

Question 1

Construct the Maclaurin series for e^x from its derivatives. Prove that it converges to e^x for every real x , and give a uniform Lagrange-remainder bound for the degree- N polynomial on $|x| \leq 1$.

Question 1 – Solution

Step 1: Compute the coefficients.

Every derivative of $f(x) = e^x$ is e^x , so $f^{(n)}(0) = 1$. Hence

$$P_N(x) = \sum_{n=0}^N \frac{x^n}{n!}, \quad \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}.$$

Step 2: Bound the remainder.

Taylor's theorem gives

$$R_N(x) = \frac{e^c x^{N+1}}{(N+1)!}$$

for some c between 0 and x . If $|x| \leq 1$, then $c \leq 1$, so

$$\boxed{|R_N(x)| \leq \frac{e|x|^{N+1}}{(N+1)!} \leq \frac{e}{(N+1)!}}.$$

Step 3: Justify equality with the function.

For any fixed real x , $e^c \leq e^{|x|}$ and $e^{|x|}|x|^{N+1}/(N+1)! \rightarrow 0$ by the Ratio Test. Thus $R_N(x) \rightarrow 0$ for every x , proving the series represents e^x on all of $\mathbb{R}$.

Question 2

Derive the Maclaurin series for $\sin x$ from its derivative cycle. Explain why only odd powers occur, give the degree-7 polynomial, and state the convergence domain.

Question 2 – Solution

Step 1: List the derivative cycle.

$$\sin x, \quad \cos x, \quad -\sin x, \quad -\cos x, \quad \sin x, \dots$$

At $x = 0$, the values are $0, 1, 0, -1, 0, \dots$. All even-order derivatives vanish, while the odd derivatives alternate between 1 and -1 .

Step 2: Form the series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

The requested polynomial is

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

Step 3: Establish convergence.

The absolute-value ratio of consecutive displayed terms is

$$\frac{x^2}{(2n+2)(2n+3)} \rightarrow 0$$

for every fixed x . Taylor's remainder is bounded by $|x|^{m+1}/(m+1)!$ because all derivatives of sine have magnitude at most 1, so it tends to 0. Thus the series equals $\sin x$ for every real x .

Question 3

Derive the Maclaurin series for $\cos x$ from its derivative cycle. Explain why every odd coefficient vanishes, give the degree-6 polynomial, and state the convergence domain.

Question 3 – Solution

Step 1: Evaluate the derivative cycle at zero.

$$\cos x, \quad -\sin x, \quad -\cos x, \quad \sin x, \dots$$

gives values $1, 0, -1, 0, 1, \dots$ at 0. Therefore every odd-order coefficient is zero, and the even coefficients alternate in sign.

Step 2: Write the series and polynomial.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

Step 3: Justify convergence to cosine.

Consecutive absolute terms have ratio

$$\frac{x^2}{(2n+1)(2n+2)} \rightarrow 0,$$

so the series converges for all x . Since every derivative of cosine has magnitude at most 1, the Lagrange remainder is at most $|x|^{m+1}/(m+1)! \rightarrow 0$. Hence the series equals $\cos x$ on $(-\infty, \infty)$.

Question 4

Derive the Maclaurin series for $\log(1 + x)$ by integrating a geometric series. Give its first five nonzero terms and determine its exact interval of convergence.

Question 4 – Solution**Step 1: Expand the derivative.**For $|t| < 1$,

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n.$$

Step 2: Integrate from 0 to x .

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}}.$$

The first five nonzero terms are

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}.$$

Step 3: Determine endpoints.

The interior radius is 1. At $x = 1$, the alternating harmonic series converges to $\log 2$. At $x = -1$, the series becomes $-\sum 1/n$ and diverges.

Conclusion.The interval is $\boxed{(-1, 1]}$.

Question 5

Obtain the Maclaurin series and degree-4 Taylor polynomial for e^{2x} by substitution into the series for e^u . State the convergence domain and justify it.

Question 5 – Solution

Step 1: Start with the parent series.

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} \quad (u \in \mathbb{R}).$$

Set $u = 2x$:

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

Step 2: Extract the requested polynomial.

$$P_4(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4.$$

Step 3: Check convergence.

For $a_n = (2x)^n/n!$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2|x|}{n+1} \rightarrow 0$$

for every real x . Hence the radius is infinite and the series represents e^{2x} on all of $\mathbb{R}$.

Question 6

Use $\cosh x = (e^x + e^{-x})/2$ to derive its Maclaurin series. Explain why all odd powers cancel, give the degree-6 polynomial, and state the convergence domain.

Question 6 – Solution

Step 1: Expand both exponentials.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

When the series are added, odd-power coefficients cancel and even-power coefficients double.

Step 2: Divide by two.

$$\boxed{\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}, \quad P_6(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}.$$

Step 3: State convergence.

Both exponential series converge absolutely for every real x , so their sum and scalar multiple do also. Thus the Maclaurin series for $\cosh x$ has infinite radius and represents the function on $(-\infty, \infty)$.

Question 7

Derive the Maclaurin series for xe^x by multiplying a known series by x . Give two equivalent index forms, the first five nonzero terms, and the convergence domain.

Question 7 – Solution

Step 1: Multiply the exponential series.

$$xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}}.$$

Multiplication by x raises every exponent by one but leaves the coefficient denominator tied to the original index.

Step 2: Reindex if powers x^k are preferred.

Set $k = n + 1$:

$$xe^x = \boxed{\sum_{k=1}^{\infty} \frac{x^k}{(k-1)!}}.$$

The first five nonzero terms are

$$x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \frac{x^5}{24}.$$

Step 3: State convergence.

Multiplying an everywhere-convergent series by x does not change its infinite radius. Therefore the identity holds for every real x .

Question 8

Derive the Maclaurin series for $f(x) = 1/(1 - x)$ from Taylor coefficients, then connect it with the finite geometric-sum identity. Determine the exact interval of convergence.

Question 8 – Solution**Step 1: Compute Taylor coefficients.**

Repeated differentiation gives

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \quad f^{(n)}(0) = n!.$$

Thus the Maclaurin coefficient is $f^{(n)}(0)/n! = 1$, and

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}.$$

Step 2: Verify using finite sums.

$$S_N = 1 + x + \cdots + x^N = \frac{1 - x^{N+1}}{1 - x}.$$

If $|x| < 1$, then $x^{N+1} \rightarrow 0$, so $S_N \rightarrow 1/(1-x)$. This proves the Taylor series and geometric identity are the same construction.

Step 3: Check endpoints.

At $x = 1$ the terms are 1; at $x = -1$ they alternate between ± 1 . Both diverge by the nth-term test.

Conclusion.

$R = 1$ and the interval is $\boxed{(-1, 1)}$.

Question 9

Use the generalized binomial coefficients to find the first four nonzero terms of the Maclaurin series for $\sqrt{1+x}$. State the radius and determine convergence at both endpoints.

Question 9 – Solution

Step 1: Use the binomial pattern.

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \quad \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

With $\alpha = 1/2$,

$$\binom{1/2}{0} = 1, \quad \binom{1/2}{1} = \frac{1}{2}, \quad \binom{1/2}{2} = -\frac{1}{8}, \quad \binom{1/2}{3} = \frac{1}{16}.$$

Therefore

$$\boxed{\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots}.$$

Step 2: State radius and endpoints.

The binomial series has $R = 1$. Its coefficients have magnitude asymptotic to a constant times $n^{-3/2}$, so it converges absolutely at both $x = 1$ and $x = -1$.

Conclusion.

The interval is $\boxed{[-1, 1]}$, with sums $\sqrt{2}$ and 0 at the endpoints.

Question 10

Use complex exponentials to find the first four nonzero Maclaurin terms of $e^x \cos x$. Give a general coefficient formula and state the convergence domain.

Question 10 – Solution

Step 1: Encode the product as a real part.

$$e^x \cos x = \operatorname{Re} (e^x e^{ix}) = \operatorname{Re} (e^{(1+i)x}).$$

Step 2: Expand the complex exponential.

$$e^{(1+i)x} = \sum_{n=0}^{\infty} \frac{(1+i)^n x^n}{n!},$$

so the real coefficient of x^n is $\operatorname{Re}((1+i)^n)/n!$. Since $1+i = \sqrt{2}e^{i\pi/4}$,

$$e^x \cos x = \sum_{n=0}^{\infty} \frac{2^{n/2} \cos(n\pi/4)}{n!} x^n.$$

Step 3: Extract nonzero terms.

The x^2 coefficient vanishes. The first four nonzero terms are

$$\boxed{1 + x - \frac{x^3}{3} - \frac{x^4}{6}} \quad \left(\text{next term } -\frac{x^5}{30} \right).$$

The factorial denominator gives infinite radius, so the series represents the function for every real x .