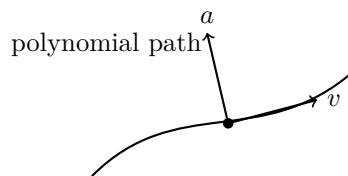


Question 1

A particle has position $\vec{r}(t) = \langle t^2 - 2t, t^3 - 3t, 2t \rangle$, $t \geq 0$. Find velocity, acceleration, and speed at $t = 2$. Determine whether it is speeding up or slowing down then.

Question 1 – Solution

Strategy Differentiate componentwise and use the sign of $\vec{v} \cdot \vec{a}$.



Kinematics

$$\vec{v} = \langle 2t - 2, 3t^2 - 3, 2 \rangle, \quad \vec{a} = \langle 2, 6t, 0 \rangle.$$

At $t = 2$, $\vec{v} = \langle 2, 9, 2 \rangle$, $\vec{a} = \langle 2, 12, 0 \rangle$, and

$\text{speed} = \|\vec{v}\| = \sqrt{89}.$

Speeding test $\vec{v} \cdot \vec{a} = 4 + 108 = 112 > 0$, so speed is increasing. The particle is

speeding up

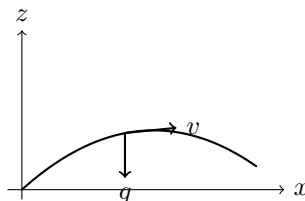
 at $t = 2$.

Question 2

A projectile is launched from $\langle 0, 0, 5 \rangle$ with initial velocity $\langle 12, 5, 20 \rangle$ ft/s. With gravity $\vec{a} = \langle 0, 0, -32 \rangle$ ft/s², find its position, time of impact with $z = 0$, and horizontal range vector.

Question 2 – Solution

Strategy Integrate constant acceleration and impose the ground condition on the vertical coordinate.



Position

$$\vec{r}(t) = \langle 12t, 5t, 5 + 20t - 16t^2 \rangle.$$

Impact Solve $5 + 20t - 16t^2 = 0$. The positive root is

$$t_* = \frac{5 + \sqrt{45}}{8} = \frac{5 + 3\sqrt{5}}{8}.$$

Horizontal range At impact,

$$\langle x(t_*), y(t_*) \rangle = \frac{5 + 3\sqrt{5}}{8} \langle 12, 5 \rangle.$$

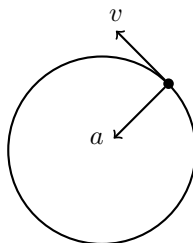
The horizontal velocity is constant because gravity has no horizontal component.

Question 3

A particle moves on the circle $\vec{r}(t) = \langle R \cos(\omega t), R \sin(\omega t) \rangle$, where $R, \omega > 0$. Find velocity, acceleration, speed, and show that acceleration points toward the center with magnitude v^2/R .

Question 3 – Solution

Strategy Differentiate twice and compare acceleration with the position vector.



Derivatives

$$\vec{v} = R\omega \langle -\sin \omega t, \cos \omega t \rangle, \quad \vec{a} = -R\omega^2 \langle \cos \omega t, \sin \omega t \rangle = -\omega^2 \vec{r}.$$

Thus $\boxed{v = R\omega}$ and acceleration is radially inward.

Magnitude

$$\|\vec{a}\| = R\omega^2 = \frac{(R\omega)^2}{R} = \boxed{\frac{v^2}{R}}.$$

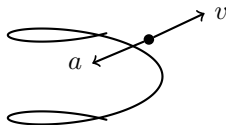
Velocity is tangent to the circle and perpendicular to the centripetal acceleration.

Question 4

For the helix $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$, find velocity and acceleration. Prove that its speed is constant and decide whether acceleration is entirely normal.

Question 4 – Solution

Strategy Compute $\vec{v} \cdot \vec{a}$ after differentiating.



Derivatives

$$\vec{v} = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \vec{a} = \langle -3 \cos t, -3 \sin t, 0 \rangle.$$

The speed is $\|\vec{v}\| = 5$, independent of t .

Decomposition Since

$$\vec{v} \cdot \vec{a} = 9 \sin t \cos t - 9 \sin t \cos t = 0,$$

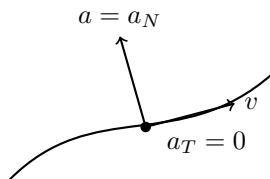
acceleration is perpendicular to velocity. Hence $a_T = 0$ and all acceleration is normal, with $a_N = \|\vec{a}\| = 3$.

Question 5

At an instant, a particle has velocity $\vec{v} = \langle 2, 1, 2 \rangle$ and acceleration $\vec{a} = \langle 1, 2, -2 \rangle$. Find the scalar tangential and normal components a_T, a_N . State what each says about the motion.

Question 5 – Solution

Strategy Use $a_T = (\vec{v} \cdot \vec{a})/\|\vec{v}\|$ and $a_N = \sqrt{\|\vec{a}\|^2 - a_T^2}$.



Components Here $\|\vec{v}\| = 3$, $\|\vec{a}\| = 3$, and $\vec{v} \cdot \vec{a} = 0$. Thus

$$\boxed{a_T = 0}, \quad \boxed{a_N = 3}.$$

Interpretation The instantaneous speed is not changing because the tangential component is zero. The full acceleration changes the direction of motion; it is perpendicular to the velocity.

Question 6

Prove that a particle with nonzero velocity is speeding up exactly when $\vec{v} \cdot \vec{a} > 0$, and slowing down exactly when $\vec{v} \cdot \vec{a} < 0$.

Question 6 – Solution

Strategy Differentiate speed through its square to avoid differentiating a square root prematurely.

$$v \cdot a > 0$$



Derivation Let $v = \|\vec{v}\| > 0$. Then

$$v^2 = \vec{v} \cdot \vec{v}.$$

Differentiating gives $2v \, dv/dt = 2\vec{v} \cdot \vec{a}$, so

$$\boxed{\frac{dv}{dt} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{v}\|}}.$$

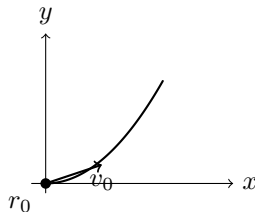
Conclusion The denominator is positive. Therefore the sign of the speed derivative is exactly the sign of $\vec{v} \cdot \vec{a}$: positive means speeding up, negative means slowing down, and zero means instantaneously constant speed.

Question 7

A particle has acceleration $\vec{a}(t) = \langle 2, 6t, e^t \rangle$, initial velocity $\vec{v}(0) = \langle 1, -1, 0 \rangle$, and initial position $\vec{r}(0) = \langle 2, 0, 3 \rangle$. Determine $\vec{v}(t)$ and $\vec{r}(t)$.

Question 7 – Solution

Strategy Integrate twice and use each vector initial condition to determine constants.



Velocity

$$\vec{v}(t) = \langle 2t + C_1, 3t^2 + C_2, e^t + C_3 \rangle.$$

Using $\vec{v}(0) = \langle 1, -1, 0 \rangle$ gives

$$\boxed{\vec{v}(t) = \langle 2t + 1, 3t^2 - 1, e^t - 1 \rangle}.$$

Position Integrating again and applying $\vec{r}(0) = \langle 2, 0, 3 \rangle$,

$$\boxed{\vec{r}(t) = \langle t^2 + t + 2, t^3 - t, e^t - t + 2 \rangle}.$$

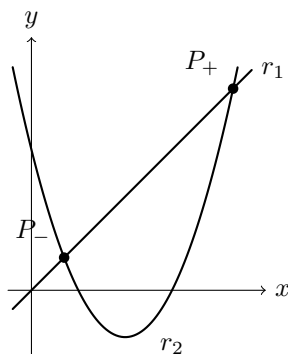
Verification Differentiating the final position recovers $\vec{v}$, and differentiating once more recovers the given acceleration.

Question 8

Particles follow $\vec{r}_1(t) = \langle t, t, 0 \rangle$ and $\vec{r}_2(t) = \langle 2 - t, t^2 - 1, 0 \rangle$. Do their paths intersect? Do the particles collide for $t \geq 0$? Clearly distinguish the two questions.

Question 8 – Solution

Strategy Path intersection may use different parameter values; collision requires equal positions at the same time.



Path intersection Write the first path as $y = x$. On the second, $x = 2 - u$ and $y = u^2 - 1$. Thus

$$u^2 + u - 3 = 0, \quad u = \frac{-1 \pm \sqrt{13}}{2}.$$

The full paths intersect at

$$\left(\frac{5 \mp \sqrt{13}}{2}, \frac{5 \mp \sqrt{13}}{2}, 0 \right).$$

If both paths are restricted to nonnegative times, only the point with $5 - \sqrt{13}$ remains.

Collision test At the same time t , equality of x -coordinates requires $t = 2 - t$, so $t = 1$. But

$$\vec{r}_1(1) = \langle 1, 1, 0 \rangle, \quad \vec{r}_2(1) = \langle 1, 0, 0 \rangle.$$

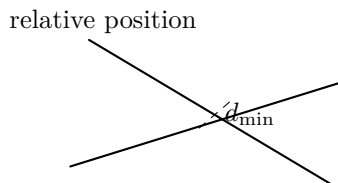
Therefore the particles do not collide. The geometric intersections occur at different parameter times.

Question 9

Two particles move according to $\vec{r}_1(t) = \langle t, 2t, 0 \rangle$ and $\vec{r}_2(t) = \langle 4 - t, 1, 2t \rangle$ for $t \geq 0$. Find the time and value of their minimum separation.

Question 9 – Solution

Strategy Minimize squared distance; it has the same minimizer as distance and avoids a square root.



Relative position

$$\vec{d}(t) = \vec{r}_1 - \vec{r}_2 = \langle 2t - 4, 2t - 1, -2t \rangle.$$

Thus

$$D^2(t) = (2t - 4)^2 + (2t - 1)^2 + (-2t)^2 = 12t^2 - 20t + 17.$$

Minimum The vertex occurs at $t = 20/24 = 5/6$. Then

$$D^2\left(\frac{5}{6}\right) = \frac{26}{3}, \quad \boxed{D_{\min} = \sqrt{\frac{26}{3}}}.$$

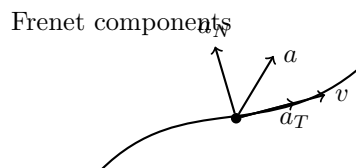
Because the quadratic opens upward and $5/6 \geq 0$, this is the required global minimum.

Question 10

For a regular motion with speed $v = ds/dt$, prove the Frenet acceleration formula $\vec{a} = (dv/dt)\mathbf{T} + \kappa v^2\mathbf{N}$. Identify the geometric role of each term.

Question 10 – Solution

Strategy Write velocity as speed times unit tangent and apply the product and chain rules.



Derivation

$$\vec{v} = v\mathbf{T}.$$

Therefore

$$\vec{a} = \frac{d}{dt}(v\mathbf{T}) = \frac{dv}{dt}\mathbf{T} + v\frac{d\mathbf{T}}{ds}\frac{ds}{dt}.$$

Since $d\mathbf{T}/ds = \kappa\mathbf{N}$ and $ds/dt = v$,

$$\boxed{\vec{a} = \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N}}.$$

Meaning The tangential term changes speed. The normal term changes direction and points toward the local center of curvature; its magnitude grows quadratically with speed.