

Question 1

Approximate $e^{0.1}$ using the cubic Maclaurin polynomial for e^x . Then use the Lagrange remainder to give a rigorous upper bound for the absolute error.

Question 1 – Solution

Step 1: Write the cubic polynomial.

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}.$$

Step 2: Evaluate at $x = 0.1$.

$$\begin{aligned} P_3(0.1) &= 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{6} \\ &= 1 + 0.1 + 0.005 + 0.0001666667 \\ &= \boxed{1.1051666667}. \end{aligned}$$

Step 3: Bound the error.

Taylor's theorem gives

$$R_3(0.1) = \frac{e^c(0.1)^4}{4!} \quad \text{for some } 0 < c < 0.1.$$

Because $e^c \leq e^{0.1}$,

$$|R_3(0.1)| \leq \frac{e^{0.1}(0.1)^4}{24} < 4.61 \times 10^{-6}.$$

Thus $e^{0.1}$ lies within 0.00000461 of the stated approximation.

Question 2

Approximate $\sin(0.2)$ so that correct rounding through four decimal places is guaranteed. Use the lowest-degree odd Maclaurin polynomial whose error bound is below 5×10^{-5} .

Question 2 – Solution**Step 1: Check the linear approximation.**

For $x = 0.2$, the sine series is alternating with decreasing term magnitudes. Using only x gives

$$|R| \leq \frac{0.2^3}{3!} = 0.001333\ldots > 5 \times 10^{-5},$$

so the degree-1 polynomial does not provide the requested guarantee.

Step 2: Use the cubic polynomial.

$$P_3(x) = x - \frac{x^3}{3!}, \quad P_3(0.2) = 0.2 - \frac{0.008}{6} = \boxed{0.1986666667}.$$

Step 3: Bound the first omitted term.

$$|R| \leq \frac{0.2^5}{5!} = \frac{0.00032}{120} \approx 2.67 \times 10^{-6} < 5 \times 10^{-5}.$$

The alternating sign gives $0.1986666666 < \sin(0.2) < 0.1986693334$; both endpoints round to 0.1987. Therefore the smallest qualifying odd polynomial is cubic, and $\boxed{\sin(0.2) \approx 0.1987}$ to four decimal places.

Question 3

Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$ by using a Taylor expansion. Identify why the constant and linear terms disappear and which coefficient determines the limit.

Question 3 – Solution

Step 1: Expand far enough to survive the cancellation.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4).$$

Step 2: Subtract the specified terms.

$$e^x - 1 - x = \frac{x^2}{2} + \frac{x^3}{6} + O(x^4).$$

The constant 1 and linear term x cancel exactly.

Step 3: Divide by x^2 .

$$\frac{e^x - 1 - x}{x^2} = \frac{1}{2} + \frac{x}{6} + O(x^2).$$

As $x \rightarrow 0$, every remaining positive-power term tends to zero.

Conclusion.

$$\boxed{\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}}.$$

The answer is the first Taylor coefficient not removed by the numerator's subtraction.

Question 4

Solve the initial-value problem $y' = y$, $y(0) = 1$ by assuming a power-series solution $y = \sum_{n=0}^{\infty} a_n x^n$. Derive the coefficient recursion and identify the resulting function.

Question 4 – Solution

Step 1: Differentiate and align powers.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

The original series is $y = \sum_{n=0}^{\infty} a_n x^n$.

Step 2: Match coefficients in $y' = y$.

$$(n+1)a_{n+1} = a_n, \quad a_{n+1} = \frac{a_n}{n+1}.$$

The initial condition gives $a_0 = y(0) = 1$. Repeated use of the recursion yields

$$a_1 = 1, \quad a_2 = \frac{1}{2!}, \quad a_3 = \frac{1}{3!}, \quad \text{and in general} \quad a_n = \frac{1}{n!}.$$

Conclusion.

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{e^x}.$$

The series has infinite radius by the Ratio Test, so the solution is valid for every real x .

Question 5

Estimate $I = \int_0^{0.5} e^{-x^2} dx$ by integrating the first four terms of the Maclaurin series for e^{-x^2} . Give a rigorous error bound from the first omitted integrated term.

Question 5 – Solution

Step 1: Substitute into the exponential series.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots.$$

For $0 \leq x \leq 0.5$, the alternating term magnitudes decrease.

Step 2: Integrate through the x^6 term.

$$\begin{aligned} I &\approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} \right]_0^{0.5} \\ &= 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42} \\ &= \boxed{0.4612723214}. \end{aligned}$$

Step 3: Bound the integrated remainder.

The omitted integrand has magnitude at most $x^8/4!$, and the integrated alternating remainder satisfies

$$|R| \leq \int_0^{0.5} \frac{x^8}{4!} dx = \frac{0.5^9}{9 \cdot 4!} < 9.05 \times 10^{-6}.$$

Thus the estimate is accurate to within 0.00000905.

Question 6

For $0 < x \leq 1$, compare the linear approximation $L(x) = x$ with $\log(1 + x)$. Decide whether L overestimates or underestimates, and use the alternating series to bound the absolute error.

Question 6 – Solution

Step 1: Expand the logarithm.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad 0 < x \leq 1.$$

The first neglected term after the linear term is negative.

Step 2: Determine the direction.

Subtract the function from the linear approximation:

$$x - \log(1+x) = \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots.$$

This is positive by the Alternating Series Test. Therefore x is an **overestimate** of $\log(1+x)$.

Step 3: Bound the error.

The alternating remainder after the first term is no larger than the first omitted magnitude:

$$0 < x - \log(1+x) \leq \frac{x^2}{2}.$$

For example, at $x = 0.2$, the error is at most 0.02; the actual error is approximately 0.01768.

Question 7

Derive one forward Euler step for $y' = f(t, y)$ from Taylor's theorem. Identify the discarded term. Then apply one step with $h = 0.1$ to $y' = y$, $y(0) = 1$, and bound its local error.

Question 7 – Solution**Step 1: Expand the exact solution in time.**Taylor's theorem about t_n gives

$$y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(\xi_n)$$

for some ξ_n between t_n and $t_n + h$.**Step 2: Replace the derivative using the ODE.**Since $y'(t_n) = f(t_n, y(t_n))$, dropping the quadratic remainder produces Euler's update

$$\boxed{y_{n+1} = y_n + hf(t_n, y_n)}.$$

The one-step local truncation error is $h^2y''(\xi_n)/2$, hence is order h^2 .**Step 3: Apply the method.**For $y' = y$, $y_0 = 1$, and $h = 0.1$,

$$y_1 = 1 + 0.1(1) = \boxed{1.1}.$$

The exact value is $e^{0.1}$. On $[0, 0.1]$, $y''(t) = e^t \leq e^{0.1}$, so

$$|e^{0.1} - 1.1| \leq \frac{e^{0.1}(0.1)^2}{2} < 0.00553.$$

Question 8

The small-angle model replaces $\sin \theta$ by θ when θ is measured in radians. At $\theta = 0.2$, compute the relative error $|\theta - \sin \theta|/|\sin \theta|$ and explain it using the sine series.

Question 8 – Solution

Step 1: Use the local series.

$$\sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots.$$

Thus the leading absolute error in replacing $\sin \theta$ by θ is approximately $\theta^3/6$, and the leading relative error is approximately $\theta^2/6$.

Step 2: Evaluate the exact relative error.

At $\theta = 0.2$ radians,

$$\sin(0.2) \approx 0.1986693308, \quad 0.2 - \sin(0.2) \approx 0.0013306692.$$

Therefore

$$\frac{0.2 - \sin(0.2)}{\sin(0.2)} \approx 0.0066979 = \boxed{0.6698\%}.$$

Step 3: Compare with the leading prediction.

$$\frac{\theta^2}{6} = \frac{0.04}{6} \approx 0.0066667 = 0.6667\%,$$

which is close to the exact relative error. This quantifies why the approximation is useful but not exact.

Question 9

A principal P earns annual rate r compounded n times per year, so $A_n = P(1 + r/n)^n$. Use the series for $\log(1 + u)$ to derive the continuous-compounding limit as $n \rightarrow \infty$.

Question 9 – Solution**Step 1: Take a logarithm.**For sufficiently large n , $1 + r/n > 0$, and

$$\log\left(\frac{A_n}{P}\right) = n \log\left(1 + \frac{r}{n}\right).$$

Step 2: Insert the logarithmic expansion.

$$\begin{aligned} n \log\left(1 + \frac{r}{n}\right) &= n \left(\frac{r}{n} - \frac{r^2}{2n^2} + \frac{r^3}{3n^3} + O(n^{-4}) \right) \\ &= r - \frac{r^2}{2n} + \frac{r^3}{3n^2} + O(n^{-3}) \longrightarrow r. \end{aligned}$$

Step 3: Exponentiate.

Continuity of the exponential function gives

$$\frac{A_n}{P} \longrightarrow e^r, \quad \boxed{A_n \longrightarrow P e^r}.$$

The correction $-r^2/(2n)$ also explains the approach rate. For positive r , discrete compounding approaches the continuous value from below.

Question 10

Construct the quadratic Maclaurin model for the Gaussian e^{-x^2} . Determine where that polynomial is positive, and give an error bound for $|x| \leq 1$ using the first omitted term.

Question 10 – Solution

Step 1: Substitute into the exponential series.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots.$$

Keeping terms through degree 2 gives

$$\boxed{Q_2(x) = 1 - x^2}.$$

Step 2: Analyze the model's sign.

$$Q_2(x) > 0 \iff 1 - x^2 > 0 \iff |x| < 1.$$

It vanishes at $x = \pm 1$ and becomes negative for $|x| > 1$, whereas the true Gaussian is positive for every real x . This shows the model is local.

Step 3: Bound the local error.

For $|x| \leq 1$, the series in x^2 alternates with decreasing magnitudes. Therefore

$$\boxed{|e^{-x^2} - (1 - x^2)| \leq \frac{x^4}{2}}.$$

The error is order x^4 near the origin.