

Question 1

Consider the Basel series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

- (a) Prove that the series converges using the p -series test.
- (b) State Euler's classical evaluation of its exact sum and give a decimal approximation.
- (c) Use an integral estimate to bound the remainder after N terms, and explain why convergence alone does not produce the exact constant.

Question 1 – Solution**Step 1: Classify the series.**

This is a p -series $\sum 1/n^p$ with $p = 2 > 1$. Therefore it converges. This test establishes existence of a finite sum but does not determine that sum.

Step 2: State the classical special value.

Euler's solution of the Basel problem gives the non-elementary identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \approx 1.644934.$$

The equality with $\pi^2/6$ requires additional machinery, such as Euler's product for $\sin x$ or Fourier series; it does not follow from the p -series test alone.

Step 3: Quantify convergence.

Because $f(x) = 1/x^2$ is positive and decreasing,

$$\int_{N+1}^{\infty} \frac{dx}{x^2} \leq R_N \leq \int_N^{\infty} \frac{dx}{x^2},$$

so

$$\frac{1}{N+1} \leq R_N \leq \frac{1}{N}.$$

Thus the partial sums approach $\pi^2/6$ from below, with an error of order $1/N$.

Question 2

Consider $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

- (a) Prove convergence and compare its speed with the Basel series.
- (b) State Euler's classical value $\zeta(4)$.
- (c) Using that value, compute the even-indexed and odd-indexed fourth-power subseries separately.

Question 2 – Solution**Step 1: Establish convergence.**

The series is a p -series with $p = 4 > 1$, so it converges. Moreover, $1/n^4 \leq 1/n^2$ for $n \geq 1$, so its tail decays substantially faster than the Basel-series tail.

Step 2: State the special value accurately.

Euler's classical evaluation is

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.082323.$$

As with $\zeta(2)$, the p -series test proves convergence but not this exact constant.

Step 3: Extract the even contribution.

For even indices $n = 2k$,

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^4} = \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{1440}.$$

Step 4: Extract the odd contribution.

Subtracting the even terms from the full series gives

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{90} - \frac{\pi^4}{1440} = \frac{\pi^4}{96}.$$

The even and odd pieces recombine to $\pi^4/90$, providing an algebraic check.

Question 3

Consider $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

- (a) Identify the geometric data and derive the N th partial sum.
- (b) Find the sum and exact remainder.
- (c) Interpret the result as repeatedly adding half of the remaining distance to 2.

Question 3 – Solution**Step 1: Identify the series.**

The expansion is $1 + 1/2 + 1/4 + \cdots$, so the first term is $a = 1$ and the common ratio is $r = 1/2$. Since $|r| < 1$, the series converges.

Step 2: Derive the finite partial sum.

With indices 0 through N , there are $N + 1$ terms. The finite geometric formula gives

$$s_N = \frac{1 - (1/2)^{N+1}}{1 - 1/2} = 2 - 2^{-N}.$$

Because $2^{-N} \rightarrow 0$,

$$S = \lim_{N \rightarrow \infty} s_N = 2.$$

Step 3: Compute and interpret the remainder.

The exact gap is $R_N = S - s_N = 2^{-N}$. Starting at $s_0 = 1$, each new term equals one-half of the current distance to 2; the gap is halved at every step. The process never exceeds 2, but it can get arbitrarily close.

Question 4

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

- (a) Find the partial-fraction decomposition.
- (b) Display the cancellation in the N th partial sum.
- (c) Find the sum and exact remainder after N terms.

Question 4 – Solution**Step 1: Decompose one summand.**

Write $1/[n(n+1)] = A/n + B/(n+1)$. Multiplication by $n(n+1)$ gives $1 = A(n+1) + Bn$, so $A = 1$ and $B = -1$. Hence

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Step 2: Telescope a finite sum.

$$\begin{aligned} s_N &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}. \end{aligned}$$

All interior reciprocal terms cancel; only the two boundary terms remain.

Step 3: Pass to the limit and measure the error.

Since $1/(N+1) \rightarrow 0$, the series sums to $S = 1$. The exact remainder is

$$R_N = S - s_N = \frac{1}{N+1}.$$

As a check, $s_N - s_{N-1} = 1/[N(N+1)]$, the original N th term.

Question 5

Consider the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

- (a) Verify the Alternating Series Test hypotheses.
- (b) Decide whether the convergence is absolute or conditional.
- (c) Derive the value $\ln 2$ from the power series for $\ln(1+x)$ and state an error bound.

Question 5 – Solution**Step 1: Prove convergence.**

Let $b_n = 1/n$. Then $b_n > 0$, $b_{n+1} < b_n$, and $b_n \rightarrow 0$. Therefore the Alternating Series Test proves that $\sum (-1)^{n-1} b_n$ converges.

Step 2: Test absolute convergence.

Taking absolute values gives

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n},$$

the divergent harmonic series. Hence the original series is conditionally convergent, not absolutely convergent.

Step 3: Identify the sum.

For $-1 < x \leq 1$, the endpoint-valid power-series identity is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

Setting $x = 1$ gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2.$$

Step 4: Quantify the approximation.

The Alternating Series Estimation Theorem gives $|S - s_N| \leq b_{N+1} = 1/(N+1)$. Odd partial sums lie above $\ln 2$ and even partial sums lie below it.

Question 6

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

- (a) Explain why this selects exactly the odd-indexed terms of the Basel series.
- (b) Compute the even-indexed contribution by scaling $\zeta(2)$.
- (c) Subtract to obtain the odd-square sum and verify that the two pieces recombine correctly.

Question 6 – Solution**Step 1: Partition the Basel series.**

Every positive integer is either even or odd, and the terms are positive, so

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Step 2: Evaluate the even contribution.

Using Euler's value $\zeta(2) = \pi^2/6$,

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}.$$

Step 3: Subtract to isolate the odd terms.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \boxed{\frac{\pi^2}{8}}.$$

Step 4: Verify the decomposition.

The even portion is one-fourth of the full sum, while the odd portion is three-fourths. Their sum is $(1/4 + 3/4)\zeta(2) = \zeta(2)$, as required.

Question 7

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$.

- (a) Factor the constant scale from the summand.
- (b) Use the Basel sum to evaluate the series.
- (c) Relate the answer to the odd-square contribution and the full series.

Question 7 – Solution

Step 1: Simplify the scaled denominator.

Since $(2n)^2 = 4n^2$,

$$\frac{1}{(2n)^2} = \frac{1}{4} \frac{1}{n^2}.$$

Step 2: Use linearity of convergent series.

The Basel series converges, so its constant factor may be pulled outside:

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} = \boxed{\frac{\pi^2}{24}}.$$

Step 3: Interpret the result.

The substitution $m = 2n$ selects precisely the even-indexed Basel terms. Their total is one-fourth of the complete sum because squaring the index contributes the scale factor $2^2 = 4$.

Step 4: Check against the odd contribution.

Adding the odd-square value $\pi^2/8$ gives

$$\frac{\pi^2}{24} + \frac{\pi^2}{8} = \frac{\pi^2}{6},$$

recovering the full Basel sum.

Question 8

Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

- (a) Differentiate the geometric power series inside its interval of convergence.
- (b) Align the powers and evaluate at $x = 1/2$.
- (c) Find the finite partial sum and exact remainder.

Question 8 – Solution

Step 1: Begin with a generating function.

For $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Power series may be differentiated term by term at interior points, giving

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Step 2: Match and evaluate the desired series.

Multiplying by x yields

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Since $|1/2| < 1$, substitution is valid:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = \boxed{2}.$$

Step 3: Quantify the finite approximation.

Differentiating the finite geometric identity gives

$$s_N = \sum_{n=1}^N \frac{n}{2^n} = 2 - \frac{N+2}{2^N}, \quad R_N = \frac{N+2}{2^N}.$$

The remainder is positive and tends to zero, so the partial sums approach 2 from below.

Question 9

Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

- (a) Differentiate the geometric generating function twice.
- (b) Explain why $n^2 = n(n-1) + n$ requires restoring a first-derivative contribution.
- (c) Derive a closed generating function and evaluate it at $x = 1/2$.

Question 9 – Solution**Step 1: Differentiate twice.**

For $|x| < 1$, $(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n$. Two derivatives give

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}.$$

Multiplying by x^2 produces

$$\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

Step 2: Restore the missing linear part.

Because $n^2 = n(n-1) + n$, add the first-derivative identity $\sum nx^n = x/(1-x)^2$:

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3}.$$

Step 3: Evaluate at the requested point.

Since $x = 1/2$ lies inside the interval of convergence,

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{(1/2)(3/2)}{(1/2)^3} = \boxed{6}.$$

Omitting the $\sum nx^n$ term would compute $n(n-1)$ rather than n^2 .

Question 10

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

- (a) Find the partial-fraction decomposition.
- (b) Display the two-step telescoping cancellation in a finite partial sum.
- (c) Find the sum and exact remainder after N terms.

Question 10 – Solution

Step 1: Decompose the gap-two denominator.

Solving $1/[n(n+2)] = A/n + B/(n+2)$ gives $A = 1/2$ and $B = -1/2$. Thus

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

Step 2: Telescope the finite sum.

$$\begin{aligned} s_N &= \frac{1}{2} \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \right) - \left(\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N+2} \right) \right] \\ &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right). \end{aligned}$$

The cancellation forms two interlaced chains because the indices differ by two.

Step 3: Take the limit.

Both ending reciprocals tend to zero, so

$$S = \frac{1}{2} \left(1 + \frac{1}{2} \right) = \boxed{\frac{3}{4}}.$$

Step 4: Find the exact remainder.

$$R_N = S - s_N = \frac{1}{2(N+1)} + \frac{1}{2(N+2)}.$$

As a check, subtracting consecutive partial-sum formulas reproduces $1/[N(N+2)]$.