

Question 1

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n}$.

- (a) Identify the dominant power and select a standard benchmark.
- (b) Prove a direct-comparison inequality with the correct direction.
- (c) Classify the series and confirm the result by limit comparison.

State why the benchmark converges or diverges.

Question 1 – Solution

Step 1: Select the benchmark.

The denominator behaves like n^2 , so use $b_n = 1/n^2$, a convergent p -series with $p = 2$.

Step 2: Prove direct comparison.

Since $n^2 + 3n \geq n^2 > 0$,

$$0 < \frac{1}{n^2 + 3n} \leq \frac{1}{n^2}.$$

The Direct Comparison Test proves convergence.

Step 3: Verify by limit comparison.

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 + 3n)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{1 + 3/n} = 1.$$

The finite positive limit confirms that both series share the same verdict.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}$.

- (a) Use dominant powers to select a harmonic benchmark.
- (b) Compute the Limit Comparison Test ratio.
- (c) Classify the series and give an independent direct lower comparison.

Question 2 – Solution

Step 1: Select a benchmark.

The leading-term quotient is $2n/n^2 = 2/n$, so choose $b_n = 1/n$. All terms are positive.

Step 2: Compute the ratio limit.

$$L = \lim_{n \rightarrow \infty} \frac{(2n+1)/(n^2+1)}{1/n} = \lim_{n \rightarrow \infty} \frac{2n^2+n}{n^2+1} = 2.$$

Because $0 < L < \infty$ and the harmonic series diverges, the given series diverges.

Step 3: Check the direction directly.

For $n \geq 1$, $n^2 + 1 \leq 2n^2$ and $2n + 1 \geq 2n$, so

$$\frac{2n+1}{n^2+1} \geq \frac{2n}{2n^2} = \frac{1}{n}.$$

A series larger than the divergent harmonic series must diverge.

Question 3

Consider $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$ (radians).

- (a) Establish a pointwise trigonometric bound.
- (b) Compare with a standard p -series.
- (c) Classify the series and explain why no averaging argument is needed.

Question 3 – Solution

Step 1: Bound the numerator.

For every real number t , $0 \leq \sin^2 t \leq 1$. Hence for each positive integer n ,

$$0 \leq \frac{\sin^2 n}{n^2} \leq \frac{1}{n^2}.$$

Step 2: Classify the benchmark.

The series $\sum 1/n^2$ converges because it is a p -series with $p = 2 > 1$.

Step 3: Apply Direct Comparison.

The given nonnegative series is bounded term by term above by a convergent series, so it converges. No information about the average value or periodic distribution of $\sin^2 n$ is required.

Question 4

Consider $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$.

- (a) Select a benchmark using dominant powers.
- (b) Perform limit comparison.
- (c) Prove divergence again using a direct lower bound.

Question 4 – Solution

Step 1: Select the harmonic benchmark.

Since $\sqrt{n^2 + 1}$ behaves like n , choose $b_n = 1/n$.

Step 2: Compute the ratio.

$$\lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2 + 1}}{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + 1/n^2}} = 1.$$

The finite positive ratio and divergence of $\sum 1/n$ imply divergence.

Step 3: Give a direct lower bound.

Since $n^2 + 1 \leq 2n^2$ for $n \geq 1$,

$$\frac{1}{\sqrt{n^2 + 1}} \geq \frac{1}{\sqrt{2}n}.$$

The smaller comparison series diverges, independently confirming the verdict.

Question 5

Consider $\sum_{n=1}^{\infty} \frac{n^3}{n^5 + 4}$.

- (a) Choose a benchmark from the dominant powers.
- (b) Compute the limiting ratio and verify all Limit Comparison Test conditions.
- (c) Classify the series and give a direct upper bound.

Question 5 – Solution

Step 1: Select the benchmark.

The leading quotient $n^3/n^5 = 1/n^2$ suggests $b_n = 1/n^2$, a convergent p -series.

Step 2: Compute the ratio limit.

$$\lim_{n \rightarrow \infty} \frac{n^3/(n^5 + 4)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^5}{n^5 + 4} = 1.$$

All terms are positive and the limit lies in $(0, \infty)$, so Limit Comparison proves convergence.

Step 3: Verify directly.

Since $n^5 + 4 \geq n^5$,

$$0 < \frac{n^3}{n^5 + 4} \leq \frac{1}{n^2}.$$

This upper comparison with a convergent series gives the same conclusion.

Question 6

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2 - n + 1}$.

- (a) Prove useful upper and lower quadratic bounds for the denominator.
- (b) Convert them into bounds for the summand, noting reciprocal inequality directions.
- (c) Classify the series and confirm by limit comparison.

Question 6 – Solution

Step 1: Bound the denominator.

Clearly $n^2 - n + 1 \leq n^2$. Also

$$n^2 - n + 1 \geq \frac{n^2}{2} \iff n^2 - 2n + 2 = (n - 1)^2 + 1 \geq 0.$$

Thus $n^2/2 \leq n^2 - n + 1 \leq n^2$.

Step 2: Take reciprocals carefully.

All expressions are positive, so reciprocals reverse the inequalities:

$$\frac{1}{n^2} \leq \frac{1}{n^2 - n + 1} \leq \frac{2}{n^2}.$$

The upper bound is the decisive one for convergence. Since $\sum 2/n^2$ converges, Direct Comparison proves convergence.

Step 3: Confirm asymptotically.

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 - n + 1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/n + 1/n^2} = 1.$$

Question 7

Consider $\sum_{n=1}^{\infty} \frac{3^n + 1}{5^n - 2}$.

- (a) Select a geometric benchmark from the dominant exponential terms.
- (b) Compute the Limit Comparison Test ratio.
- (c) Classify the series and identify the benchmark's common ratio.

Question 7 – Solution

Step 1: Select a geometric benchmark.

The dominant quotient is $3^n/5^n = (3/5)^n$. Let $b_n = (3/5)^n$, whose common ratio $3/5$ has absolute value less than 1; hence $\sum b_n$ converges.

Step 2: Compute the limiting ratio.

$$\frac{(3^n + 1)/(5^n - 2)}{(3/5)^n} = \frac{1 + 3^{-n}}{1 - 2 \cdot 5^{-n}} \rightarrow 1.$$

The original terms and benchmark are positive for $n \geq 1$, and the ratio limit is finite and positive.

Step 3: Apply Limit Comparison.

The two series have the same convergence behavior. Since the benchmark is geometric and convergent,

$$\boxed{\sum_{n=1}^{\infty} \frac{3^n + 1}{5^n - 2} \text{ converges.}}$$

Question 8

Consider $\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$.

- (a) Prove a direct lower bound using $\sqrt{n} \leq n$.
- (b) Use the correct comparison direction to classify the series.
- (c) Confirm the conclusion by limit comparison with $1/n$.

Question 8 – Solution

Step 1: Bound the denominator.

For $n \geq 1$, $\sqrt{n} \leq n$, so

$$n + \sqrt{n} \leq 2n.$$

Because both denominators are positive, taking reciprocals reverses the inequality:

$$\frac{1}{n + \sqrt{n}} \geq \frac{1}{2n}.$$

Step 2: Apply Direct Comparison.

The smaller series $\sum 1/(2n)$ diverges, so the larger given positive-term series diverges.

Step 3: Confirm by limit comparison.

$$\lim_{n \rightarrow \infty} \frac{1/(n + \sqrt{n})}{1/n} = \lim_{n \rightarrow \infty} \frac{1}{1 + 1/\sqrt{n}} = 1.$$

This finite positive limit confirms harmonic tail behavior.

Question 9

Consider $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$.

- (a) Prove a usable power bound for $\ln n$.
- (b) Compare the summand with a convergent p -series.
- (c) Explain why limit comparison with $1/n^3$ is inconclusive.

Question 9 – Solution

Step 1: Bound the logarithm.

We claim $\ln x \leq \sqrt{x}$ for $x \geq 1$. Let $h(x) = \sqrt{x} - \ln x$. Then

$$h'(x) = \frac{\sqrt{x} - 2}{2x},$$

so the minimum occurs at $x = 4$, where $h(4) = 2 - \ln 4 > 0$. The claim follows.

Step 2: Compare directly.

For $n \geq 1$,

$$0 \leq \frac{\ln n}{n^3} \leq \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}.$$

The benchmark is a convergent p -series with $p = 5/2 > 1$, so Direct Comparison proves convergence.

Step 3: Explain benchmark choice.

Comparison with $1/n^3$ by ratios gives $(\ln n/n^3)/(1/n^3) = \ln n \rightarrow \infty$, outside the standard finite-positive Limit Comparison case. A slightly larger convergent power benchmark is more useful.

Question 10

Consider $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

- (a) Show why limit comparison with $1/n$ and $1/n^p$ does not immediately decide the series.
- (b) Select a natural logarithmic benchmark or use the Integral Test.
- (c) Classify the series and state an explicit tail bound.

Question 10 – Solution

Step 1: Audit ordinary power benchmarks.

Relative to $1/n$,

$$\frac{1/[n(\ln n)^2]}{1/n} = \frac{1}{(\ln n)^2} \rightarrow 0,$$

which does not transfer the harmonic series' divergence. For any $p > 1$, the ratio to $1/n^p$ is $n^{p-1}/(\ln n)^2 \rightarrow \infty$, which does not transfer that p -series' convergence.

Step 2: Use the natural logarithmic test.

Let $f(x) = 1/[x(\ln x)^2]$. It is positive, continuous, and decreasing for $x \geq 2$. With $u = \ln x$,

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty u^{-2} du = \frac{1}{\ln 2} < \infty.$$

Thus the series converges by the Integral Test.

Step 3: State the tail estimate.

$$\boxed{\frac{1}{\ln(N+1)} \leq R_N \leq \frac{1}{\ln N}}.$$

The correct benchmark retains the logarithm rather than discarding it.