

Question 1

Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

- (a) Verify every Alternating Series Test hypothesis.
- (b) Test absolute convergence and classify the series.
- (c) State its familiar value and give the error bound after N terms.

Question 1 – Solution

Step 1: Verify the AST hypotheses.

Write $b_n = 1/n$. Then $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. Hence the series converges by the Alternating Series Test.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n},$$

which diverges. Therefore convergence is conditional.

Step 3: State the value and error.

The power series for $\ln(1+x)$ at $x=1$ gives $S = \ln 2$. The Alternating Series Estimation Theorem gives

$$|S - s_N| \leq b_{N+1} = \frac{1}{N+1}.$$

Odd and even partial sums approach $\ln 2$ from opposite sides.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

- (a) Verify that the Alternating Series Test applies.
- (b) Determine whether the series converges absolutely or conditionally.
- (c) Give an error bound for s_N and explain which conclusion is stronger.

Question 2 – Solution**Step 1: Apply the AST.**

For $b_n = 1/n^2$, we have $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. Thus the alternating series converges.

Step 2: Test absolute convergence.

$$\sum \left| \frac{(-1)^n}{n^2} \right| = \sum \frac{1}{n^2},$$

a convergent p -series with $p = 2$. Hence the original series converges absolutely, a stronger conclusion than the AST alone provides.

Step 3: Bound the alternating error.

$$|S - s_N| \leq \frac{1}{(N+1)^2}.$$

In fact $S = -\pi^2/12$, though this exact value is not needed for classification.

Question 3

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{n}$.

- (a) Evaluate the limit of the term magnitude.
- (b) Apply the nth-term test before considering the AST.
- (c) Explain exactly which AST condition fails.

Question 3 – Solution

Step 1: Simplify the magnitude.

$$b_n = \frac{n+1}{n} = 1 + \frac{1}{n} \rightarrow 1.$$

Thus the series terms $a_n = (-1)^n b_n$ alternate near ± 1 and do not approach zero.

Step 2: Apply the nth-term test.

Every convergent series must satisfy $a_n \rightarrow 0$. Here the even subsequence tends to 1 and the odd subsequence tends to -1 , so the term sequence has no limit and the series diverges.

Step 3: Audit the AST.

Although b_n is positive and decreasing, the required condition $b_n \rightarrow 0$ fails. Alternating signs cannot compensate for terms of persistent size.

Question 4

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

- (a) Verify the AST hypotheses and classify absolute versus conditional convergence.
- (b) Find the smallest N for which the next-term bound b_{N+1} is strictly below 0.01, thereby guaranteeing $|S - s_N| < 0.01$.
- (c) Explain the role of the strict inequality.

Question 4 – Solution**Step 1: Verify convergence.**

With $b_n = 1/\sqrt{n}$, positivity is clear, b_n decreases, and $b_n \rightarrow 0$. The AST proves convergence. The absolute series $\sum 1/\sqrt{n}$ diverges ($p = 1/2$), so convergence is conditional.

Step 2: Apply the error estimate.

$$|S - s_N| \leq b_{N+1} = \frac{1}{\sqrt{N+1}}.$$

To guarantee an error strictly below 0.01 it is sufficient that

$$\frac{1}{\sqrt{N+1}} < \frac{1}{100} \iff N+1 > 10000.$$

Thus the smallest integer furnished by this bound is $\boxed{N = 10000}$.

Step 3: Verify strictness.

At $N = 9999$, the bound equals 0.01, not less than it. At $N = 10000$, it is $1/\sqrt{10001} < 0.01$.

Question 5

Consider $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$.

- (a) Use a continuous derivative to prove eventual decrease of the magnitude.
- (b) Verify the zero-limit condition and apply the AST.
- (c) Test absolute convergence and classify the series.

Question 5 – Solution**Step 1: Prove eventual decrease.**

Let $f(x) = \ln x/x$. Then

$$f'(x) = \frac{1 - \ln x}{x^2} < 0 \quad (x > e).$$

Hence $b_n = \ln n/n$ decreases for integers $n \geq 3$. Eventual decrease is sufficient; finitely many initial terms do not affect convergence.

Step 2: Check the limit.

L'Hopital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0.$$

Therefore the AST proves convergence.

Step 3: Test absolute convergence.

For $n \geq 3$, $\ln n \geq 1$, so $\ln n/n \geq 1/n$. The absolute-value series diverges by comparison with the harmonic series. Thus convergence is conditional.

Question 6

Consider $\sum_{n=2}^{\infty} \frac{(-1)^n}{n + (-1)^n}$.

- (a) Explain the corrected starting index and why the usual AST magnitude is not decreasing.
- (b) Pair consecutive even and odd terms to test convergence.
- (c) Test absolute convergence and classify the series.

Question 6 – Solution

Step 1: Check the domain and AST template.

At $n = 1$ the denominator is $1 - 1 = 0$, so the series must start at 2. Magnitudes satisfy $b_{2k} = 1/(2k+1)$ and $b_{2k+1} = 1/(2k)$; hence $b_{2k+1} > b_{2k}$ infinitely often, so the standard monotonic AST does not apply.

Step 2: Pair terms.

$$a_{2k} + a_{2k+1} = \frac{1}{2k+1} - \frac{1}{2k} = -\frac{1}{2k(2k+1)}.$$

The paired series converges absolutely by comparison with $1/(4k^2)$. Since the unpaired final term tends to zero, both even and odd partial-sum subsequences approach the same limit; the original series converges.

Step 3: Test absolute convergence.

$$|a_{2k}| + |a_{2k+1}| = \frac{1}{2k+1} + \frac{1}{2k} \geq \frac{1}{2k}.$$

Thus the absolute series diverges, so convergence is conditional.

Question 7

Consider $\sum_{n=1}^{\infty} (-1)^n e^{-n}$.

- (a) Verify the AST hypotheses.
- (b) Test absolute convergence using a geometric series.
- (c) Find the exact sum and classify convergence.

Question 7 – Solution**Step 1: Verify the AST.**

With $b_n = e^{-n} > 0$, we have $b_{n+1} = e^{-1}b_n < b_n$ and $b_n \rightarrow 0$. Thus the series converges.

Step 2: Test absolute convergence.

$$\sum |(-1)^n e^{-n}| = \sum (e^{-1})^n,$$

a geometric series with ratio $e^{-1} < 1$. Hence convergence is absolute.

Step 3: Find the sum.

The original series is geometric with first term $-e^{-1}$ and ratio $-e^{-1}$:

$$S = \frac{-e^{-1}}{1 + e^{-1}} = -\frac{1}{e + 1}.$$

Question 8

Consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$.

- (a) Verify decrease without differentiating.
- (b) Apply the AST.
- (c) Test absolute convergence and classify the series.

Question 8 – Solution**Step 1: Verify monotonicity.**

Since $n + 2 > n + 1$ and logarithm is strictly increasing,

$$\ln(n + 2) > \ln(n + 1) > 0.$$

Taking positive reciprocals reverses the inequality, so $b_{n+1} < b_n$ for $b_n = 1/\ln(n + 1)$.

Step 2: Check the limit and apply AST.

Because $\ln(n + 1) \rightarrow \infty$, $b_n \rightarrow 0$. Thus the alternating series converges.

Step 3: Test absolute convergence.

Since $\ln(n + 1) \leq n$ for $n \geq 1$,

$$\frac{1}{\ln(n + 1)} \geq \frac{1}{n}.$$

The absolute series diverges by comparison with the harmonic series. Therefore convergence is conditional.

Question 9

Consider $\sum_{n=1}^{\infty} (-1)^n \sin(1/n)$.

- (a) Verify the AST hypotheses using monotonicity of sine.
- (b) Test absolute convergence by limit comparison with $1/n$.
- (c) Classify the series and state the alternating error bound.

Question 9 – Solution**Step 1: Verify the AST.**

For $n \geq 1$, $1/n \in (0, 1]$ and $b_n = \sin(1/n) > 0$. Since $1/(n+1) < 1/n$ and sine is increasing on $[0, 1]$, $b_{n+1} < b_n$. Continuity gives $b_n \rightarrow \sin 0 = 0$. Thus the series converges.

Step 2: Test absolute convergence.

$$\lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

The harmonic series diverges, so Limit Comparison shows $\sum \sin(1/n)$ diverges. Hence convergence is conditional.

Step 3: State the error bound.

$$|S - s_N| \leq \sin\left(\frac{1}{N+1}\right) < \frac{1}{N+1}.$$

Question 10

Consider $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1}$.

- (a) Use a derivative to verify decreasing magnitudes.
- (b) Apply the AST.
- (c) Test absolute convergence and classify the series.

Question 10 – Solution**Step 1: Verify monotonicity.**

Let $f(x) = x/(x^2 + 1)$. Then

$$f'(x) = \frac{1 - x^2}{(x^2 + 1)^2} \leq 0 \quad (x \geq 1).$$

Thus $b_n = n/(n^2 + 1)$ is nonincreasing and positive.

Step 2: Check the limit and apply AST.

$$b_n = \frac{1/n}{1 + 1/n^2} \longrightarrow 0.$$

Therefore the alternating series converges.

Step 3: Test absolute convergence.

$$\lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1.$$

By comparison with the divergent harmonic series, the absolute series diverges. Hence convergence is conditional.