

Question 1

Use the generalized binomial theorem to find the first four nonzero terms of $(1 + x)^{1/2}$. State the radius and determine convergence at both endpoints.

Question 1 – Solution

Step 1: State the coefficient formula.

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

Step 2: Substitute $\alpha = 1/2$.

$$\binom{1/2}{0} = 1, \quad \binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = -\frac{1}{8}, \quad \binom{1/2}{3} = \frac{1}{16}.$$

Therefore

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots}.$$

Step 3: Determine the interval.

The radius is $R = 1$. The coefficient magnitudes are asymptotic to a constant times $n^{-3/2}$, so the series converges absolutely at both $x = 1$ and $x = -1$.

Conclusion.

The interval is $\boxed{[-1, 1]}$.

Question 2

Treat $(1 + x)^{-1}$ as a generalized binomial series. Derive its coefficients, connect the result with a geometric series, and determine the exact interval of convergence.

Question 2 – Solution

Step 1: Compute the generalized coefficients.

For $n \geq 0$,

$$\binom{-1}{n} = \frac{(-1)(-2) \cdots (-n)}{n!} = (-1)^n.$$

Thus

$$(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots.$$

Step 2: Connect with geometry.

This is exactly

$$\frac{1}{1 - (-x)} = \sum_{n=0}^{\infty} (-x)^n,$$

whose common ratio is $-x$. It converges when $|-x| < 1$, so $R = 1$.

Step 3: Test endpoints.

At $x = 1$, terms are $(-1)^n$; at $x = -1$, terms are 1. Neither tends to zero.

Conclusion.

The interval is $\boxed{(-1, 1)}$.

Question 3

Find the first four nonzero terms of $(1 + x)^{-1/2}$ using generalized binomial coefficients. Explain the signs and determine the exact interval of convergence.

Question 3 – Solution**Step 1: Compute coefficients.**

$$\binom{-1/2}{1} = -\frac{1}{2}, \quad \binom{-1/2}{2} = \frac{3}{8}, \quad \binom{-1/2}{3} = -\frac{5}{16}.$$

Every factor in the numerator is negative, so the coefficient sign is $(-1)^n$. Hence

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} - \cdots.$$

Step 2: Find the radius.

The generalized binomial series has $R = 1$.

Step 3: Test endpoints.

The coefficient magnitudes satisfy

$$\left| \binom{-1/2}{n} \right| = \frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

At $x = 1$ the signs alternate, so the series converges conditionally. At $x = -1$, all terms are positive and comparable with $1/\sqrt{n}$, so it diverges.

The endpoint magnitudes $b_n = 4^{-n} \binom{2n}{n}$ decrease, since $b_{n+1}/b_n = (2n+1)/(2n+2) < 1$, and tend to zero. This verifies the alternating-test hypotheses.

Conclusion.

The interval is $\boxed{(-1, 1]}$.

Question 4

Expand $(1 - 2x)^{3/2}$ through the x^4 term. Track the substitution $u = -2x$ carefully, then find the radius and endpoint behavior.

Question 4 – Solution

Step 1: Expand in an intermediate variable.

$$(1 + u)^{3/2} = 1 + \frac{3}{2}u + \frac{3}{8}u^2 - \frac{1}{16}u^3 + \frac{3}{128}u^4 + \cdots.$$

Step 2: Substitute $u = -2x$.

$$\begin{aligned}(1 - 2x)^{3/2} &= 1 + \frac{3}{2}(-2x) + \frac{3}{8}(4x^2) - \frac{1}{16}(-8x^3) + \frac{3}{128}(16x^4) + \cdots \\ &= \boxed{1 - 3x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 + \cdots}.\end{aligned}$$

Each degree- n coefficient is multiplied by $(-2)^n$.

Step 3: Determine convergence.

The condition $|u| < 1$ gives $|x| < 1/2$, so $R = 1/2$. For exponent $3/2$, coefficient magnitudes are order $n^{-5/2}$, so both endpoint series converge absolutely.

Conclusion.

The interval is $\boxed{[-1/2, 1/2]}$.

Question 5

For $(1+x)^{1/3} = \sum_{n=0}^{\infty} c_n x^n$, derive a recurrence for c_{n+1} from c_n . Use it to compute through x^4 , then state the convergence interval.

Question 5 – Solution**Step 1: Derive the recurrence.**Since $c_n = \binom{1/3}{n}$,

$$\frac{c_{n+1}}{c_n} = \frac{(1/3) - n}{n + 1}, \quad \boxed{c_{n+1} = c_n \frac{1/3 - n}{n + 1}}, \quad c_0 = 1.$$

Step 2: Generate coefficients.

$$c_1 = \frac{1}{3}, \quad c_2 = -\frac{1}{9}, \quad c_3 = \frac{5}{81}, \quad c_4 = -\frac{10}{243}.$$

Therefore

$$\boxed{(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \cdots}.$$

Step 3: State convergence.

The radius is 1. Since the coefficient magnitudes are order $n^{-4/3}$, both endpoint series converge absolutely.

Conclusion.The interval is $\boxed{[-1, 1]}$.

Question 6

Derive the binomial series for $(1 - x)^{-2}$ and show that it agrees with differentiating the geometric series. Determine the exact interval of convergence.

Question 6 – Solution**Step 1: Use generalized coefficients.**For $n \geq 0$,

$$\binom{-2}{n} = \frac{(-2)(-3)\cdots(-n-1)}{n!} = (-1)^n(n+1).$$

Substitute $u = -x$ into $(1+u)^{-2}$:

$$\binom{-2}{n}(-x)^n = (-1)^n(n+1)(-1)^n x^n = (n+1)x^n.$$

Thus

$$(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Step 2: Verify by differentiation.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \Rightarrow \quad \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Step 3: Check convergence.

The radius is 1. At $x = 1$ terms are $n+1$; at $x = -1$ their magnitudes are $n+1$. Both endpoints diverge.

Conclusion.The interval is $\boxed{(-1, 1)}$.

Question 7

Approximate $\sqrt{1.04}$ using only the constant and linear terms of the binomial series for $\sqrt{1+x}$. Use the alternating-series structure to bound the omitted correction and report the approximation.

Question 7 – Solution

Step 1: Use the binomial expansion.

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots.$$

Set $x = 0.04$.

Step 2: Keep two terms.

$$\sqrt{1.04} \approx 1 + \frac{0.04}{2} = \boxed{1.02}.$$

Step 3: Bound the error.

Beginning with the quadratic term, the omitted terms alternate and decrease in magnitude at $x = 0.04$. Therefore

$$|R| \leq \frac{(0.04)^2}{8} = \boxed{0.0002}.$$

The first correction is negative, so the linear approximation is an overestimate. More precisely,

$$1.0198 \leq \sqrt{1.04} < 1.02.$$

The actual value, $1.0198039\dots$, is consistent with this bound.

Question 8

Let m be a nonnegative integer. Explain algebraically why the generalized binomial series for $(1+x)^m$ terminates, and recover the ordinary finite Binomial Theorem. Illustrate with $m = 4$.

Question 8 – Solution

Step 1: Examine the generalized coefficient.

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

If $n = m + 1$, the numerator includes the factor $m - m = 0$. For every $n > m$, it still includes that zero factor. Hence

$$\binom{m}{n} = 0 \quad (n > m).$$

Step 2: Recover the finite theorem.

The formally infinite series therefore reduces to

$$(1+x)^m = \sum_{n=0}^m \binom{m}{n} x^n.$$

Because this is a polynomial identity, it is valid for every real (and complex) x ; there is no finite radius restriction.

Step 3: Illustrate with $m = 4$.

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The next coefficient is $\binom{4}{5} = 0$, so no higher powers occur.

Question 9

Find the coefficient of x^4 in the Maclaurin series for $(1+x)^{1/2}$ directly from the generalized binomial formula. Do not compute the lower-degree coefficients.

Question 9 – Solution**Step 1: Identify the needed coefficient.**

In

$$(1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n,$$

the coefficient of x^4 is exactly $\binom{1/2}{4}$.**Step 2: Evaluate only that coefficient.**

$$\begin{aligned} \binom{1/2}{4} &= \frac{(1/2)(1/2-1)(1/2-2)(1/2-3)}{4!} \\ &= \frac{(1/2)(-1/2)(-3/2)(-5/2)}{24} \\ &= \frac{-15}{16 \cdot 24} = \boxed{-\frac{5}{128}}. \end{aligned}$$

There are three negative numerator factors, so the negative sign is expected.

Conclusion.The x^4 term is $\boxed{-5x^4/128}$. Computing preceding terms is unnecessary because generalized binomial coefficients directly target any desired power.

Question 10

Find the binomial series, radius, and exact interval of convergence for $(1 + 3x)^{-1/2}$. Test $x = -1/3$ and $x = 1/3$ separately.

Question 10 – Solution

Step 1: Substitute into the parent series.

$$(1 + u)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} u^n.$$

With $u = 3x$,

$$(1 + 3x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} 3^n x^n = 1 - \frac{3}{2}x + \frac{27}{8}x^2 - \frac{135}{16}x^3 + \cdots.$$

Step 2: Find the radius.

The condition $|u| < 1$ becomes $|3x| < 1$, so $R = 1/3$.

Step 3: Test endpoints.

At $x = 1/3$, $u = 1$, and the terms alternate with magnitude asymptotic to $1/\sqrt{\pi n}$; the series converges conditionally. At $x = -1/3$, $u = -1$ cancels the coefficient signs, leaving positive terms comparable with $1/\sqrt{n}$; it diverges.

The endpoint magnitudes $b_n = 4^{-n} \binom{2n}{n}$ decrease, since $b_{n+1}/b_n = (2n+1)/(2n+2) < 1$, and tend to zero. This verifies the alternating-test hypotheses.

Conclusion.

The interval is $\boxed{(-1/3, 1/3]}$.