

Question 1

Define $a_1 = 2$ and $a_{n+1} = \frac{1}{2} \left(a_n + \frac{3}{a_n} \right)$.

- (a) Explain why this is Newton's method applied to $x^2 - 3 = 0$.
- (b) Prove that $a_n \geq \sqrt{3}$ for every n , and prove that (a_n) is decreasing.
- (c) Conclude that the sequence converges and determine its limit.

Question 1 – Solution**Step 1: Identify the algorithm.**

For $f(x) = x^2 - 3$, Newton's method uses $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Since $f'(x) = 2x$, the update becomes

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 3}{2x} = \frac{1}{2} \left(x + \frac{3}{x} \right),$$

which is exactly the given recurrence.

Step 2: Establish positivity and a lower bound.

All terms are positive because $a_1 = 2 > 0$ and, whenever $a_n > 0$, both a_n and $3/a_n$ are positive. More precisely,

$$a_{n+1} - \sqrt{3} = \frac{(a_n - \sqrt{3})^2}{2a_n} \geq 0,$$

The right side is nonnegative, so $a_{n+1} \geq \sqrt{3}$. Since $a_1 \geq \sqrt{3}$, induction proves $a_n \geq \sqrt{3}$ for every n .

Step 3: Prove monotonicity.

Using the lower bound,

$$a_{n+1} - a_n = \frac{3 - a_n^2}{2a_n} \leq 0,$$

Thus $a_{n+1} \leq a_n$, so (a_n) is decreasing.

Step 4: Prove convergence before solving for the limit.

The sequence is decreasing and bounded below by $\sqrt{3}$. The Monotone Convergence Theorem therefore guarantees $a_n \rightarrow L$ for some $L \geq \sqrt{3}$. Continuity and $L > 0$ allow passage to the limit:

$$L = \frac{1}{2} \left(L + \frac{3}{L} \right), \quad L^2 = 3.$$

The equation gives $L = \pm\sqrt{3}$, and the established positive lower bound selects $L = \sqrt{3}$.

Question 2

Let (b_n) be a bounded real sequence and define $a_n = \sup\{b_k : k \geq n\}$.

- (a) Explain why every a_n exists.
- (b) Prove that (a_n) is decreasing and bounded, and hence convergent.
- (c) State what its limit represents, and compute it when $b_n = (-1)^n + 1/n$.

Question 2 – Solution

Step 1: Verify that each supremum exists.

Let $T_n = \{b_k : k \geq n\}$. Each tail is nonempty, and because (b_n) is bounded, T_n is bounded above. The least-upper-bound property of $\mathbb{R}$ therefore guarantees that $a_n = \sup T_n$ exists.

Step 2: Prove monotonicity.

Removing the first element of a tail cannot increase its set of possible values. Formally, $T_{n+1} \subseteq T_n$, so

$$a_{n+1} = \sup T_{n+1} \leq \sup T_n = a_n.$$

Thus (a_n) is decreasing.

Step 3: Establish a lower bound and convergence.

If m is any lower bound for (b_n) , then $m \leq b_k \leq a_n$ for every $k \geq n$, so $m \leq a_n$. Hence (a_n) is bounded below. The Monotone Convergence Theorem now gives convergence, and its limit is called the limit superior:

$$\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} b_n.$$

Step 4: Apply the construction to the example.

For $b_n = (-1)^n + 1/n$, even terms equal $1 + 1/(2j)$ and odd terms are negative. Among the even terms in a tail, the earliest one is largest. If e_n is the first even integer satisfying $e_n \geq n$, then

$$a_n = 1 + \frac{1}{e_n} \longrightarrow 1.$$

Thus $\limsup b_n = 1$. The tail-supremum process has turned oscillating bounded data into a decreasing upper envelope.

Question 3

Let $a_n = \{n\sqrt{2}\} = n\sqrt{2} - \lfloor n\sqrt{2} \rfloor$.

- (a) Prove that $0 \leq a_n < 1$, so the sequence is bounded.
- (b) Using the fact that the fractional parts of multiples of an irrational number are dense in $[0, 1]$, find two different subsequential limits.
- (c) Decide whether (a_n) converges and explain why boundedness is insufficient.

Question 3 – Solution**Step 1: Establish boundedness.**

For every real x , its fractional part satisfies $0 \leq \{x\} < 1$. Therefore $0 \leq a_n < 1$ for all n , and the sequence is bounded.

Step 2: Use irrationality and density.

Because $\sqrt{2}$ is irrational, the density theorem for irrational rotations says that the set $\{\{n\sqrt{2}\} : n \geq 1\}$ is dense in $[0, 1]$. In particular, every open subinterval contains infinitely many terms, allowing the selection of increasing subsequence indices.

Step 3: Construct two subsequences.

For each positive integer j , choose increasing indices n_j and m_j such that

$$0 < a_{n_j} < 1/j, \quad 1/2 - 1/j < a_{m_j} < 1/2 + 1/j.$$

The first inequality and Squeeze Theorem give $a_{n_j} \rightarrow 0$; the second gives $a_{m_j} \rightarrow 1/2$.

Step 4: Apply the subsequence criterion.

Every subsequence of a convergent sequence must approach the same limit as the original sequence. Since these two subsequences have different limits, (a_n) diverges. Boundedness alone guarantees neither monotonicity nor convergence.

Question 4

Let $a_n = \sum_{k=1}^n 2^{-k}$.

- (a) Derive a closed formula for a_n from the finite geometric-sum formula.
- (b) Prove that (a_n) converges and find its limit.
- (c) Find the exact remaining gap $L - a_n$ and the smallest n for which this gap is below 10^{-3} .

Question 4 – Solution

Step 1: Identify and sum the finite geometric series.

The terms form a geometric sequence with first term $a = 1/2$ and common ratio $r = 1/2$. Therefore

$$a_n = \frac{(1/2)(1 - (1/2)^n)}{1 - 1/2} = 1 - 2^{-n}.$$

Step 2: Determine convergence and the limit.

Since $2^{-n} \rightarrow 0$, the closed formula gives $a_n \rightarrow L = 1$. Also $a_{n+1} - a_n = 2^{-(n+1)} > 0$ and $a_n < 1$, so the sequence is increasing and bounded above, independently confirming convergence by the Monotone Convergence Theorem.

Step 3: Compute the exact remaining gap.

The exact error after n terms is

$$L - a_n = 2^{-n}.$$

Step 4: Solve the accuracy inequality.

We need $2^{-n} < 10^{-3}$, equivalently $2^n > 1000$. Since $2^9 = 512 \leq 1000$ but $2^{10} = 1024 > 1000$, the smallest qualifying index is $n = 10$.

Question 5

Let $a_n = \frac{n^2}{2^n}$ for $n \geq 1$.

- (a) Use a_{n+1}/a_n to determine exactly when $a_{n+1} < a_n$.
- (b) Identify the largest term and the first index from which the sequence is strictly decreasing.
- (c) Prove that $a_n \rightarrow 0$.

Question 5 – Solution**Step 1: Compare consecutive positive terms.**

Because $a_n > 0$, the inequality $a_{n+1} < a_n$ is equivalent to $a_{n+1}/a_n < 1$. Compute

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2n^2}.$$

Step 2: Solve the ratio inequality exactly.

Thus $a_{n+1} < a_n$ exactly when

$$(n+1)^2 < 2n^2 \iff n^2 - 2n - 1 > 0 \iff n > 1 + \sqrt{2}.$$

For integer $n \geq 1$, this is equivalent to $n \geq 3$. Since $a_1 = 1/2 < a_2 = 1 < a_3 = 9/8$ and $a_{n+1} < a_n$ for every $n \geq 3$, a_3 is the unique largest term and the sequence is strictly decreasing from index 3 onward.

Step 3: Prove the decreasing tail approaches zero.

A decreasing positive sequence has a limit, but the ratio bound identifies it directly. For $n \geq 7$,

$$\frac{a_{n+1}}{a_n} = \frac{1}{2} \left(1 + \frac{1}{n}\right)^2 \leq \frac{1}{2} \left(\frac{8}{7}\right)^2 = \frac{32}{49} < 1.$$

Iterating this inequality gives $0 < a_n \leq a_7(32/49)^{n-7}$ for $n \geq 7$. The right side is geometric with ratio less than 1, so it tends to zero. The Squeeze Theorem yields $a_n \rightarrow 0$.

Question 6

Let $a_n = \frac{(-1)^n}{n}$.

- (a) Prove that $a_n \rightarrow 0$ using an absolute-value estimate.
- (b) Show that the full sequence is not monotone and is not eventually monotone.
- (c) Describe the monotonic behavior of its even and odd subsequences.

Question 6 – Solution**Step 1: Prove convergence.**

Use absolute values to remove the alternating sign:

$$|a_n - 0| = \frac{1}{n} \longrightarrow 0,$$

Since $1/n \rightarrow 0$, the definition of convergence gives $a_n \rightarrow 0$. Equivalently, $-1/n \leq a_n \leq 1/n$ and the Squeeze Theorem applies.

Step 2: Test eventual monotonicity.

For every positive integer k ,

$$a_{2k} = \frac{1}{2k} > 0, \quad a_{2k+1} = -\frac{1}{2k+1} < 0.$$

Thus $a_{2k} > a_{2k+1}$ but $a_{2k+1} < a_{2k+2}$, producing infinitely many decreases and increases. Because this occurs for arbitrarily large k , no tail can be monotone.

Step 3: Analyze the two subsequences.

The even subsequence $a_{2k} = 1/(2k)$ decreases to 0, while the odd subsequence $a_{2k-1} = -1/(2k-1)$ increases to 0. Both close in on the same limit from opposite sides. This demonstrates that convergence does not require eventual monotonicity.

Question 7

Let $a_n = \sqrt{n+1} - \sqrt{n}$.

- (a) Rationalize the expression and use the result to find the limit.
- (b) Prove that (a_n) is positive and decreasing.
- (c) Show that $a_n \sim 1/(2\sqrt{n})$ and interpret this as the spacing between neighboring square roots.

Question 7 – Solution**Step 1: Rationalize the difference.**

Multiplying numerator and denominator by the conjugate avoids the indeterminate form $\infty - \infty$:

$$a_n = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Step 2: Find the limit and monotonicity.

The denominator tends to infinity, so $a_n \rightarrow 0$. It is also positive. Furthermore, $\sqrt{n+2} + \sqrt{n+1} > \sqrt{n+1} + \sqrt{n}$; taking reciprocals of positive quantities reverses the inequality and gives $a_{n+1} < a_n$. Thus the sequence decreases to zero.

Step 3: Determine the asymptotic gap size.

Compare a_n with $1/(2\sqrt{n})$ by taking their ratio:

$$\frac{a_n}{1/(2\sqrt{n})} = \frac{2\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{2}{\sqrt{1+1/n} + 1} \rightarrow 1.$$

Because the ratio tends to 1, $a_n \sim 1/(2\sqrt{n})$. Therefore neighboring square roots get closer, and their gap is approximately $1/(2\sqrt{n})$ for large n .

Question 8

Let $a_n = \int_0^1 x^n dx$ for $n \geq 1$.

- (a) Evaluate the integral and find $\lim_{n \rightarrow \infty} a_n$.
- (b) Prove directly from the integrands that (a_n) is decreasing.
- (c) Give a geometric explanation for why the areas approach zero, despite $x^n = 1$ at $x = 1$.

Question 8 – Solution

Step 1: Evaluate the area exactly.

By the power rule,

$$a_n = \left[\frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1} \rightarrow 0.$$

Step 2: Prove monotonicity directly from the integrands.

For $0 \leq x \leq 1$, $0 \leq x^{n+1} \leq x^n$, with strict inequality on $(0, 1)$. Integrating over $[0, 1]$ gives $0 < a_{n+1} < a_n$, so the areas decrease.

Step 3: Explain the geometry rigorously.

For every fixed $x \in [0, 1)$, $x^n \rightarrow 0$; only the single endpoint $x = 1$ remains at height 1, and one point has zero width and contributes no area. For a quantitative version, fix $0 < c < 1$ and split the interval:

$$\int_0^1 x^n dx = \int_0^c x^n dx + \int_c^1 x^n dx \leq c^n + (1 - c).$$

Given $\varepsilon > 0$, first choose c so that $1 - c < \varepsilon/2$. Then choose n so large that $c^n < \varepsilon/2$. The displayed bound gives $a_n < \varepsilon$, proving geometrically that the areas vanish.

Question 9

For $n \geq 2$, let $a_n = \left(1 - \frac{1}{n}\right)^{n^2}$.

- (a) Take logarithms and use $\ln(1 - x) \leq -x$ for $0 < x < 1$ to obtain a useful bound.
- (b) Prove that $a_n \rightarrow 0$.
- (c) Use $\ln(1 - x) = -x - x^2/2 + O(x^3)$ to describe the rate of decay more precisely.

Question 9 – Solution

Step 1: Take logarithms.

For $n \geq 2$, $0 < 1 - 1/n < 1$, so $a_n > 0$ and logarithms are valid:

$$\ln a_n = n^2 \ln \left(1 - \frac{1}{n} \right).$$

Step 2: Obtain a decisive upper bound.

For $0 < x < 1$, $\ln(1-x) \leq -x$. With $x = 1/n$, this gives $\ln a_n \leq -n$. Because the exponential function is increasing,

$$0 < a_n \leq e^{-n} \longrightarrow 0.$$

The Squeeze Theorem proves $a_n \rightarrow 0$.

Step 3: Refine the decay rate.

For a sharper description, substitute $x = 1/n$ into $\ln(1-x) = -x - x^2/2 + O(x^3)$:

$$\ln a_n = n^2 \left(-\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3}) \right) = -n - \frac{1}{2} + O(n^{-1}).$$

Exponentiating both sides gives

$$a_n = e^{-n-1/2+O(1/n)} \sim e^{-n-1/2}.$$

The notation $u_n \sim v_n$ means $u_n/v_n \rightarrow 1$. Thus the quadratic exponent repeats a base close to 1 often enough to produce decay essentially like e^{-n} , with the additional constant factor $e^{-1/2}$.

Question 10

Define $a_1 = 0$ and $a_{n+1} = \frac{1}{2 + a_n}$.

- (a) Show that $0 \leq a_n \leq 1/2$ for all n .
- (b) Find the unique fixed point L in this interval and prove that consecutive terms alternate around it.
- (c) Establish an error contraction and use it to prove $a_n \rightarrow L$.

Question 10 – Solution**Step 1: Establish an invariant interval.**

The bound follows by induction. The base case $a_1 = 0$ lies in $[0, 1/2]$. If $0 \leq a_n \leq 1/2$, then $2 + a_n \geq 2$, so $0 < a_{n+1} = 1/(2 + a_n) \leq 1/2$. Hence every term remains in $[0, 1/2]$.

Step 2: Find the admissible fixed point.

A fixed point satisfies

$$L = \frac{1}{2 + L} \iff L^2 + 2L - 1 = 0,$$

so $L = -1 \pm \sqrt{2}$. Only $L = \sqrt{2} - 1$ lies in the invariant interval $[0, 1/2]$.

Step 3: Explain the alternating behavior.

Let $f(x) = 1/(2 + x)$. Because $f'(x) = -1/(2 + x)^2 < 0$, f is strictly decreasing. Since $a_1 < L$, $a_2 = f(a_1) > f(L) = L$. Applying the decreasing map reverses the side of L at every iteration, so successive terms alternate across the fixed point.

Step 4: Prove convergence with an error contraction.

$$|a_{n+1} - L| = \left| \frac{1}{2 + a_n} - \frac{1}{2 + L} \right| = \frac{|a_n - L|}{(2 + a_n)(2 + L)} \leq \frac{1}{4} |a_n - L|.$$

Both denominator factors are at least 2, which justifies the factor $1/4$. Iterating gives

$$|a_n - L| \leq \left(\frac{1}{4}\right)^{n-1} |a_1 - L| \longrightarrow 0.$$

Therefore $a_n \rightarrow \sqrt{2} - 1$; the even and odd subsequences approach the same value from opposite sides.