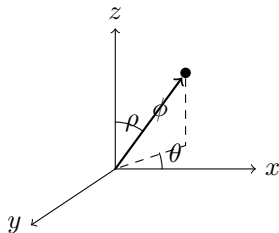


Question 1

Convert the point $(1, \sqrt{3}, 2\sqrt{3})$ to spherical coordinates, using $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$. Verify the result.

Question 1 – Solution

Strategy Find distance, azimuth, and polar angle in that order.



Coordinates

$$\rho = \sqrt{1 + 3 + 12} = 4, \quad \theta = \arctan(\sqrt{3}) = \frac{\pi}{3}.$$

Also $\cos \phi = z/\rho = \sqrt{3}/2$, so $\phi = \pi/6$. Thus

$$\boxed{(\rho, \theta, \phi) = \left(4, \frac{\pi}{3}, \frac{\pi}{6}\right)}.$$

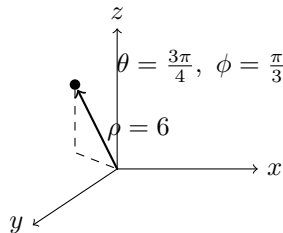
Verification Using $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$ gives $1, \sqrt{3}, 2\sqrt{3}$, respectively.

Question 2

Convert the spherical point $(\rho, \theta, \phi) = (6, 3\pi/4, \pi/3)$ to rectangular and cylindrical coordinates. Keep all values exact.

Question 2 – Solution

Strategy First compute the cylindrical radius $r = \rho \sin \phi$ and height $z = \rho \cos \phi$.



Cylindrical form

$$r = 6 \sin \frac{\pi}{3} = 3\sqrt{3}, \quad z = 6 \cos \frac{\pi}{3} = 3.$$

Hence $(r, \theta, z) = (3\sqrt{3}, 3\pi/4, 3)$.

Rectangular form

$$x = 3\sqrt{3} \cos \frac{3\pi}{4} = -\frac{3\sqrt{6}}{2}, \quad y = 3\sqrt{3} \sin \frac{3\pi}{4} = \frac{3\sqrt{6}}{2}.$$

Therefore

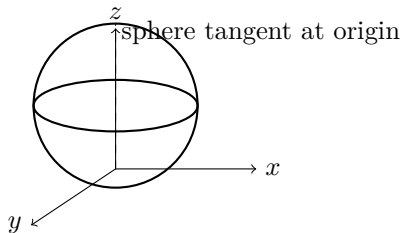
$$(x, y, z) = \left(-\frac{3\sqrt{6}}{2}, \frac{3\sqrt{6}}{2}, 3 \right).$$

Question 3

Convert the spherical equation $\rho = 4 \cos \phi$ to rectangular coordinates and identify the surface, including its center and radius. Explain why the origin belongs to it.

Question 3 – Solution

Strategy Multiply by ρ and use $\rho^2 = x^2 + y^2 + z^2$ and $\rho \cos \phi = z$.



Conversion

$$\rho^2 = 4\rho \cos \phi \implies x^2 + y^2 + z^2 = 4z.$$

Completing the square,

$$\boxed{x^2 + y^2 + (z - 2)^2 = 4}.$$

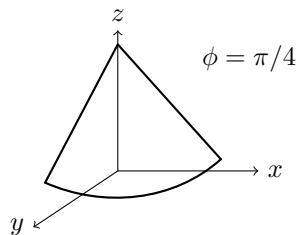
Geometry The surface is a sphere of radius 2 centered at $(0,0,2)$. It contains the origin because the distance from the origin to the center is exactly the radius. In the original spherical equation, $\rho = 0$ and $\phi = \pi/2$ represent the origin and satisfy $0 = 4 \cos(\pi/2)$.

Question 4

Identify and convert the surface $\phi = \pi/4$ to rectangular and cylindrical equations. Explain why it is only one nappe of a cone.

Question 4 – Solution

Strategy Relate the polar angle to cylindrical radius and height: $r = \rho \sin \phi$, $z = \rho \cos \phi$.



Conversion At $\phi = \pi/4$,

$$\frac{r}{z} = \tan \phi = 1,$$

so $\boxed{z = r}$ in cylindrical coordinates. Therefore

$$\boxed{z = \sqrt{x^2 + y^2}}.$$

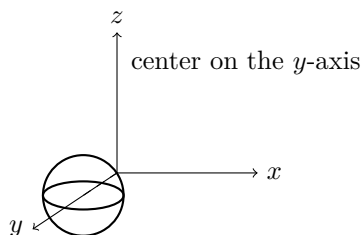
Geometry Because $\phi = \pi/4 < \pi/2$, $z = \rho \cos \phi \geq 0$. Thus this is the upper nappe only. Squaring to $z^2 = x^2 + y^2$ without retaining $z \geq 0$ would incorrectly add the lower nappe.

Question 5

Convert $\rho = 2 \sin \phi \sin \theta$ to rectangular coordinates and identify the surface. Which coordinate axis contains its center?

Question 5 – Solution

Strategy Recognize $\rho \sin \phi \sin \theta = y$, multiply the equation by ρ , and complete a square.



Conversion

$$\rho^2 = 2\rho \sin \phi \sin \theta = 2y.$$

Thus

$$x^2 + y^2 + z^2 = 2y \iff \boxed{x^2 + (y - 1)^2 + z^2 = 1}.$$

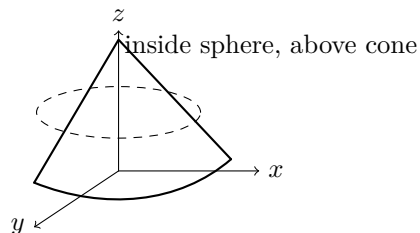
Geometry This is a sphere of radius 1 centered at $\boxed{(0, 1, 0)}$, so its center lies on the positive y -axis. The sphere is tangent to the origin.

Question 6

Describe with spherical inequalities the solid inside the sphere $\rho = 4$ and above the cone $\phi = \pi/3$. Include the full azimuthal range and explain the meaning of “above.”

Question 6 – Solution

Strategy A smaller polar angle lies closer to the positive z -axis.



Bounds Inside the sphere gives $0 \leq \rho \leq 4$. The full rotational sweep is $0 \leq \theta < 2\pi$. Above the cone means between the positive z -axis and the conical surface, so $0 \leq \phi \leq \pi/3$. Hence

$$\boxed{0 \leq \rho \leq 4, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{3}}.$$

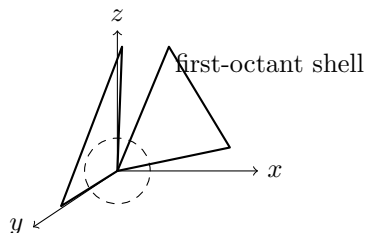
Boundary note At $\rho = 0$, angles are not unique, but including all angle values does not create additional geometric points.

Question 7

Write spherical-coordinate bounds for the first-octant portion of the shell between the spheres of radii 1 and 3. Justify the angular bounds.

Question 7 – Solution

Strategy Separate radial shell bounds from the two angular restrictions defining the first octant.



Bounds Between the spheres,

$$1 \leq \rho \leq 3.$$

The first quadrant in the xy -plane requires $0 \leq \theta \leq \pi/2$. Being above the xy -plane requires $0 \leq \phi \leq \pi/2$. Therefore

$$\boxed{1 \leq \rho \leq 3, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}}.$$

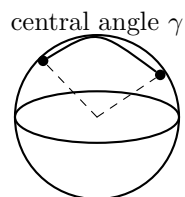
These bounds impose $x, y, z \geq 0$ while retaining every point of the radial shell in that octant.

Question 8

Two points have spherical coordinates $(\rho_1, \theta_1, \phi_1)$ and $(\rho_2, \theta_2, \phi_2)$. Derive a formula for the square of their distance using their central angle.

Question 8 – Solution

Strategy Write the spherical unit direction vectors and use the dot-product form of the law of cosines.



Direction dot product If γ is the angle between the radial directions, then

$$\cos \gamma = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2).$$

Distance With position vectors $\vec{p}_1, \vec{p}_2$,

$$d^2 = \|\vec{p}_1 - \vec{p}_2\|^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos \gamma.$$

Therefore

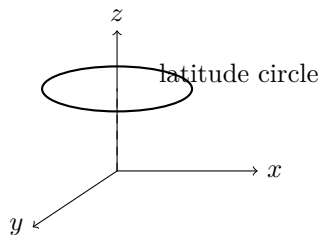
$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 [\cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2)].$$

Question 9

A curve on the sphere $\rho = R$ has constant polar angle $\phi = \alpha$, while $\theta = t$ varies from 0 to 2π . Find a rectangular parameterization and the exact length of the curve.

Question 9 – Solution

Strategy Convert the fixed spherical coordinates and recognize a horizontal circle.



Parameterization

$$\vec{r}(t) = \langle R \sin \alpha \cos t, R \sin \alpha \sin t, R \cos \alpha \rangle, \quad 0 \leq t \leq 2\pi.$$

Length Differentiation gives

$$\|\vec{r}'(t)\| = R \sin \alpha$$

for $0 \leq \alpha \leq \pi$. Hence

$$L = 2\pi R \sin \alpha.$$

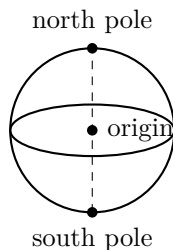
Geometry The latitude circle has Euclidean radius $R \sin \alpha$ and lies in the horizontal plane $z = R \cos \alpha$. At either pole, this radius and length are zero.

Question 10

Under the standard ranges $\rho \geq 0$, $0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$, discuss exactly where spherical coordinates fail to be unique and why.

Question 10 – Solution

Strategy Examine the conversion formulas at the origin and on the polar axis.



Origin If $\rho = 0$, then $x = y = z = 0$ for every θ and ϕ . Thus both angles are arbitrary at the origin.

Polar axis At $\phi = 0$ or $\phi = \pi$, $\sin \phi = 0$, so $x = y = 0$ regardless of θ . Therefore azimuth is arbitrary at the north and south polar rays.

Elsewhere For $\rho > 0$ and $0 < \phi < \pi$, the height uniquely determines $\phi = \arccos(z/\rho)$, and the nonzero projection (x, y) uniquely determines θ in $[0, 2\pi)$. Thus coordinates are unique away from the origin and z -axis.