

Question 1

Consider the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$.

- (a) Identify the first term and common ratio, then evaluate the series using the infinite geometric-series formula.
- (b) Derive a formula for the N th partial sum s_N and evaluate $\lim_{N \rightarrow \infty} s_N$.
- (c) Find the exact remainder $R_N = S - s_N$ and determine the smallest N for which $R_N < 10^{-4}$.

Question 1 – Solution**Step 1: Identify the geometric structure.**

Writing the first terms as $3/4 + 3/16 + 3/64 + \cdots$ shows that the first term is $a = 3/4$ and the common ratio is $r = 1/4$. Since $|r| < 1$, the infinite geometric-series formula applies:

$$S = \frac{a}{1-r} = \frac{3/4}{1-1/4} = 1.$$

Step 2: Confirm the result from partial sums.

For a geometric series, the finite sum is $s_N = a(1 - r^N)/(1 - r)$. Substitution gives

$$s_N = \frac{3}{4} \frac{1 - (1/4)^N}{1 - 1/4} = 1 - 4^{-N},$$

Since $4^{-N} \rightarrow 0$, $s_N \rightarrow 1$, confirming the result from the definition of an infinite series.

Step 3: Find and use the exact remainder.

The exact remainder after N terms is

$$R_N = S - s_N = 4^{-N}.$$

We require $4^{-N} < 10^{-4}$, equivalently $4^N > 10^4$. Since $4^6 = 4096 \leq 10^4$ but $4^7 = 16384 > 10^4$, the smallest choice is $N = 7$. The strict inequality is important.

Question 2

Consider $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

- (a) Write out the first four terms and display the cancellation.
- (b) Find a formula for the N th partial sum, identifying the surviving boundary terms.
- (c) Decide whether the series converges and, if so, find its sum and exact remainder.

Question 2 – Solution**Step 1: Form a finite partial sum.**

Never telescope directly to infinity. First write the N th partial sum:

$$\begin{aligned}s_N &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}.\end{aligned}$$

Step 2: Identify the surviving boundary terms.

Every interior fraction appears once positively and once negatively, so it cancels. Only the initial 1 and final $-1/(N+1)$ survive.

Step 3: Take the partial-sum limit.

By definition, the series converges exactly when (s_N) has a finite limit. Here

$$S = \lim_{N \rightarrow \infty} s_N = 1.$$

Step 4: Compute and verify the remainder.

The exact tail after N terms is

$$R_N = S - s_N = \frac{1}{N+1}.$$

As an algebraic check, $s_N - s_{N-1} = 1/N - 1/(N+1)$, which reproduces the original N th summand exactly.

Question 3

For a real parameter x , consider $\sum_{n=0}^{\infty} x^n$.

- (a) For $x \neq 1$, derive the finite identity $s_N = (1 - x^{N+1})/(1 - x)$.
- (b) Determine every x for which the infinite series converges and find its sum there.
- (c) Analyze $x = 1$, $x = -1$, and $|x| > 1$ separately.

Question 3 – Solution

Step 1: Derive the finite identity.

Let $s_N = 1 + x + \cdots + x^N$. Then $xs_N = x + x^2 + \cdots + x^{N+1}$. Subtracting the second equation from the first gives

$$(1 - x)s_N = 1 - x^{N+1},$$

so for $x \neq 1$ division by $1 - x$ yields

$$s_N = \frac{1 - x^{N+1}}{1 - x}.$$

Step 2: Determine the convergent parameter region.

If $|x| < 1$, then $x^{N+1} \rightarrow 0$. Therefore the partial sums converge and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

Step 3: Check the excluded cases.

At $x = 1$, $s_N = N + 1 \rightarrow \infty$. At $x = -1$, the partial sums alternate between 1 and 0 and have no limit. If $|x| > 1$, the terms x^n do not approach zero, so the nth-term test proves divergence. Thus convergence occurs exactly for $|x| < 1$; checking both endpoints separately is essential.

Question 4

Consider $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$.

- (a) Rewrite the N th partial sum as the logarithm of a product.
- (b) Simplify that product and determine whether the series converges.
- (c) Explain how the term test and the growth of the partial sums illustrate different facts.

Question 4 – Solution**Step 1: Work with a finite partial sum.**

For positive factors, $\ln u + \ln v = \ln(uv)$. Therefore

$$s_N = \ln \prod_{n=1}^N \frac{n+1}{n} = \ln \left(\frac{2}{1} \frac{3}{2} \frac{4}{3} \cdots \frac{N+1}{N} \right) = \ln(N+1).$$

Step 2: Simplify the product.

Every interior factor cancels: the numerator $2 \cdot 3 \cdots N$ cancels the same factors in the denominator, leaving $N+1$. This is multiplicative telescoping.

Step 3: Classify the series.

Since $s_N = \ln(N+1) \rightarrow \infty$, the sequence of partial sums is unbounded and the series diverges to $+\infty$.

Step 4: Interpret the nth-term condition.

The individual terms do satisfy

$$\ln \left(\frac{n+1}{n} \right) = \ln(1 + 1/n) \rightarrow 0,$$

which is necessary but not sufficient for convergence. The logarithm grows slowly, explaining why a short numerical table can look nearly stable even though the partial sums are unbounded.

Question 5

Consider $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

- (a) Find a partial-fraction decomposition that produces telescoping cancellation.
- (b) Derive the exact N th partial sum and verify it for $N = 1, 2, 3$.
- (c) Find the sum and the exact remainder after N terms.

Question 5 – Solution**Step 1: Find the partial fractions.**

Write $1/[n(n+2)] = A/n + B/(n+2)$. Multiplying by $n(n+2)$ gives $1 = A(n+2) + Bn$. Matching coefficients yields $A = 1/2$ and $B = -1/2$, so

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

Step 2: Telescope the finite partial sum.

Hence

$$s_N = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right) = \frac{3}{4} - \frac{1}{2(N+1)} - \frac{1}{2(N+2)}.$$

The surviving terms are 1, $1/2$, $-1/(N+1)$, and $-1/(N+2)$. For $N = 1, 2, 3$, the formula gives $1/3$, $11/24$, and $21/40$, matching direct addition.

Step 3: Take the limit.

Since both remaining reciprocal terms approach zero,

$$S = \frac{3}{4}.$$

Step 4: State the exact error.

Subtracting the partial sum from S gives the exact positive tail

$$R_N = S - s_N = \frac{1}{2(N+1)} + \frac{1}{2(N+2)}.$$

Question 6

Consider $\sum_{n=1}^{\infty} (-1)^n$.

- (a) List the first six terms and partial sums.
- (b) Prove that the sequence of partial sums is bounded but divergent.
- (c) Apply the nth-term test and explain why bounded partial sums alone do not imply convergence.

Question 6 – Solution**Step 1: Distinguish terms from partial sums.**

The series terms are $a_n = (-1)^n$, so the first six are $-1, 1, -1, 1, -1, 1$. Adding successively gives the partial sums

$$-1, 0, -1, 0, -1, 0, \dots$$

More precisely,

$$s_{2k} = 0, \quad s_{2k-1} = -1.$$

Step 2: Test convergence of the partial sums.

The sequence (s_N) is bounded between -1 and 0 , but $s_{2k} = 0 \rightarrow 0$ while $s_{2k-1} = -1 \rightarrow -1$. Since these subsequences have different limits, (s_N) does not converge. By definition, the series therefore diverges.

Step 3: Apply the nth-term test as a check.

The terms $(-1)^n$ do not approach zero, so the nth-term test also proves divergence immediately. Indeed, if $s_N \rightarrow S$, then $a_N = s_N - s_{N-1} \rightarrow S - S = 0$. This example shows that bounded partial sums are not enough; the partial sums must approach one number.

Question 7

Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

- (a) Differentiate the geometric-series identity to derive a formula for $\sum_{n=1}^{\infty} nx^{n-1}$ when $|x| < 1$.
- (b) Use that formula at $x = 1/2$ to evaluate the series.
- (c) Derive the finite partial sum and exact remainder after N terms.

Question 7 – Solution

Step 1: Start from the geometric series.

For $|x| < 1$, the geometric identity is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Step 2: Differentiate within the interval of convergence.

Power series may be differentiated term by term at every interior point of their interval of convergence. Thus

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Multiplying both sides by x aligns the power with the desired form:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Step 3: Substitute the required value.

Since $|1/2| < 1$, substitution is valid and gives $(1/2)/(1/2)^2 = 2$.

Step 4: Derive the finite sum and remainder.

Differentiating the finite identity $1 + x + \cdots + x^N = (1 - x^{N+1})/(1 - x)$ and multiplying by x yields

$$\sum_{n=1}^N nx^n = \frac{x - (N+1)x^{N+1} + Nx^{N+2}}{(1-x)^2}.$$

At $x = 1/2$, this simplifies to

$$s_N = 2 - \frac{N+2}{2^N}, \quad R_N = 2 - s_N = \frac{N+2}{2^N}.$$

Question 8

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$.

- (a) Reindex the series using $k = n + 1$.
- (b) Compare it with the Maclaurin series for e and identify every omitted term.
- (c) Find the sum and give a simple upper bound for the remainder after N terms.

Question 8 – Solution**Step 1: Reindex carefully.**

Let $k = n + 1$. When $n = 1$, the new index is $k = 2$, and as $n \rightarrow \infty$, so does k . Therefore

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \sum_{k=2}^{\infty} \frac{1}{k!}.$$

Step 2: Compare with the exponential series.

The Maclaurin series for $e = e^1$ is

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \sum_{k=2}^{\infty} \frac{1}{k!}.$$

The desired tail begins at $k = 2$, so the omitted terms are exactly $1/0! = 1$ and $1/1! = 1$. Therefore

$$S = e - 2.$$

Step 3: Bound the remainder.

After the original terms through $n = N$, the tail starts at $k = N + 2$. For $j \geq 0$, each new factorial factor is at least $N + 3$, so

$$(N + 2 + j)! \geq (N + 2)!(N + 3)^j.$$

Taking reciprocals and summing gives

$$0 < R_N \leq \frac{1}{(N + 2)!} \sum_{j=0}^{\infty} \frac{1}{(N + 3)^j} = \frac{N + 3}{(N + 2)(N + 2)!}.$$

This bound is explicit, positive, and tends to zero, confirming the convergence of the partial sums to $e - 2$.

Question 9

Consider $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$.

- (a) Write the first four terms and expose the telescoping cancellation.
- (b) Derive the N th partial sum and determine whether the series converges.
- (c) Reconcile the divergence with the fact that the individual terms approach zero.

Question 9 – Solution**Step 1: Form the finite partial sum.**

Write the N th partial sum before passing to infinity:

$$\begin{aligned}s_N &= (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{N+1} - \sqrt{N}) \\ &= \sqrt{N+1} - 1.\end{aligned}$$

Step 2: Analyze the partial-sum limit.

All interior square-root terms cancel, leaving only the first negative boundary term and final positive boundary term. Because $\sqrt{N+1} - 1 \rightarrow \infty$, the partial sums are unbounded and the series diverges to $+\infty$.

Step 3: Check the n th-term condition.

The summand does approach zero. Rationalization gives

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0.$$

Step 4: Reconcile the two facts.

There is no contradiction: the n th-term condition is necessary, not sufficient. Here every increment is positive and shrinks to zero, but the accumulated total still grows without bound. Since the denominator above is asymptotic to $2\sqrt{n}$, each increment is asymptotic to $1/(2\sqrt{n})$, consistent with a divergent p -series scale.

Question 10

For a real parameter r , consider $\sum_{n=1}^{\infty} r^{2n}$.

- (a) Identify the first term and common ratio as functions of r .
- (b) Determine all r for which the series converges and find its sum.
- (c) Analyze the boundary values $r = \pm 1$ and the region $|r| > 1$ using the n th-term test.

Question 10 – Solution**Step 1: Identify the geometric data.**

Expanding the first terms gives

$$r^2 + r^4 + r^6 + \dots,$$

Thus the first term is $a = r^2$ and the common ratio is $q = r^2$. A geometric series converges precisely when $|q| < 1$, so

$$|r^2| < 1 \iff |r| < 1.$$

Step 2: Find the sum on the convergence interval.For $|r| < 1$, the geometric formula gives

$$S(r) = \frac{r^2}{1 - r^2}.$$

The finite partial sum provides a direct verification:

$$s_N = \frac{r^2(1 - r^{2N})}{1 - r^2} \quad (r^2 \neq 1),$$

When $|r| < 1$, $r^{2N} \rightarrow 0$, so $s_N \rightarrow r^2/(1 - r^2)$.**Step 3: Check the endpoints and exterior region.**

At both $r = 1$ and $r = -1$, every term r^{2n} equals 1, so $s_N = N \rightarrow \infty$. If $|r| > 1$, then r^{2n} grows rather than approaches zero; the nth-term test proves divergence. Thus the convergence set is exactly $(-1, 1)$.