

Question 1

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$.

- (a) Test the absolute-value series first.
- (b) Classify the original series and identify the strongest efficient theorem.
- (c) Give an upper bound for the absolute tail after N terms.

Question 1 – Solution

Step 1: Remove the signs.

$$\sum \left| \frac{(-1)^n}{n^{3/2}} \right| = \sum \frac{1}{n^{3/2}}.$$

This is a convergent p -series because $p = 3/2 > 1$.

Step 2: Classify convergence.

Absolute convergence implies ordinary convergence, so the original series converges absolutely. Although the Alternating Series Test also proves ordinary convergence, it gives a weaker conclusion and is unnecessary here.

Step 3: Bound the absolute tail.

Since $x^{-3/2}$ is positive and decreasing,

$$\sum_{n=N+1}^{\infty} \frac{1}{n^{3/2}} \leq \int_N^{\infty} x^{-3/2} dx = \frac{2}{\sqrt{N}}.$$

This also bounds the signed remainder in magnitude.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

- (a) Prove ordinary convergence.
- (b) Test absolute convergence separately.
- (c) Classify the series, state its sum, and give an alternating error bound.

Question 2 – Solution

Step 1: Prove ordinary convergence.

For $b_n = 1/n$, we have $b_n > 0$, b_n decreases, and $b_n \rightarrow 0$. The Alternating Series Test therefore proves convergence.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n},$$

which diverges. Thus the original series is conditionally convergent.

Step 3: State the value and error.

This sign convention is the negative alternating harmonic series, so

$$S = -\ln 2, \quad |S - s_N| \leq \frac{1}{N+1}.$$

Absolute convergence fails because removing the cancellation exposes the harmonic series.

Question 3

Consider $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$.

- (a) Bound the absolute value of each term.
- (b) Apply Direct Comparison to classify absolute convergence.
- (c) Explain what this implies about the original oscillatory series.

Question 3 – Solution

Step 1: Remove oscillation by an inequality.

Since $|\cos n| \leq 1$,

$$0 \leq \left| \frac{\cos n}{n^2} \right| \leq \frac{1}{n^2}.$$

Step 2: Compare absolute values.

The benchmark $\sum 1/n^2$ converges. Direct Comparison gives $\sum |\cos n|/n^2 < \infty$.

Step 3: State the strongest conclusion.

The series converges absolutely, and therefore converges ordinarily. No cancellation or special information about the values of $\cos n$ is needed.

Question 4

Consider $\sum_{n=1}^{\infty} \frac{\sin n}{n}$.

- (a) Prove ordinary convergence using Dirichlet's Test.
- (b) Prove the absolute-value series diverges.
- (c) Classify the series precisely.

Question 4 – Solution

Step 1: Bound the numerator partial sums.

From the finite geometric sum of e^{in} ,

$$\left| \sum_{n=1}^N e^{in} \right| \leq \frac{2}{|1 - e^i|}.$$

Taking imaginary parts shows that $\sum_{n=1}^N \sin n$ is uniformly bounded.

Step 2: Apply Dirichlet's Test.

The sequence $1/n$ decreases to zero. Hence $\sum (\sin n)/n$ converges.

Step 3: Rule out absolute convergence.

Since $0 \leq |\sin n| \leq 1$, we have $|\sin n| \geq \sin^2 n$. Section 4-4 established that $\sum \sin^2 n/n$ diverges; therefore

$$\sum \frac{|\sin n|}{n} \geq \sum \frac{\sin^2 n}{n} = \infty.$$

Thus the series is conditionally convergent.

Question 5

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

- (a) Verify ordinary convergence.
- (b) Test the absolute-value series with the p -series rule.
- (c) Classify convergence and give an error bound.

Question 5 – Solution

Step 1: Apply the AST.

With $b_n = 1/\sqrt{n}$, positivity, decrease, and $b_n \rightarrow 0$ all hold. Thus the series converges.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum \frac{1}{n^{1/2}},$$

which diverges because $p = 1/2 \leq 1$.

Step 3: Classify and estimate.

The convergence is conditional, and

$$|S - s_N| \leq \frac{1}{\sqrt{N+1}}.$$

Question 6

Consider $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$.

- (a) Verify ordinary convergence with the AST.
- (b) Test absolute convergence using the logarithmic benchmark.
- (c) Classify the series and state an alternating error estimate.

Question 6 – Solution**Step 1: Verify the AST.**

Let $b_n = 1/(n \ln n)$. For $n \geq 2$ it is positive. Its denominator is increasing, so b_n decreases, and $b_n \rightarrow 0$. Thus the alternating series converges.

Step 2: Test absolute convergence.

The absolute series is $\sum 1/(n \ln n)$. For $f(x) = 1/(x \ln x)$,

$$\int_2^b f(x) dx = \ln(\ln b) - \ln(\ln 2) \rightarrow \infty.$$

The Integral Test proves divergence.

Step 3: Classify and estimate.

The original series is conditionally convergent, with

$$|S - s_N| \leq \frac{1}{(N+1) \ln(N+1)}.$$

Question 7

Consider $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$.

- (a) Apply the Ratio Test to the absolute-value series.
- (b) Classify convergence.
- (c) Identify the exact sum using the exponential series.

Question 7 – Solution**Step 1: Test absolute values.**Let $b_n = 2^n/n!$. Then

$$\frac{b_{n+1}}{b_n} = \frac{2}{n+1} \rightarrow 0 < 1.$$

The Ratio Test proves that $\sum b_n$ converges.**Step 2: Classify.**

Therefore the signed series converges absolutely. More generally, replacing 2 by any fixed real base c gives ratio $|c|/(n+1) \rightarrow 0$.

Step 3: Find the sum.From $e^x = \sum_{n=0}^{\infty} x^n/n!$, substitute $x = -2$:

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} = e^{-2}.$$

Question 8

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n)!}$.

- (a) Simplify the ratio of consecutive absolute terms.
- (b) Apply the Ratio Test and classify convergence.
- (c) Explain the effect of the doubled factorial.

Question 8 – Solution

Step 1: Form the absolute ratio.

For $b_n = n!/(2n)!$,

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)!}{(2n+2)!} \frac{(2n)!}{n!} = \frac{n+1}{(2n+2)(2n+1)} = \frac{1}{2(2n+1)}.$$

Step 2: Apply the Ratio Test.

The ratio tends to $0 < 1$, so $\sum b_n$ converges. Hence the original series converges absolutely.

Step 3: Interpret.

Each step in $(2n)!$ introduces two new denominator factors but only one new numerator factor. This drives the ratio rapidly to zero.

Question 9

Consider $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^{-n}$.

- (a) Find the limit of the magnitude.
- (b) Apply the nth-term test.
- (c) Explain why alternating signs cannot rescue the series.

Question 9 – Solution

Step 1: Find the magnitude limit.

The standard exponential limit gives

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e, \quad \left(1 + \frac{1}{n}\right)^{-n} \rightarrow e^{-1} \neq 0.$$

Step 2: Examine the signed terms.

Even-indexed terms approach e^{-1} and odd-indexed terms approach $-e^{-1}$. Thus the terms do not approach zero.

Step 3: Apply the nth-term test.

A necessary condition for series convergence fails, so the series diverges. The AST is unavailable because its magnitude condition $b_n \rightarrow 0$ fails.

Question 10

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n^2}$.

- (a) Prove a useful power bound for $\ln n$.
- (b) Use it to test absolute convergence.
- (c) Classify the original series and explain why the AST is not needed.

Question 10 – Solution**Step 1: Dominate the logarithm.**

For $x \geq 1$, $\ln x \leq \sqrt{x}$. One proof minimizes $h(x) = \sqrt{x} - \ln x$; its minimum is $h(4) = 2 - \ln 4 > 0$.

Step 2: Test absolute convergence.

$$0 \leq \frac{\ln n}{n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}.$$

The upper benchmark is a convergent p -series, so Direct Comparison proves convergence of the absolute-value series.

Step 3: Classify.

The original series converges absolutely. The AST could prove ordinary convergence after checking eventual decrease, but absolute comparison is faster and stronger.