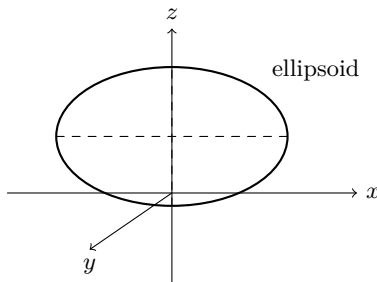


Question 1

Put $4x^2 + y^2 + 9z^2 - 8x + 6y - 36z + 4 = 0$ in standard form. Classify the surface and give its center, semiaxes, and coordinate-direction extreme points.

Question 1 – Solution

Strategy Complete the square separately in x, y, z , then divide by the resulting constant.



Standard form Completing squares gives

$$4(x - 1)^2 + (y + 3)^2 + 9(z - 2)^2 = 45,$$

so

$$\frac{(x - 1)^2}{45/4} + \frac{(y + 3)^2}{45} + \frac{(z - 2)^2}{5} = 1.$$

Geometry This is an ellipsoid centered at $(1, -3, 2)$ with semiaxes $3\sqrt{5}/2$, $3\sqrt{5}$, and $\sqrt{5}$. Its axial extreme points are

$$(1 \pm 3\sqrt{5}/2, -3, 2), \quad (1, -3 \pm 3\sqrt{5}, 2), \quad (1, -3, 2 \pm \sqrt{5}).$$

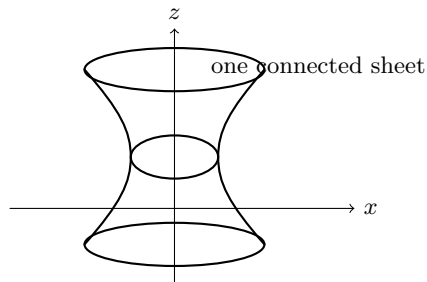
Verification Substitution of any listed endpoint makes exactly one normalized square equal to 1.

Question 2

Analyze $x^2 + 4y^2 - z^2 - 2x + 16y + 4z - 12 = 0$. Put it in standard form, classify it, identify its axis, and describe the traces $z = 2$ and $z = 2 + k$.

Question 2 – Solution

Strategy Complete squares and inspect the signs. Then substitute fixed z -levels.



Standard form Simplification gives

$$(x-1)^2 + 4(y+2)^2 - (z-2)^2 = 25, \quad \boxed{\frac{(x-1)^2}{25} + \frac{(y+2)^2}{25/4} - \frac{(z-2)^2}{25} = 1}.$$

It is a hyperboloid of one sheet centered at $(1, -2, 2)$ with axis parallel to z .

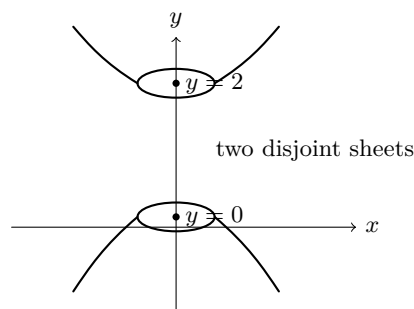
Traces At $z = 2$ the waist ellipse is $(x-1)^2 + 4(y+2)^2 = 25$. At $z = 2+k$ it becomes $(x-1)^2 + 4(y+2)^2 = 25 + k^2$, so ellipses expand as $|k|$ grows.

Question 3

For $9x^2 - 4y^2 + 9z^2 + 36x + 8y - 18z + 45 = 0$, find the standard form, classify the surface, and determine the vertices and which coordinate planes fail to intersect it.

Question 3 – Solution

Strategy Complete squares; a single negative term with right side -1 is more clearly read after multiplying by -1 .



Standard form Completing squares gives

$$9(x+2)^2 - 4(y-1)^2 + 9(z-1)^2 = -4,$$

hence

$$\frac{(y-1)^2}{1} - \frac{(x+2)^2}{4/9} - \frac{(z-1)^2}{4/9} = 1.$$

This is a hyperboloid of two sheets with axis parallel to y .

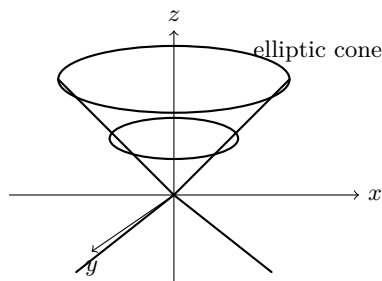
Vertices and coordinate traces The vertices are $(-2, 0, 1), (-2, 2, 1)$. All three coordinate planes intersect the surface: $y = 0$ contains the first vertex, while $x = 0$ and $z = 0$ give hyperbolas. The plane $y = 1$ does not intersect the surface, but it is not a coordinate plane.

Question 4

Classify the surface $4x^2 + y^2 - 9z^2 = 0$. Describe its traces in the planes $z = k$, $x = k$, and $y = k$, and find the lines obtained in the trace $y = 0$.

Question 4 – Solution

Strategy A homogeneous quadratic with two positive squares and one negative square describes a cone.



Classification The equation is $4x^2 + y^2 = 9z^2$, an elliptic cone with vertex at the origin and axis the z -axis.

Traces For $z = k \neq 0$, $4x^2 + y^2 = 9k^2$ is an ellipse; for $z = 0$ it is only the vertex. For fixed $x = k$ or $y = k$, the traces are hyperbolas when the fixed value is nonzero and intersecting lines when it is zero.

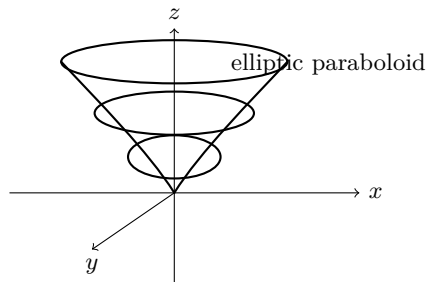
$y = 0$ Then $4x^2 = 9z^2$, so $z = \pm \frac{2}{3}x$.

Question 5

The surface $z = x^2 + 4y^2 - 2x + 8y + 7$ is cut by the plane $z = 10$. Identify the surface, locate its vertex, and find the semiaxes and area of the resulting planar region.

Question 5 – Solution

Strategy Complete squares in x, y ; a horizontal cut of an elliptic paraboloid is an ellipse.



Standard form We obtain

$$z = (x - 1)^2 + 4(y + 1)^2 + 2.$$

Thus the surface is an elliptic paraboloid opening upward with vertex $(1, -1, 2)$.

Cut at $z = 10$ The boundary satisfies $(x - 1)^2 + 4(y + 1)^2 = 8$, or

$$\frac{(x - 1)^2}{8} + \frac{(y + 1)^2}{2} = 1.$$

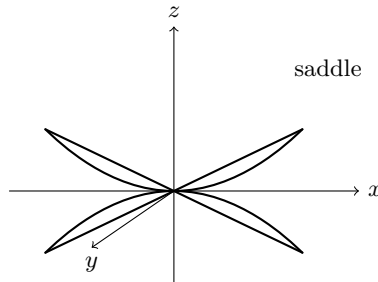
Its semiaxes are $2\sqrt{2}$ and $\sqrt{2}$, so the enclosed area is $\pi(2\sqrt{2})(\sqrt{2}) = 4\pi$.

Question 6

Analyze the hyperbolic paraboloid $z = x^2 - 4y^2$. Find the traces in $x = 0$, $y = 0$, $z = 0$, and $z = k \neq 0$. Explain how these traces establish the saddle shape.

Question 6 – Solution

Strategy Substitute each fixed coordinate and classify the resulting planar curve.



Vertical traces In $y = 0$, $z = x^2$ opens upward. In $x = 0$, $z = -4y^2$ opens downward. Their opposite concavities create the saddle.

Horizontal traces At $z = 0$, $x^2 - 4y^2 = 0$, so $x = \pm 2y$, two intersecting lines. For $k > 0$, $x^2 - 4y^2 = k$ is a hyperbola opening in the x -direction; for $k < 0$, it opens in the y -direction.

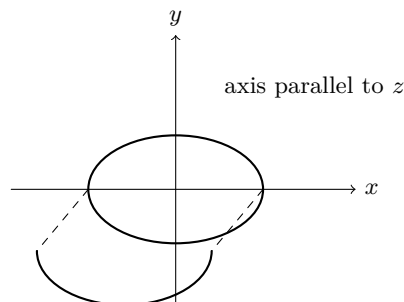
Conclusion The origin is neither a local maximum nor minimum because nearby traces rise in one direction and fall in the perpendicular direction.

Question 7

Classify $9x^2 + 4y^2 - 18x + 16y - 11 = 0$, describe its axis, and find the area of a cross-section perpendicular to that axis. Does the surface intersect the z -axis?

Question 7 – Solution

Strategy Because z is absent, any planar curve in xy is extruded parallel to the z -axis.



Standard form Completing squares yields

$$9(x-1)^2 + 4(y+2)^2 = 36, \quad \boxed{\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1}.$$

This is an elliptic cylinder parallel to the z -axis.

Cross-section Its semiaxes are 2 and 3, so each perpendicular cross-section has area $\boxed{6\pi}$.

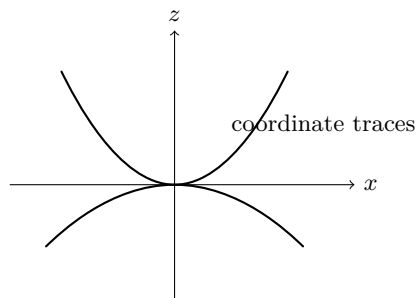
z -axis test At $x = y = 0$, the left side of the original equation is $-11 \neq 0$, so the cylinder does not meet the z -axis.

Question 8

A quadric has the traces $z = x^2$ in the plane $y = 0$ and $z = -y^2$ in the plane $x = 0$. Every horizontal trace $z = k \neq 0$ is a hyperbola. Determine a simplest equation for the surface and explain whether the trace information uniquely fixes all coefficients.

Question 8 – Solution

Strategy Combine the two vertical traces into a quadratic having the required restrictions.



Simplest model The equation

$$z = x^2 - y^2$$

has exactly the stated coordinate-plane traces and horizontal hyperbolas. It is a hyperbolic paraboloid.

Uniqueness issue The given coordinate traces force the coefficients of x^2 and y^2 in a diagonal model, but without an assumption excluding a mixed term, $z = x^2 - y^2 + cxy$ has the same traces in $x = 0$ and $y = 0$. Its horizontal traces remain hyperbolic when the quadratic form is indefinite.

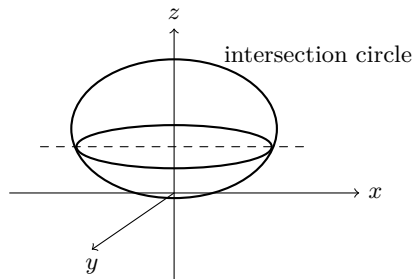
Conclusion The simplest axis-aligned equation is determined, but the data do not uniquely exclude rotated saddles.

Question 9

Find and classify the curve of intersection of the sphere $x^2 + y^2 + z^2 = 25$ with the plane $x + 2y + 2z = 9$. Find the curve's center and radius, and determine whether the plane is tangent, secant, or disjoint.

Question 9 – Solution

Strategy The circle center is the perpendicular projection of the sphere center onto the plane.



Plane distance The origin-to-plane distance is $d = 9/\sqrt{1+4+4} = 3$. Since $3 < 5$, the plane is secant.

Center and radius The foot from the origin lies along $\langle 1, 2, 2 \rangle$ and is

$$C = \frac{9}{9} \langle 1, 2, 2 \rangle = \boxed{(1, 2, 2)}.$$

By the right triangle from sphere center to circle edge, $r^2 = 5^2 - 3^2 = 16$, so $\boxed{r = 4}$.

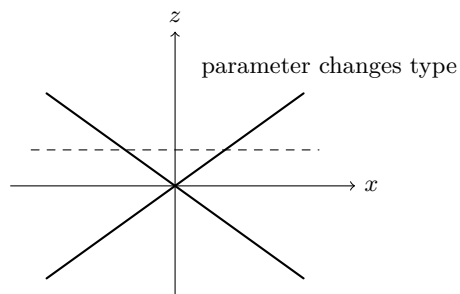
Classification The intersection is a circle in the given plane, centered at $(1, 2, 2)$ with radius 4.

Question 10

Classify the family $x^2 + y^2 - z^2 = \lambda$ for $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Describe how the topology changes as λ passes through zero and give the vertices when they exist.

Question 10 – Solution

Strategy Divide by $|\lambda|$ and inspect which side has the single positive term.



$\lambda > 0$ The form $x^2/\lambda + y^2/\lambda - z^2/\lambda = 1$ is a hyperboloid of one sheet, connected, with axis z .

$\lambda = 0$ The equation $x^2 + y^2 = z^2$ is a double circular cone with vertex at the origin.

$\lambda < 0$ Writing $z^2/|\lambda| - x^2/|\lambda| - y^2/|\lambda| = 1$ gives a hyperboloid of two sheets. Its vertices are $(0, 0, \pm\sqrt{|\lambda|})$.

Transition At zero, the one-sheet surface pinches to a cone and separates into two components.