

Question 1

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

- (a) Define a continuous function f with $f(n) = 1/n^{3/2}$ and verify that it is positive, continuous, and decreasing on the required interval.
- (b) Use the Integral Test to determine whether the series converges or diverges.
- (c) If s_N is the N th partial sum and $R_N = S - s_N$, derive both an upper and a lower bound for R_N .

Show the improper-integral limit and all antiderivative work.

Question 1 – Solution**Step 1: Choose the continuous function.**

Let $f(x) = x^{-3/2}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{3}{2}x^{-5/2} < 0,$$

so it is decreasing. Also $f(n) = 1/n^{3/2}$.

Step 2: Evaluate the improper integral.

$$\int_1^{\infty} x^{-3/2} dx = \lim_{b \rightarrow \infty} \left[-2x^{-1/2} \right]_1^b = 2.$$

The integral converges, so the series converges by the Integral Test.

Step 3: Bound the remainder.

For a positive decreasing f ,

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx.$$

Therefore

$$\boxed{\frac{2}{\sqrt{N+1}} \leq R_N \leq \frac{2}{\sqrt{N}}}.$$

The test proves convergence; these inequalities additionally measure the truncation error.

Question 2

Consider the logarithmic series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

- (a) Explain why the series begins at $n = 2$, and verify all three Integral Test hypotheses for $f(x) = 1/(x \ln x)$.
- (b) Evaluate the associated improper integral using $u = \ln x$.
- (c) State the convergence or divergence conclusion and describe the growth rate of the truncated integral.

The n th-term test alone is not a complete solution.

Question 2 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = 1/(x \ln x)$ for $x \geq 2$. It is continuous and positive because $x > 1$. Moreover,

$$f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2} < 0,$$

so f is decreasing, and $f(n)$ equals the summand.

Step 2: Evaluate the improper integral.

With $u = \ln x$ and $du = dx/x$,

$$\int_2^b \frac{dx}{x \ln x} = \int_{\ln 2}^{\ln b} \frac{du}{u} = \ln(\ln b) - \ln(\ln 2).$$

As $b \rightarrow \infty$, $\ln(\ln b) \rightarrow \infty$. Thus the improper integral diverges.

Step 3: Apply the test.

All hypotheses hold, so the Integral Test gives

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

The growth scale $\ln \ln b$ explains why divergence is extremely slow.

Question 3

Consider

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

- (a) Verify that $f(x) = 1/[x(\ln x)^2]$ is positive, continuous, and decreasing for $x \geq 2$.
- (b) Apply the Integral Test, showing the substitution and evaluation of the improper integral.
- (c) Derive two-sided Integral Test bounds for the remainder R_N .
- (d) Briefly compare the result with $\sum 1/(n \ln n)$.

Question 3 – Solution**Step 1: Verify the hypotheses.**

Define $f(x) = 1/[x(\ln x)^2]$ for $x \geq 2$. It is continuous and positive, and

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3} < 0,$$

so it decreases.

Step 2: Evaluate the integral.

Let $u = \ln x$, $du = dx/x$:

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty u^{-2} du = \frac{1}{\ln 2} < \infty.$$

Hence the series converges. The extra logarithm power changes $\int du/u$ into the convergent p -integral $\int du/u^2$.

Step 3: Estimate the remainder.

$$\boxed{\frac{1}{\ln(N+1)} \leq R_N \leq \frac{1}{\ln N}}.$$

These bounds follow by integrating from $N+1$ and N , respectively.

Question 4

Consider

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}.$$

- (a) For $f(x) = x/(x^2 + 1)^2$, verify positivity, continuity, and decreasing behavior on $[1, \infty)$.
- (b) Evaluate the associated improper integral using an appropriate substitution.
- (c) Determine convergence and obtain upper and lower bounds for R_N .

Clearly distinguish the value of the integral from the value of the series.

Question 4 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = x/(x^2 + 1)^2$ for $x \geq 1$. It is continuous and positive. Differentiation gives

$$f'(x) = \frac{1 - 3x^2}{(x^2 + 1)^3} < 0 \quad (x \geq 1),$$

so it decreases.

Step 2: Evaluate the integral.

With $u = x^2 + 1$ and $du = 2x \, dx$,

$$\int_1^\infty \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2} \int_2^\infty u^{-2} \, du = \frac{1}{4}.$$

The integral converges, so the series converges.

Step 3: Bound the tail.

Since

$$\int_A^\infty \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2(A^2 + 1)},$$

the remainder satisfies

$$\boxed{\frac{1}{2((N+1)^2 + 1)} \leq R_N \leq \frac{1}{2(N^2 + 1)}}.$$

Question 5

Consider the nested-logarithm series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}.$$

- (a) Explain why starting at $n = 3$ makes the terms positive and well defined.
- (b) Verify the Integral Test hypotheses for the corresponding function.
- (c) Evaluate the improper integral using two successive logarithmic substitutions.
- (d) State the verdict and identify the slow growth expression responsible for it.

Question 5 – Solution

Step 1: Check the domain and hypotheses.

Start at $n = 3$ so that $\ln n > 1$ and $\ln(\ln n) > 0$. Let

$$f(x) = \frac{1}{x \ln x \ln(\ln x)} \quad (x \geq 3).$$

The denominator is positive, continuous, and strictly increasing, so f is positive, continuous, and decreasing.

Step 2: Perform two substitutions.

First let $u = \ln x$, $du = dx/x$; then let $v = \ln u$, $dv = du/u$:

$$\int_3^b \frac{dx}{x \ln x \ln(\ln x)} = \int_{\ln 3}^{\ln b} \frac{du}{u \ln u} = [\ln(\ln u)]_{\ln 3}^{\ln b}.$$

Thus the antiderivative grows as $\ln(\ln(\ln b))$, which tends to infinity.

Step 3: Conclude.

By the Integral Test,

$$\boxed{\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)} \text{ diverges.}}$$

This is another borderline series: three nested logarithms still grow without bound.

Question 6

Consider the rapidly decaying series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

- (a) Verify that $f(x) = xe^{-x^2}$ satisfies the Integral Test hypotheses on $[1, \infty)$.
- (b) Evaluate the improper integral with a substitution and classify the series.
- (c) Derive upper and lower bounds for the remainder R_N .
- (d) Explain what the bound says about the speed of convergence.

Question 6 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = xe^{-x^2}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = e^{-x^2}(1 - 2x^2) < 0 \quad (x \geq 1),$$

so it decreases.

Step 2: Evaluate the improper integral.

With $u = x^2$ and $du = 2x \, dx$,

$$\int_1^\infty xe^{-x^2} \, dx = \frac{1}{2} \int_1^\infty e^{-u} \, du = \frac{1}{2e}.$$

Thus the series converges by the Integral Test.

Step 3: Bound the remainder.

Since $\int_A^\infty xe^{-x^2} \, dx = \frac{1}{2}e^{-A^2}$,

$$\boxed{\frac{1}{2}e^{-(N+1)^2} \leq R_N \leq \frac{1}{2}e^{-N^2}}.$$

The Gaussian factor makes the tail decrease exceptionally rapidly.

Question 7

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}.$$

- (a) Verify positivity, continuity, and decreasing behavior for $f(x) = 1/(x^2 + 4)$.
- (b) Use the Integral Test to classify the series, showing the arctangent antiderivative and improper limit.
- (c) Write explicit upper and lower bounds for the remainder after N terms.

Your remainder bounds should contain N and inverse tangent functions.

Question 7 – Solution

Step 1: Verify the hypotheses.

Define $f(x) = 1/(x^2 + 4)$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{2x}{(x^2 + 4)^2} < 0,$$

so it decreases.

Step 2: Test the improper integral.

$$\int_1^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \left[\arctan \left(\frac{x}{2} \right) \right]_1^\infty = \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{1}{2} \right) < \infty.$$

Therefore the series converges.

Step 3: Estimate the remainder.

For $A > 0$,

$$\int_A^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{A}{2} \right).$$

Consequently,

$$\frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{N+1}{2} \right) \leq R_N \leq \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{N}{2} \right).$$

Question 8

Consider

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}.$$

- (a) Verify the Integral Test hypotheses for $f(x) = \ln x/x^2$ on $[2, \infty)$.
- (b) Evaluate the improper integral using integration by parts.
- (c) Determine convergence and derive two-sided bounds for R_N .
- (d) Explain how the result illustrates polynomial growth dominating logarithmic growth.

Question 8 – Solution

Step 1: Verify the hypotheses on the tail.

Let $f(x) = \ln x/x^2$ for $x \geq 2$. It is continuous and positive. Also

$$f'(x) = \frac{1 - 2 \ln x}{x^3} < 0 \quad (x \geq 2),$$

so it decreases.

Step 2: Integrate by parts.

Take $u = \ln x$ and $dv = x^{-2}dx$. Then $du = dx/x$ and $v = -1/x$:

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Hence

$$\int_2^\infty \frac{\ln x}{x^2} dx = \frac{\ln 2 + 1}{2} < \infty,$$

and the series converges.

Step 3: Estimate the remainder.

Since the tail integral from A equals $(\ln A + 1)/A$,

$$\boxed{\frac{\ln(N+1) + 1}{N+1} \leq R_N \leq \frac{\ln N + 1}{N}}.$$

Quadratic denominator growth dominates the logarithmic numerator.

Question 9

Consider the shifted p -series

$$\sum_{n=1}^{\infty} \frac{1}{(n+3)^{5/4}}.$$

- (a) Verify the Integral Test hypotheses for $f(x) = (x+3)^{-5/4}$.
- (b) Evaluate the associated improper integral and classify the series.
- (c) Obtain two-sided bounds for R_N .
- (d) Explain why a fixed horizontal shift does not alter the underlying p -series verdict.

Question 9 – Solution**Step 1: Verify the hypotheses.**

Set $f(x) = (x + 3)^{-5/4}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{5}{4}(x + 3)^{-9/4} < 0,$$

so it decreases.

Step 2: Evaluate the integral.

$$\int_1^{\infty} (x + 3)^{-5/4} dx = \left[-4(x + 3)^{-1/4} \right]_1^{\infty} = \frac{4}{\sqrt{2}} < \infty.$$

Thus the shifted p -series converges. The horizontal shift changes initial values but not the tail exponent $p = 5/4 > 1$.

Step 3: Bound the remainder.

$$\boxed{\frac{4}{(N + 4)^{1/4}} \leq R_N \leq \frac{4}{(N + 3)^{1/4}}}.$$

Although convergent, the exponent is close to 1, so the error decreases slowly.

Question 10

Consider

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}.$$

- (a) Verify the domain and all Integral Test hypotheses for $f(x) = 1/[x\sqrt{\ln x}]$.
- (b) Evaluate the improper integral using $u = \ln x$.
- (c) Determine convergence or divergence.
- (d) Place this series within the family $\sum 1/[n(\ln n)^p]$ and state the threshold value of p .

Question 10 – Solution

Step 1: Verify the hypotheses.

Let $f(x) = 1/[x\sqrt{\ln x}]$ for $x \geq 2$. It is continuous and positive. Its denominator $x\sqrt{\ln x}$ is increasing, so f is decreasing. Equivalently,

$$f'(x) = -\frac{2\ln x + 1}{2x^2(\ln x)^{3/2}} < 0.$$

Step 2: Evaluate the improper integral.

Put $u = \ln x$, so $du = dx/x$:

$$\int_2^b \frac{dx}{x\sqrt{\ln x}} = \int_{\ln 2}^{\ln b} u^{-1/2} du = 2\sqrt{\ln b} - 2\sqrt{\ln 2}.$$

This tends to infinity, so the series diverges by the Integral Test.

Step 3: Identify the threshold.

More generally,

$$\sum \frac{1}{n(\ln n)^p}$$

converges for $p > 1$ and diverges for $p \leq 1$. Here $p = 1/2$, which lies on the divergent side.