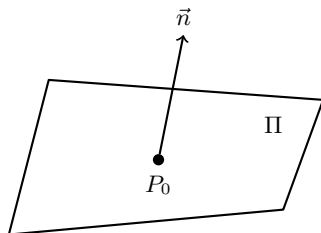


Question 1

Find scalar and vector equations of the plane through $P_0 = (2, -1, 3)$ with normal vector $\vec{n} = \langle 4, 2, -1 \rangle$. Find its intercepts with the coordinate axes.

Question 1 – Solution

Strategy Use $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$, then set two coordinates equal to zero for each intercept.



Equations The vector equation is $\langle 4, 2, -1 \rangle \cdot \langle x - 2, y + 1, z - 3 \rangle = 0$, which simplifies to $4x + 2y - z = 3$.

Intercepts The x -, y -, and z -intercepts are

$$(3/4, 0, 0), \quad (0, 3/2, 0), \quad (0, 0, -3).$$

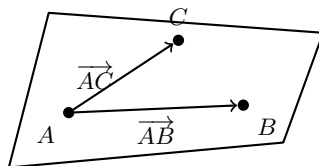
Verification Substituting P_0 gives $8 - 2 - 3 = 3$, and each intercept satisfies the scalar equation.

Question 2

Find an equation of the plane through $A = (1, 0, 2)$, $B = (3, -1, 4)$, and $C = (0, 2, 5)$. Determine whether the origin and $D = (2, 1, 3)$ lie on the same side of the plane.

Question 2 – Solution

Strategy Cross two independent displacement vectors to obtain a normal. Compare signs after substituting test points.



Plane Here $\overrightarrow{AB} = \langle 2, -1, 2 \rangle$ and $\overrightarrow{AC} = \langle -1, 2, 3 \rangle$, so $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -7, -8, 3 \rangle$. Thus

$$\boxed{7x + 8y - 3z = 1}.$$

Side test Write $F(x, y, z) = 7x + 8y - 3z - 1$. Then $F(0, 0, 0) = -1$, while $F(2, 1, 3) = 12$. The signs differ, so the points lie on opposite sides.

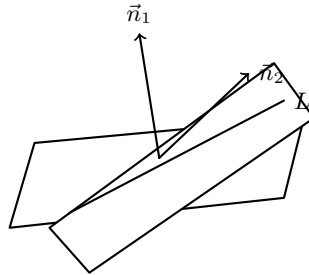
Verification Each of A, B, C satisfies the plane equation.

Question 3

Find the line of intersection of the planes $\Pi_1 : x + 2y - z = 3$ and $\Pi_2 : 2x - y + 2z = 1$. Also find the acute dihedral angle between the planes.

Question 3 – Solution

Strategy The intersection direction is perpendicular to both normals. Obtain one point by fixing a convenient coordinate.



Intersection line With $\vec{n}_1 = \langle 1, 2, -1 \rangle$ and $\vec{n}_2 = \langle 2, -1, 2 \rangle$,

$$\vec{n}_1 \times \vec{n}_2 = \langle 3, -4, -5 \rangle.$$

Setting $z = 0$ gives $x + 2y = 3$, $2x - y = 1$, hence $(x, y) = (1, 1)$. Therefore

$$\vec{r} = \langle 1, 1, 0 \rangle + t \langle 3, -4, -5 \rangle.$$

Angle Since $\vec{n}_1 \cdot \vec{n}_2 = -2$, $\|\vec{n}_1\| = \sqrt{6}$, and $\|\vec{n}_2\| = 3$,

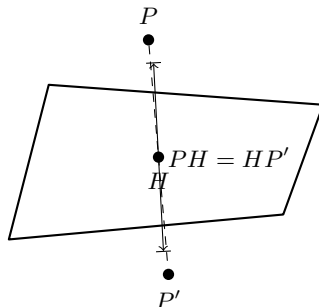
$$\theta = \cos^{-1} \left(\frac{2}{3\sqrt{6}} \right) \approx 74.2^\circ.$$

Question 4

Find the distance from $P = (4, -1, 5)$ to the plane $2x - y + 2z = 3$, the perpendicular foot H , and the reflection P' of P across the plane.

Question 4 – Solution

Strategy Move from P along the normal $\vec{n} = \langle 2, -1, 2 \rangle$.



Distance The signed numerator is $2(4) - (-1) + 2(5) - 3 = 16$, and $\|\vec{n}\| = 3$, so $d = 16/3$.

Foot For $H = P - \lambda\vec{n}$, impose the plane equation. Since $\lambda = 16/9$,

$$H = \left(4/9, 7/9, 13/9 \right).$$

Reflection Since H is the midpoint of PP' , $P' = 2H - P = \left(-28/9, 23/9, -19/9 \right)$.

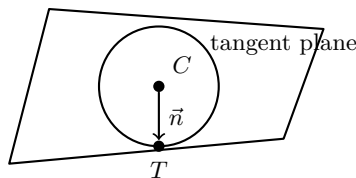
Verification H satisfies the plane equation, and $P - H$ is parallel to $\vec{n}$.

Question 5

The sphere $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 36$ contains the point $T = (3, 2, 7)$. Find the tangent plane at T and the line normal to the sphere there.

Question 5 – Solution

Strategy The radius from the center to T is normal to the tangent plane.



Normal direction The center is $C = (1, -2, 3)$ and $T - C = \langle 2, 4, 4 \rangle = 2\langle 1, 2, 2 \rangle$.

Tangent plane Using normal $\langle 1, 2, 2 \rangle$ through T ,

$$\boxed{(x - 3) + 2(y - 2) + 2(z - 7) = 0}, \quad \text{or} \quad \boxed{x + 2y + 2z = 21}.$$

Normal line $\boxed{\vec{r} = \langle 3, 2, 7 \rangle + t \langle 1, 2, 2 \rangle}.$

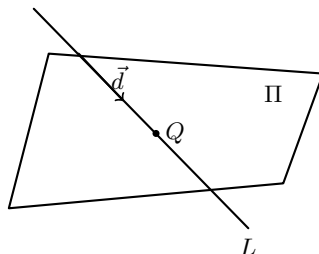
Verification $|T - C| = 6$, so T is indeed on the sphere.

Question 6

Classify the relationship between the line $L : \vec{r} = \langle 1, 2, 0 \rangle + t \langle 2, -1, 3 \rangle$ and the plane $\Pi : 3x + y - z = 5$. If they intersect, find the point and angle at which the line meets the plane.

Question 6 – Solution

Strategy Substitute the line into the plane. The line-plane angle is complementary to the angle between the direction and the normal.



Intersection Substitution gives $3(1 + 2t) + (2 - t) - 3t = 5$, so $2t = 0$ and $t = 0$. Thus $Q = \boxed{(1, 2, 0)}$.

Angle With $\vec{d} = \langle 2, -1, 3 \rangle$ and $\vec{n} = \langle 3, 1, -1 \rangle$,

$$\sin \alpha = \frac{|\vec{d} \cdot \vec{n}|}{\|\vec{d}\| \|\vec{n}\|} = \frac{2}{\sqrt{154}}.$$

Hence $\boxed{\alpha = \sin^{-1}(2/\sqrt{154}) \approx 9.27^\circ}$.

Verification Since $\vec{d} \cdot \vec{n} \neq 0$, the line is not parallel to the plane.

Question 7

Find the plane in the family containing the intersection line of $x + y + z = 2$ and $2x - y + 3z = 1$ that is perpendicular to the plane $x - 2y + z = 4$.

Question 7 – Solution

Strategy Apart from $2x - y + 3z = 1$ itself, every plane through the common line has equation $(x + y + z - 2) + \lambda(2x - y + 3z - 1) = 0$. The omitted plane has normal dot product $7 \neq 0$ with the given normal, so it cannot qualify. Choose λ so the family normal is perpendicular to the given normal.

pencil of planes



Normal condition The family normal is $\langle 1 + 2\lambda, 1 - \lambda, 1 + 3\lambda \rangle$. Dotting with $\langle 1, -2, 1 \rangle$ gives 7λ , so $\lambda = 0$.

Result The required plane is therefore simply

$$\boxed{x + y + z = 2}.$$

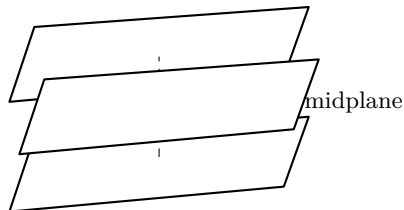
Verification Its normal $\langle 1, 1, 1 \rangle$ has dot product $1 - 2 + 1 = 0$ with $\langle 1, -2, 1 \rangle$, and it contains the stated intersection line.

Question 8

Find the distance between the parallel planes $2x - y + 2z = 3$ and $4x - 2y + 4z = 21$. Find the plane exactly halfway between them.

Question 8 – Solution

Strategy Normalize the equations to identical left sides before comparing constants.



Distance Divide the second equation by 2 to get $2x - y + 2z = 21/2$. Therefore

$$d = \frac{|21/2 - 3|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{15/2}{3} = \boxed{5/2}.$$

Midplane Average the normalized constants: $(3 + 21/2)/2 = 27/4$. Hence

$$\boxed{2x - y + 2z = 27/4}.$$

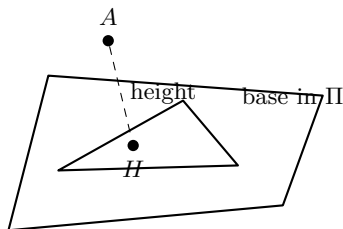
Verification Its constant differs from each neighboring constant by $15/4$, corresponding to distance $5/4$.

Question 9

A tetrahedron has base vertices $B = (0, 0, 0)$, $C = (4, 0, 0)$, $D = (1, 3, 0)$ and apex $A = (2, 1, 6)$. Find the base plane, the height, and the volume. Then find the plane parallel to the base that cuts the tetrahedron at half its height.

Question 9 – Solution

Strategy The base is horizontal. Volume is one third of base area times perpendicular height.



Base and height The base plane is $z = 0$. Its triangular area is $\frac{1}{2}(4)(3) = 6$, and the apex is 6 units from the plane.

Volume Thus $V = \frac{1}{3}(6)(6) = 12$.

Half-height plane The plane parallel to $z = 0$ halfway from the base to the apex is $z = 3$.

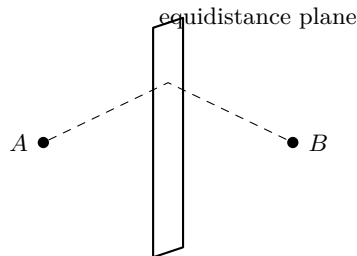
Verification Distance from $z = 3$ to both $z = 0$ and A 's level $z = 6$ is 3.

Question 10

Prove that the set of points equidistant from two distinct points A and B is a plane perpendicular to $\overline{AB}$ through its midpoint. Then find that plane for $A = (1, -2, 4)$ and $B = (5, 0, -2)$.

Question 10 – Solution

Strategy Expand $\|\vec{r} - \vec{a}\|^2 = \|\vec{r} - \vec{b}\|^2$ and cancel $\vec{r} \cdot \vec{r}$.



General proof Expansion gives

$$2(\vec{b} - \vec{a}) \cdot \vec{r} = \|\vec{b}\|^2 - \|\vec{a}\|^2.$$

This is a plane with normal $\vec{b} - \vec{a}$. Substituting $(\vec{a} + \vec{b})/2$ verifies that the midpoint lies on it.

Application Here $\vec{b} - \vec{a} = \langle 4, 2, -6 \rangle$ and the midpoint is $(3, -1, 1)$. Thus

$$4(x - 3) + 2(y + 1) - 6(z - 1) = 0,$$

or $\boxed{2x + y - 3z = 2}$.

Verification Direct substitution of the midpoint gives $6 - 1 - 3 = 2$.