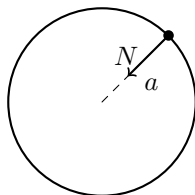


Question 1

Starting from $\vec{r}(t) = \langle a \cos t, a \sin t \rangle$, where $a > 0$, derive the curvature of a circle. Interpret the result in terms of turning sharpness and radius of curvature.

Question 1 – Solution

Strategy Use $\kappa = \|\vec{r}' \times \vec{r}''\| / \|\vec{r}'\|^3$, viewing planar vectors as three-dimensional vectors with zero third component.



Calculation $\vec{r}' = \langle -a \sin t, a \cos t, 0 \rangle$ and $\vec{r}'' = \langle -a \cos t, -a \sin t, 0 \rangle$. Thus $\|\vec{r}'\| = a$ and $\|\vec{r}' \times \vec{r}''\| = a^2$. Therefore

$$\boxed{\kappa = \frac{a^2}{a^3} = \frac{1}{a}}, \quad \boxed{\rho = \frac{1}{\kappa} = a}.$$

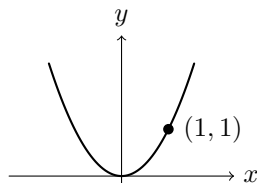
Interpretation Small circles turn more sharply and have larger curvature; the radius of curvature ρ is exactly the circle's geometric radius.

Question 2

Find the curvature of the parabola $\vec{r}(t) = \langle t, t^2 \rangle$ for general t , and evaluate it at the point $(1, 1)$. Where is the curvature largest?

Question 2 – Solution

Strategy Apply the planar curvature formula and then analyze its denominator.



Calculation With $x' = 1$, $y' = 2t$, $x'' = 0$, and $y'' = 2$,

$$\kappa(t) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} = \boxed{\frac{2}{(1 + 4t^2)^{3/2}}}.$$

At $t = 1$, $\boxed{\kappa = 2/(5\sqrt{5})}$.

Maximum The numerator is constant and the denominator is minimized at $t = 0$. Hence the maximum curvature is $\kappa(0) = 2$, at the vertex.

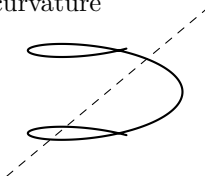
Question 3

For the helix $\vec{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle$, compute the curvature for all t . Explain geometrically why a constant answer is reasonable.

Question 3 – Solution

Strategy Compute speed and the magnitude of $\vec{r}' \times \vec{r}''$ separately.

constant curvature



Calculation

$$\vec{r}' = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \vec{r}'' = \langle -3 \cos t, -3 \sin t, 0 \rangle,$$

$$\vec{r}' \times \vec{r}'' = \langle 12 \sin t, -12 \cos t, 9 \rangle.$$

Thus $\|\vec{r}'\| = 5$ and $\|\vec{r}' \times \vec{r}''\| = 15$, giving

$$\kappa = \frac{15}{5^3} = \frac{3}{25}.$$

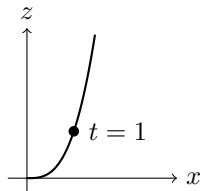
Geometry Every point of a circular helix has the same local shape: rotation about and translation along its axis carry one point to another, so its bending is uniform.

Question 4

Compute the curvature of the twisted cubic $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $t = 1$. Show the cross product and simplify the exact value.

Question 4 – Solution

Strategy Use the parameter-independent cross-product formula.



Derivatives

$$\vec{r}' = \langle 1, 2t, 3t^2 \rangle, \quad \vec{r}'' = \langle 0, 2, 6t \rangle,$$

$$\vec{r}' \times \vec{r}'' = \langle 6t^2, -6t, 2 \rangle.$$

At $t = 1$, its magnitude is $2\sqrt{19}$, while $\|\vec{r}'(1)\| = \sqrt{14}$. Therefore

$$\kappa(1) = \frac{2\sqrt{19}}{(\sqrt{14})^3} = \frac{\sqrt{19}}{7\sqrt{14}}.$$

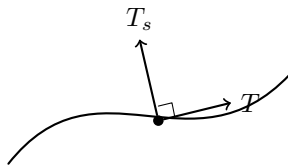
Check The cross product is nonzero, so the tangent is genuinely changing direction and the curvature is positive.

Question 5

Let $\vec{r}(s)$ be a twice-differentiable unit-speed curve. Prove that $\kappa(s) = \|\vec{r}''(s)\|$ and that the acceleration $\vec{r}''(s)$ is perpendicular to the tangent.

Question 5 – Solution

Strategy Differentiate the unit-speed identity $\vec{r}' \cdot \vec{r}' = 1$.



Orthogonality Because $\|\vec{r}'\| = 1$,

$$\vec{r}' \cdot \vec{r}' = 1 \implies 2\vec{r}' \cdot \vec{r}'' = 0.$$

Thus $\vec{r}'' \perp \vec{r}' = \mathbf{T}$.

Curvature For an arc-length parameter, $\mathbf{T} = d\vec{r}/ds$. By definition,

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d^2\vec{r}}{ds^2} \right\|.$$

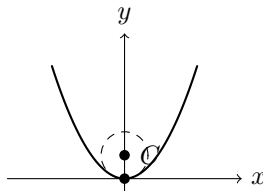
Interpretation Unit speed removes tangential acceleration. Any remaining second derivative measures only the rate at which the direction turns.

Question 6

Find the osculating circle of the parabola $y = x^2$ at its vertex. State its center, radius, and equation, and explain which side of the tangent contains the center.

Question 6 – Solution

Strategy Use curvature for the radius and the principal normal for the center direction.



Curvature and normal From Question 2, $\kappa(0) = 2$, so $\rho = 1/\kappa = 1/2$. At the vertex, the unit tangent points right and the parabola bends upward, so $\mathbf{N} = \langle 0, 1 \rangle$.

Center and equation

$$C = (0, 0) + \frac{1}{2} \langle 0, 1 \rangle = \left(0, \frac{1}{2} \right),$$

$$\boxed{x^2 + \left(y - \frac{1}{2} \right)^2 = \frac{1}{4}}.$$

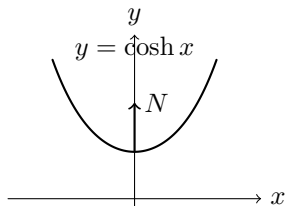
Geometry The center lies on the concave side of the curve, above the horizontal tangent. The circle has second-order contact with the parabola at the vertex.

Question 7

For the catenary $y = \cosh x$, find $\kappa(x)$ and the radius of curvature $\rho(x)$. Where is the curve most sharply bent, and what happens as $|x| \rightarrow \infty$?

Question 7 – Solution

Strategy Use the graph formula $\kappa = |y''|/(1 + y'^2)^{3/2}$ and a hyperbolic identity.



Calculation Since $y' = \sinh x$, $y'' = \cosh x$, and $1 + \sinh^2 x = \cosh^2 x$,

$$\kappa(x) = \frac{\cosh x}{\cosh^3 x} = \operatorname{sech}^2 x, \quad \rho(x) = \cosh^2 x.$$

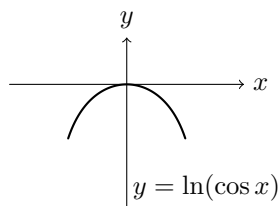
Analysis Curvature is largest at $x = 0$, where $\kappa = 1$. As $|x| \rightarrow \infty$, $\kappa \rightarrow 0$ and $\rho \rightarrow \infty$, so the arms become locally less sharply bent.

Question 8

On $-\frac{\pi}{2} < x < \frac{\pi}{2}$, determine the curvature of $y = \ln(\cos x)$. Explain why the stated domain makes the simplification unambiguous.

Question 8 – Solution

Strategy Differentiate twice and reduce the denominator using $1 + \tan^2 x = \sec^2 x$.



Calculation

$$y' = -\tan x, \quad y'' = -\sec^2 x.$$

Therefore

$$\kappa(x) = \frac{\sec^2 x}{(1 + \tan^2 x)^{3/2}} = \frac{\sec^2 x}{|\sec x|^3}.$$

On the stated interval, $\cos x > 0$ and $\sec x > 0$, so

$$\boxed{\kappa(x) = \cos x}.$$

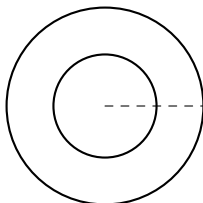
Behavior The maximum curvature is 1 at $x = 0$, while $\kappa \rightarrow 0$ as the graph approaches either vertical asymptote.

Question 9

A regular curve $\vec{r}(t)$ is enlarged by a factor $c > 0$ to form $\vec{q}(t) = c\vec{r}(t)$. Prove that $\kappa_q = \kappa_r/c$. What happens to the radius of curvature?

Question 9 – Solution

Strategy Track how first derivatives, second derivatives, cross products, and norms scale.



dilation by c lowers curvature by $1/c$

Scaling

$$\vec{q}' = c\vec{r}', \quad \vec{q}'' = c\vec{r}'', \quad \|\vec{q}' \times \vec{q}''\| = c^2 \|\vec{r}' \times \vec{r}''\|.$$

Also $\|\vec{q}'\|^3 = c^3 \|\vec{r}'\|^3$. Hence

$$\kappa_q = \frac{c^2}{c^3} \kappa_r = \frac{\kappa_r}{c}.$$

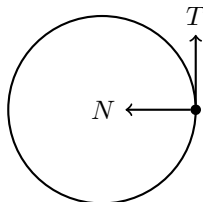
Radius Since $\rho = 1/\kappa$, $\boxed{\rho_q = c\rho_r}$. Enlargement makes every local osculating circle c times larger and the curve correspondingly less sharply bent.

Question 10

The circle of radius 2 is parameterized nonuniformly by $\vec{r}(t) = \langle 2 \cos(t^3), 2 \sin(t^3) \rangle$ for $t > 0$. Compute its curvature directly and explain why the changing speed does not affect the result.

Question 10 – Solution

Strategy Let $\theta = t^3$ and use the turning-angle relation $\kappa = \|d\mathbf{T}/dt\|/(ds/dt)$.



nonuniform parameter, same curvature

Tangent and rates Since $\theta' = 3t^2 > 0$,

$$\mathbf{T} = \langle -\sin \theta, \cos \theta \rangle, \quad \left\| \frac{d\mathbf{T}}{dt} \right\| = 3t^2.$$

The speed is $ds/dt = \|\vec{r}'\| = 2(3t^2) = 6t^2$.

Curvature

$$\kappa = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{3t^2}{6t^2} = \frac{1}{2}.$$

Interpretation Curvature measures direction change per unit distance, not per unit time. Both direction-change rate and speed acquire the same parameter factor, which cancels.