

Question 1

Determine whether $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges or diverges.

- (a) Apply the nth-term test and explain why its result is inconclusive.
- (b) Group terms between consecutive perfect squares and find a lower bound for each block.
- (c) Use the block estimates to classify the series.

Question 1 – Solution**Step 1: Check the necessary condition.**

Let $a_n = 1/\sqrt{n}$. Since $\sqrt{n} \rightarrow \infty$, $a_n \rightarrow 0$. This necessary condition is satisfied, so the n th-term test is inconclusive; it can prove divergence when the limit is nonzero, but it can never prove convergence.

Step 2: Choose a grouping strategy.

The terms decrease too slowly for a geometric comparison. Grouping between consecutive squares makes their sizes and the number of terms in each block easy to control. For $k \geq 1$, consider $k^2 \leq n < (k+1)^2$. There are

$$(k+1)^2 - k^2 = 2k + 1$$

terms, and every one satisfies $\sqrt{n} < k+1$, hence $1/\sqrt{n} > 1/(k+1)$. Therefore the k th block has sum greater than

$$\frac{2k+1}{k+1} > 1.$$

Step 3: Convert the block estimate into divergence.

The partial sum through $(m+1)^2 - 1$ contains m such blocks and consequently exceeds m . Since m can be arbitrarily large, the increasing sequence of partial sums is unbounded.

Step 4: State and check the conclusion.

Therefore

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges.}}$$

This agrees with the p -series rule: $p = 1/2 \leq 1$.

Question 2

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

- (a) Verify that the terms approach zero and explain why that does not settle convergence.
- (b) Group the indices into dyadic blocks $2^k \leq n < 2^{k+1}$.
- (c) Find a lower bound for each block and use it to classify the series.

Question 2 – Solution**Step 1: Check the terms.**

For $n \geq 2$, $a_n = 1/(n \ln n) > 0$, and $n \ln n \rightarrow \infty$. Thus $a_n \rightarrow 0$. The n th-term test is inconclusive, so a tail estimate is required.

Step 2: Form dyadic blocks.

Use the block $2^k \leq n < 2^{k+1}$. It contains exactly 2^k integers. Within this block,

$$n < 2^{k+1}, \quad \ln n < (k+1) \ln 2,$$

and hence

$$\frac{1}{n \ln n} > \frac{1}{2^{k+1}(k+1) \ln 2}.$$

Step 3: Estimate one complete block.

Because the block contains 2^k terms, its sum is greater than

$$2^k \frac{1}{2^{k+1}(k+1) \ln 2} = \frac{1}{2(k+1) \ln 2}.$$

Step 4: Compare the block series.

Summing these lower bounds over k produces the constant multiple $(2 \ln 2)^{-1} \sum 1/(k+1)$ of the divergent harmonic series. Hence the original partial sums are unbounded and

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

Step 5: Interpret the slow growth.

The hidden continuous substitution is $u = \ln x$, since $du = dx/x$ and $\int dx/(x \ln x) = \ln(\ln x) + C$. This explains the exceptionally slow divergence; a large numerical cutoff can misleadingly suggest stabilization.

Question 3

Determine whether $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n}$ converges or diverges (angles in radians).

- (a) Explain why $0 \leq \sin^2 n/n \leq 1/n$ does not determine convergence.
- (b) Use $\sin^2 n = (1 - \cos 2n)/2$ to split the partial sums.
- (c) Show that $\sum \cos(2n)/n$ converges by the Dirichlet Test, then classify the original series.

Question 3 – Solution

Step 1: Audit the obvious comparison.

The terms satisfy $0 \leq \sin^2 n/n \leq 1/n$ and approach zero. However, an upper comparison with a divergent series is logically inconclusive: a smaller positive series may converge or diverge.

Step 2: Separate the average and oscillatory parts.

Using $\sin^2 n = (1 - \cos 2n)/2$ gives

$$s_N = \frac{1}{2} \sum_{n=1}^N \frac{1}{n} - \frac{1}{2} \sum_{n=1}^N \frac{\cos(2n)}{n}.$$

Step 3: Control the oscillatory series.

To apply Dirichlet's Test to $\sum \cos(2n)/n$, verify both hypotheses. First, the partial sums of $\cos(2n)$ are bounded. Indeed,

$$\sum_{n=1}^N e^{2in} = e^{2i} \frac{1 - e^{2iN}}{1 - e^{2i}},$$

whose magnitude is at most $2/|1 - e^{2i}|$; taking real parts preserves boundedness. Second, $1/n$ is positive, decreasing, and tends to zero. Dirichlet's Test therefore shows that $\sum \cos(2n)/n$ converges.

Step 4: Combine the two parts.

The first part of s_N is one-half of the harmonic partial sum and tends to $+\infty$, while the oscillatory part approaches a finite limit. Subtracting a bounded quantity cannot stop the harmonic growth. Hence

$$\boxed{\sum_{n=1}^{\infty} \frac{\sin^2 n}{n} \text{ diverges.}}$$

The needed extra structure is that $\sin^2 n$ has positive average $1/2$, not merely that it is bounded.

Question 4

Determine whether $\sum_{n=1}^{\infty} \frac{2n+1}{n^3+1}$ converges or diverges.

- (a) Identify the reciprocal-power benchmark suggested by the leading terms.
- (b) Compute the limit-comparison ratio with that benchmark.
- (c) State the conclusion and give a direct upper comparison valid for $n \geq 1$.

Question 4 – Solution**Step 1: Identify a benchmark.**

The terms are positive. Comparing the highest powers gives $(2n)/(n^3) = 2/n^2$, so the natural benchmark is the convergent p -series $b_n = 1/n^2$.

Step 2: Compute the limit-comparison ratio.

$$\lim_{n \rightarrow \infty} \frac{(2n+1)/(n^3+1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{2n^3+n^2}{n^3+1} = 2.$$

The ratio limit is 2, which is finite and strictly positive. These are exactly the hypotheses of the Limit Comparison Test. Since $\sum 1/n^2$ converges ($p = 2 > 1$), the given series converges.

Step 3: Verify with a direct inequality.

For $n \geq 1$, $2n+1 \leq 3n$ and $n^3+1 \geq n^3$, so

$$0 < \frac{2n+1}{n^3+1} \leq \frac{3}{n^2}.$$

Because $\sum 3/n^2$ converges, the Direct Comparison Test independently confirms the conclusion. The lower-order terms affect numerical values but not tail behavior.

Question 5

Determine whether $\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$ converges or diverges.

- (a) Simplify the factorial quotient completely before selecting a test.
- (b) Express the N th partial sum using harmonic numbers.
- (c) Classify the series and explain why its factorial notation is deceptive.

Question 5 – Solution

Step 1: Simplify before selecting a test.

Use $(n+1)! = (n+1)n!$ to cancel the common factorial:

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

Step 2: Rewrite the partial sums.

Thus

$$s_N = \sum_{n=1}^N \frac{1}{n+1} = \sum_{k=2}^{N+1} \frac{1}{k} = H_{N+1} - 1,$$

where H_m is the m th harmonic number.

Step 3: Apply the known harmonic behavior.

Because $H_{N+1} \rightarrow \infty$, subtracting the fixed number 1 does not change the divergence. Therefore the partial sums are unbounded and

$$\boxed{\sum_{n=1}^{\infty} \frac{n!}{(n+1)!} \text{ diverges.}}$$

Step 4: Interpret and verify.

The terms $1/(n+1)$ do approach zero, so the n th-term test alone is inconclusive. The factorial notation suggests rapid growth, but all of $n!$ cancels, leaving exactly a shifted harmonic series. Removing or shifting finitely many harmonic terms never changes convergence or divergence.

Question 6

Determine whether $\sum_{n=1}^{\infty} e^{-\sqrt{n}}$ converges or diverges.

- (a) Verify the necessary nth-term condition.
- (b) Group indices into blocks $k^2 \leq n < (k+1)^2$ and bound the sum of each block from above.
- (c) Compare the block bounds with a convergent series.

Question 6 – Solution

Step 1: Check the necessary condition.

Let $a_n = e^{-\sqrt{n}}$. Because $\sqrt{n} \rightarrow \infty$, the exponent tends to $-\infty$ and $a_n \rightarrow 0$. The nth-term condition holds but does not prove convergence.

Step 2: Organize the tail into square blocks.

In the block $k^2 \leq n < (k+1)^2$, there are exactly $2k+1$ indices. Also $\sqrt{n} \geq k$, so the decreasing exponential satisfies $e^{-\sqrt{n}} \leq e^{-k}$. Therefore

$$\sum_{n=k^2}^{(k+1)^2-1} e^{-\sqrt{n}} \leq (2k+1)e^{-k}.$$

Step 3: Classify the series of block bounds.

It remains to check $\sum_{k=1}^{\infty} (2k+1)e^{-k}$. Let $q = e^{-1} < 1$. Then

$$2 \sum_{k=1}^{\infty} kq^k + \sum_{k=1}^{\infty} q^k,$$

Both series converge: $\sum q^k$ is geometric, and $\sum kq^k = q/(1-q)^2$ follows by differentiating the geometric series.

Step 4: Conclude by comparison.

The original nonnegative series is bounded above block by block by a convergent series, so its increasing partial sums are bounded and

$$\sum_{n=1}^{\infty} e^{-\sqrt{n}} \text{ converges.}$$

Question 7

Determine whether $\sum_{n=1}^{\infty} \arctan n$ converges or diverges.

- (a) Evaluate the limit of the summand.
- (b) Apply the nth-term test and describe the growth of the partial sums.
- (c) Explain why no comparison or ratio test is needed.

Question 7 – Solution

Step 1: Apply the nth-term test first.

The summand has limit

$$\lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2} \neq 0.$$

Because the limit is not zero, the series diverges immediately by the nth-term test.

Step 2: Describe how the partial sums diverge.

Since $\arctan n \rightarrow \pi/2 > 1$, the definition of a limit guarantees an N_0 such that $\arctan n > 1$ for all $n \geq N_0$. Then for $N \geq N_0$,

$$s_N \geq s_{N_0-1} + (N - N_0 + 1),$$

so $s_N \rightarrow +\infty$ at least linearly.

Step 3: Justify why the test is decisive.

Every convergent series must have terms tending to zero: if $s_N \rightarrow S$, then $a_N = s_N - s_{N-1} \rightarrow S - S = 0$. Since this necessary condition fails, comparison, ratio, or integral tests are unnecessary.

Question 8

Determine whether $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$ converges or diverges.

- (a) Verify that the terms approach zero.
- (b) Use $0 \leq 1 - \cos x \leq x^2/2$ to obtain a direct comparison.
- (c) Confirm the quadratic scale by computing a limit-comparison ratio.

Question 8 – Solution**Step 1: Check the terms.**

Since $1/n \rightarrow 0$ and cosine is continuous, $1 - \cos(1/n) \rightarrow 1 - \cos 0 = 0$. The necessary condition holds, so a convergence test is still needed.

Step 2: Build a direct comparison.

The standard small-angle inequality

$$0 \leq 1 - \cos x \leq \frac{x^2}{2}$$

holds for every real x . Substituting $x = 1/n$ yields

$$0 \leq 1 - \cos(1/n) \leq \frac{1}{2n^2}.$$

Both sides are nonnegative, and $\sum 1/(2n^2)$ is a convergent p -series with $p = 2 > 1$. The Direct Comparison Test therefore proves convergence.

Step 3: Confirm the asymptotic scale.

The standard Taylor limit confirms more precisely that

$$\lim_{n \rightarrow \infty} \frac{1 - \cos(1/n)}{1/n^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

Thus the summand is asymptotic to $1/(2n^2)$. The finite positive ratio also satisfies the hypotheses of the Limit Comparison Test, independently confirming the conclusion.

Question 9

Determine whether $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ converges or diverges.

- (a) Verify the nth-term condition.
- (b) Perform a limit comparison with the harmonic series.
- (c) Give a direct lower comparison that independently proves divergence.

Question 9 – Solution

Step 1: Check the terms and select a benchmark.

Dividing by n^2 shows $n/(n^2 + 1) = (1/n)/(1 + 1/n^2) \rightarrow 0$, so the n th-term test is inconclusive. The dominant-power behavior suggests the harmonic benchmark $b_n = 1/n$.

Step 2: Perform limit comparison.

$$\lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1.$$

The limit equals 1, which is finite and strictly positive. Since both series have positive terms and $\sum 1/n$ diverges, the Limit Comparison Test proves that the given series diverges.

Step 3: Verify with direct comparison.

For $n \geq 1$, $n^2 + 1 \leq 2n^2$, so taking positive reciprocals and multiplying by n gives

$$\frac{n}{n^2 + 1} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

The smaller series $\sum 1/(2n)$ diverges; therefore the larger given positive-term series also diverges. This direction of comparison is essential when proving divergence.

Question 10

Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2 + \sin n}$ converges or diverges.

- (a) Show that the denominator is positive for every $n \geq 1$.
- (b) Use $-1 \leq \sin n \leq 1$ to produce a simple upper comparison for the tail.
- (c) Confirm the classification by limit comparison with $1/n^2$.

Question 10 – Solution**Step 1: Check positivity and the domain.**

For $n = 1$, $1 + \sin 1 > 0$. For $n \geq 2$, use $\sin n \geq -1$ to obtain

$$n^2 + \sin n \geq n^2 - 1 > 0,$$

so every term is defined and positive.

Step 2: Create an upper bound for the tail.

For $n \geq 2$,

$$n^2 - 1 \geq \frac{3}{4}n^2,$$

because $n^2 \geq 4$. Hence

$$0 < \frac{1}{n^2 + \sin n} \leq \frac{1}{n^2 - 1} \leq \frac{4}{3n^2}.$$

The comparison series is a constant multiple of the convergent p -series $\sum 1/n^2$. The Direct Comparison Test therefore proves convergence of the tail, and adding the finite first term cannot change that conclusion.

Step 3: Confirm the dominant behavior.

Limit comparison gives the sharper asymptotic statement

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 + \sin n)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + \sin n} = 1,$$

because dividing numerator and denominator by n^2 produces $1/(1 + \sin n/n^2)$, and $|\sin n|/n^2 \leq 1/n^2 \rightarrow 0$. The ratio limit 1 confirms that the oscillation in the denominator is asymptotically negligible.