $c = 3 - 1 = 2$, so the curve is $x - y^2 = 2$.

Sign region The function is positive precisely when $x > y^2$, the region to the right of the parabola $x = y^2$.

Question 4

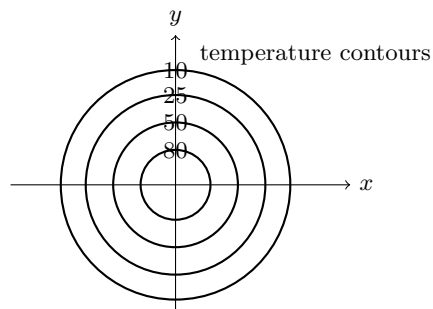
A thin plate has temperature

$$T(x, y) = 100e^{-(x^2+4y^2)/8} \quad (^\circ\text{C}).$$

Find the 50° isotherm, its semiaxes and enclosed area. Determine the warmest point and explain the contour spacing as temperature decreases.

Question 4 – Solution

Strategy Set T equal to a constant and take logarithms.



Isotherm Setting $T = 50$ gives $e^{-(x^2+4y^2)/8} = 1/2$, hence

$$x^2 + 4y^2 = 8 \ln 2.$$

The semiaxes are $\sqrt{8 \ln 2}$ and $\sqrt{2 \ln 2}$, so the area is $\boxed{4\pi \ln 2}$.

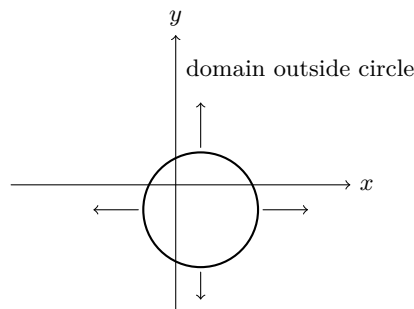
Maximum and spacing The maximum is 100° at $(0,0)$. For level $0 < c < 100$, $x^2 + 4y^2 = 8 \ln(100/c)$; lower temperatures correspond to larger ellipses, with nonlinear spacing caused by exponential decay.

Question 5

Let $f(u, v) = \sqrt{u - v}$ and $g(x, y) = (x^2 + y, x - y^2)$. Find $(f \circ g)(x, y)$ and its domain. Determine the boundary and whether the point $(1, 1)$ is interior, exterior, or on the boundary.

Question 5 – Solution

Strategy Substitute both components of g into f and require a nonnegative radicand.



Composition We obtain

$$(f \circ g)(x, y) = \sqrt{x^2 + y - x + y^2}.$$

Thus the domain is $x^2 - x + y^2 + y \geq 0$, or

$$\boxed{(x - 1/2)^2 + (y + 1/2)^2 \geq 1/2}.$$

It is the exterior of a disk, including its circular boundary.

Point test At $(1, 1)$ the original radicand is $2 > 0$, so the point is interior to the domain.

Question 6

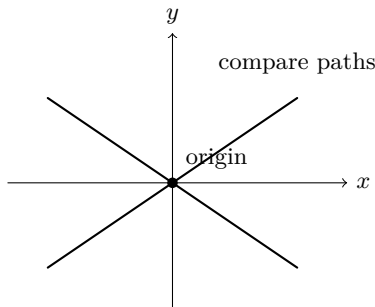
Investigate

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$$

Evaluate along at least three families of paths and explain why the evidence is decisive.

Question 6 – Solution

Strategy A multivariable limit must have the same value along every path.



Path tests Along $y = 0$, the expression equals 1; along $x = 0$, it equals -1 ; and along $y = mx$, it equals

$$\frac{1 - m^2}{1 + m^2},$$

which varies continuously with m .

Conclusion Since even two paths give different values, the limit does not exist.

Why decisive Path agreement can only suggest a limit, but one disagreement proves nonexistence because the definition requires uniform behavior over all approaches.

Question 7

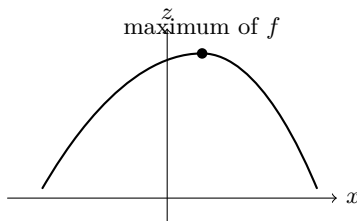
Without using derivatives, find the range and all global extrema of

$$f(x, y) = 7 - 2x^2 - 2xy - y^2 + 8x + 6y.$$

Hint: rewrite the quadratic as a sum of squares.

Question 7 – Solution

Strategy Complete squares in a way that respects the mixed term.



Rewrite Since $2x^2 + 2xy + y^2 = (x + y)^2 + x^2$,

$$f = 7 - (x + y)^2 - x^2 + 8x + 6y.$$

Set $u = x + y$. Then $y = u - x$ and $f = 7 - u^2 - x^2 + 2x + 6u = 17 - (x - 1)^2 - (u - 3)^2$.

Conclusion Therefore

$$\boxed{f(x, y) = 17 - (x - 1)^2 - (x + y - 3)^2}.$$

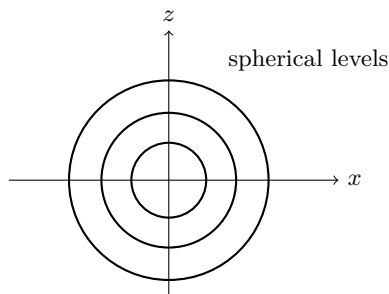
The global maximum is 17 at $x = 1$, $x + y = 3$, hence $(1, 2)$. There is no minimum, and the range is $(-\infty, 17]$.

Question 8

For $F(x, y, z) = \sqrt{(x-1)^2 + (y+2)^2 + (z-3)^2}$, interpret F geometrically. Describe every level surface $F = c$, determine the range, and find the level surface through $(4, 2, 3)$.

Question 8 – Solution

Strategy Recognize the three-dimensional distance formula.



Interpretation F is distance from $C = (1, -2, 3)$, so its range is $[0, \infty)$.

Levels For $c > 0$, $F = c$ is the sphere

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = c^2.$$

For $c = 0$ it is the single point C ; for $c < 0$ it is empty.

Requested surface The point $(4, 2, 3)$ is distance $\sqrt{3^2 + 4^2} = 5$ from C , so its level surface is the sphere of radius 5 .

Question 9

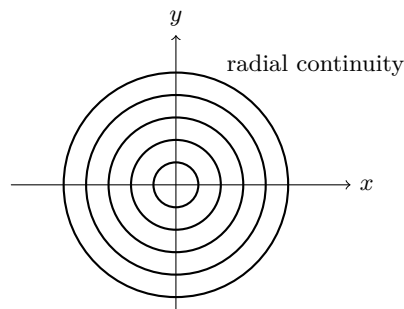
Define

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Prove directly that f is continuous at the origin, despite the direction-dependent factor in its polar form.

Question 9 – Solution

Strategy Bound the absolute value by a constant times the distance to the origin.



Estimate For $(x, y) \neq (0, 0)$,

$$|f(x, y)| = |x| \frac{|x^2 - y^2|}{x^2 + y^2} \leq |x| \leq \sqrt{x^2 + y^2}.$$

Therefore $|f(x, y) - 0| \leq r$, where r is the distance to the origin.

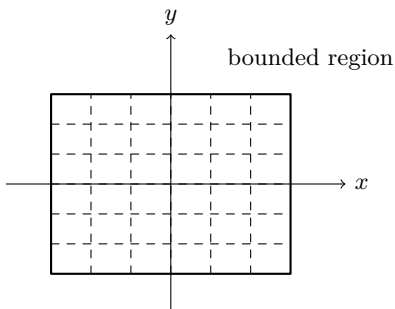
Epsilon argument Given $\varepsilon > 0$, choose $\delta = \varepsilon$. If $0 < r < \delta$, then $|f(x, y) - f(0, 0)| < \varepsilon$. Hence $\lim f = 0 = f(0, 0)$ and f is continuous.

Question 10

Let $f(x, y) = \frac{xy}{1 + x^2 + y^2}$ on the closed rectangle $R = [-2, 2] \times [-3, 3]$. Prove that $|f| < 1/2$ everywhere, and explain why f must attain a maximum and minimum on R even without finding them.

Question 10 – Solution

Strategy Compare $2|xy|$ with $x^2 + y^2$, then use continuity on a closed bounded domain.



Bound From $(|x| - |y|)^2 \geq 0$, $2|xy| \leq x^2 + y^2$. Hence

$$|f(x, y)| \leq \frac{x^2 + y^2}{2(1 + x^2 + y^2)} < \boxed{\frac{1}{2}}.$$

The strict inequality comes from the added 1 in the denominator.

Existence of extrema The denominator never vanishes, so f is continuous. The rectangle R is closed and bounded (compact). By the Extreme Value Theorem, f attains both an absolute maximum and absolute minimum on R .