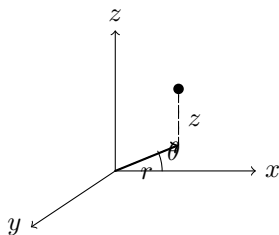


Question 1

Convert the rectangular point $(-3, 3\sqrt{3}, 5)$ to cylindrical coordinates using the convention $r \geq 0$ and $0 \leq \theta < 2\pi$. Verify by converting back.

Question 1 – Solution

Strategy Compute r first, then place the angle in the correct quadrant.



Conversion

$$r = \sqrt{(-3)^2 + (3\sqrt{3})^2} = 6.$$

Because $x < 0$ and $y > 0$, the point lies in Quadrant II. With $\tan \theta = -\sqrt{3}$, $\theta = 2\pi/3$. Hence

$$(r, \theta, z) = \left(6, \frac{2\pi}{3}, 5\right).$$

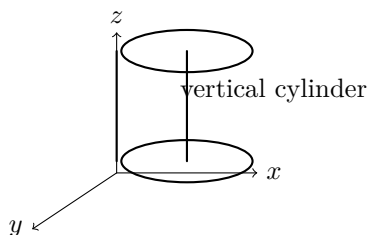
Verification $x = 6 \cos(2\pi/3) = -3$, $y = 6 \sin(2\pi/3) = 3\sqrt{3}$, and $z = 5$.

Question 2

Rewrite the surface $x^2 + y^2 = 4x$ in cylindrical coordinates. Identify the surface geometrically, including its axis and radius.

Question 2 – Solution

Strategy Replace $x^2 + y^2$ by r^2 and x by $r \cos \theta$.



Cylindrical equation

$$r^2 = 4r \cos \theta.$$

The equation $r = 4 \cos \theta$ describes the non-axis points; $r = 0$ is already included, so

$$\boxed{r = 4 \cos \theta}.$$

Geometry In rectangular form,

$$(x - 2)^2 + y^2 = 4.$$

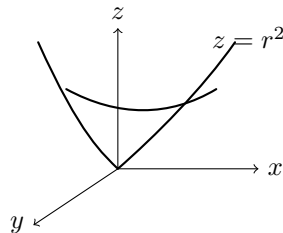
Thus the surface is a vertical circular cylinder of radius 2, whose axis is the line $\boxed{x = 2, y = 0}$ parallel to the z -axis.

Question 3

Convert the paraboloid $z = x^2 + y^2$ to cylindrical coordinates. Describe its traces for fixed z and for fixed θ , and explain why the new equation reveals its symmetry.

Question 3 – Solution

Strategy Use $x^2 + y^2 = r^2$ and interpret one coordinate at a time.



Equation

$$\boxed{z = r^2}, \quad r \geq 0.$$

Horizontal traces For $z = c \geq 0$, $r = \sqrt{c}$, a circle centered on the z -axis. There are no points for $c < 0$.

Vertical traces Fixing θ selects a vertical half-plane through the z -axis; within it, $z = r^2$ is one half of an upward-opening parabola (including its vertex).

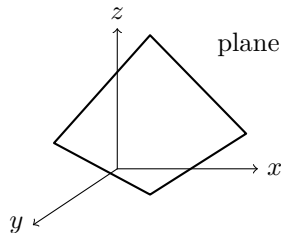
Symmetry The equation contains no θ , so rotating any point about the z -axis preserves the surface.

Question 4

Express the plane $x + y + z = 4$ in cylindrical coordinates. Solve for r where possible and state the angles at which that solved form breaks down.

Question 4 – Solution

Strategy Substitute $x = r \cos \theta$ and $y = r \sin \theta$.



Equation

$$r(\cos \theta + \sin \theta) + z = 4.$$

If $\cos \theta + \sin \theta \neq 0$, then

$$r = \frac{4 - z}{\cos \theta + \sin \theta}.$$

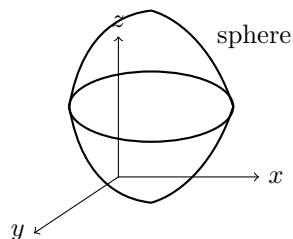
Exceptional angles $\cos \theta + \sin \theta = 0$ when $\theta = 3\pi/4$ or $7\pi/4$ modulo 2π . At those directions, the original cylindrical equation reduces to $z = 4$; dividing by the zero angular factor would incorrectly discard or invent points.

Question 5

Convert $x^2 + y^2 + z^2 = 6z$ to cylindrical coordinates and identify the surface. Describe the ranges of r and z on it.

Question 5 – Solution

Strategy Complete the square in z after replacing $x^2 + y^2$ by r^2 .



Equation

$$r^2 + z^2 = 6z \iff \boxed{r^2 + (z - 3)^2 = 9}.$$

Geometry This is a sphere of radius 3 centered at $(0, 0, 3)$. It is independent of θ , reflecting rotational symmetry about the z -axis.

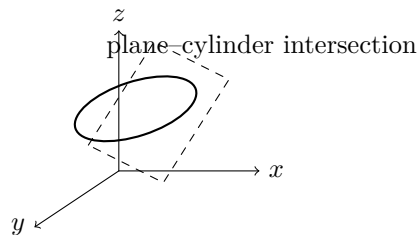
Ranges Since $r^2 \leq 9$, $\boxed{0 \leq r \leq 3}$. The vertical extent is $\boxed{0 \leq z \leq 6}$. At a fixed height z , the horizontal trace has radius $\sqrt{9 - (z - 3)^2}$.

Question 6

Find a one-parameter vector representation for the curve where the cylinder $x^2 + y^2 = 4$ meets the plane $z = x + y$. State an interval that traces the curve exactly once.

Question 6 – Solution

Strategy The cylinder fixes $r = 2$; use $\theta = t$ and substitute into the plane.



Coordinates

$$x = 2 \cos t, \quad y = 2 \sin t, \quad z = x + y = 2 \cos t + 2 \sin t.$$

Therefore

$$\boxed{\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 2 \cos t + 2 \sin t \rangle}, \quad 0 \leq t < 2\pi.$$

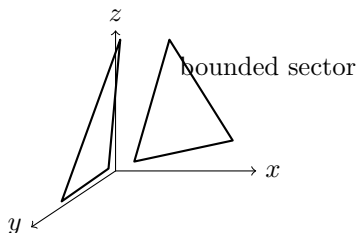
Verification The first two components satisfy $x^2 + y^2 = 4$, and the third is their sum. The half-open interval prevents duplicating the starting point.

Question 7

Describe in cylindrical inequalities the solid in the first octant that lies between the cylinders $r = 1$ and $r = 3$, above $z = 0$, and below the cone $z = 4 - r$. Explain each bound.

Question 7 – Solution

Strategy Translate the geometric restrictions directly into angular, radial, and vertical bounds.



Bounds First octant in the xy -plane gives $0 \leq \theta \leq \pi/2$. Between the cylinders gives $1 \leq r \leq 3$. At each (r, θ) , the solid lies from the xy -plane up to the cone:

$$\boxed{0 \leq \theta \leq \frac{\pi}{2}, \quad 1 \leq r \leq 3, \quad 0 \leq z \leq 4 - r.}$$

Consistency On $1 \leq r \leq 3$, the upper bound $4 - r$ remains positive. The cone is rotationally symmetric, while the first-octant restriction selects only a quarter-sector.

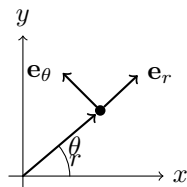
Question 8

A moving point has cylindrical coordinates $r(t), \theta(t), z(t)$. Derive its rectangular velocity and prove that its speed satisfies

$$\|\vec{v}\|^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2.$$

Question 8 – Solution

Strategy Differentiate $x = r \cos \theta$ and $y = r \sin \theta$, then simplify the sum of squares.



Velocity

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta, \quad \dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta, \quad \dot{z} = \dot{z}.$$

Speed identity Expanding $\dot{x}^2 + \dot{y}^2$ makes the cross terms cancel, while $\sin^2 \theta + \cos^2 \theta = 1$. Therefore

$$\|\vec{v}\|^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2.$$

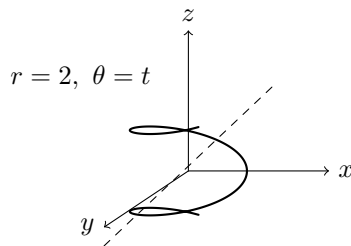
The radial, angular, and vertical velocity directions are mutually perpendicular.

Question 9

A curve is given in cylindrical form by $r = 2$, $\theta = t$, and $z = 3t$. Convert it to rectangular vector form, identify the curve, and find the vertical rise per complete revolution.

Question 9 – Solution

Strategy Substitute into $x = r \cos \theta$, $y = r \sin \theta$ and examine the change when t increases by 2π .



Rectangular form

$$\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 3t \rangle.$$

It is a circular helix of radius 2 around the z -axis.

Pitch One complete revolution corresponds to $\Delta t = 2\pi$. Therefore

$$\Delta z = 3(2\pi) = 6\pi.$$

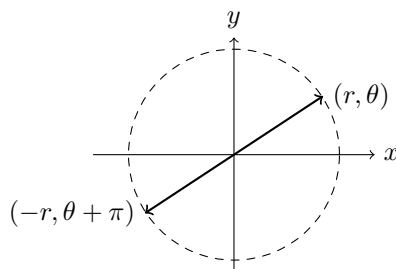
The constant r keeps the curve on a cylinder, while the linear z -coordinate produces uniform vertical advance.

Question 10

Explain why cylindrical coordinates are not unique if negative values of r are allowed. Prove the identity relating (r, θ, z) and $(-r, \theta + \pi, z)$, then state a standard uniqueness convention and its remaining exception.

Question 10 – Solution

Strategy Compare the rectangular coordinates produced by the two triples.



Identity

$$(-r) \cos(\theta + \pi) = (-r)(-\cos \theta) = r \cos \theta,$$

$$(-r) \sin(\theta + \pi) = (-r)(-\sin \theta) = r \sin \theta.$$

The z -coordinate is unchanged, so

$$\boxed{(r, \theta, z) \sim (-r, \theta + \pi, z)}.$$

Convention Requiring $r \geq 0$ and $0 \leq \theta < 2\pi$ makes coordinates unique whenever $r > 0$. On the z -axis, $r = 0$ and every angle represents the same point, so θ remains intrinsically undetermined there.