

Question 1

Starting from the geometric-series formula, find a power-series representation centered at 0 for $f(x) = 1/(1 - x)$. State its radius and interval of convergence, including endpoint tests.

Question 1 – Solution

Step 1: Use the geometric prototype.

For a geometric series,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad (|r| < 1).$$

Set $r = x$ to obtain

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n}.$$

Step 2: Carry over the convergence condition.

The requirement $|r| < 1$ becomes $|x| < 1$, so the radius is $R = 1$.

Step 3: Test the endpoints.

At $x = 1$, the series is $\sum 1$; at $x = -1$, it is $\sum (-1)^n$. In both cases the terms fail to approach zero.

Conclusion.

The interval is $\boxed{(-1, 1)}$. The identity holds only there, even though the rational function is defined at some points outside that interval.

Question 2

Obtain a power series centered at 0 for $f(x) = 1/(1+x)$ by substituting into the geometric series. State its radius and interval of convergence and test both endpoints directly.

Question 2 – Solution

Step 1: Make the sign substitution.

Replace x by $-x$ in $1/(1-x) = \sum x^n$:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Step 2: Translate the restriction.

The geometric condition is $|-x| < 1$, equivalent to $|x| < 1$, so $R = 1$.

Step 3: Test the endpoints.

- At $x = 1$, the series is $\sum (-1)^n$, which diverges by the nth-term test.
- At $x = -1$, it is $\sum 1$, which diverges.

Conclusion.

The interval is $\boxed{(-1, 1)}$. Although $1/(1+x)$ is finite at $x = 1$, its Taylor series about 0 does not converge there.

Question 3

Differentiate the geometric series term by term to find a power series for $1/(1-x)^2$. Reindex it in powers x^n and state its full interval of convergence.

Question 3 – Solution**Step 1: Begin inside the radius.**For $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

A power series may be differentiated term by term at every point strictly inside its radius.

Step 2: Differentiate.

$$\frac{d}{dx}(1-x)^{-1} = (1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Let $k = n - 1$:

$$\boxed{\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k}.$$

Step 3: Check the boundary.Differentiation preserves the radius $R = 1$, but endpoints must be retested. At $x = 1$ the terms are $k + 1$; at $x = -1$ their magnitudes are $k + 1$. Neither tends to zero.**Conclusion.**The interval is $\boxed{(-1, 1)}$.

Question 4

Derive a power series for $\log(1 + x)$ by integrating a geometric series. Determine the constant of integration and the exact interval of convergence.

Question 4 – Solution**Step 1: Expand the derivative.**For $|t| < 1$,

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n.$$

Step 2: Integrate from 0 to x .Term-by-term integration is valid when $|x| < 1$:

$$\begin{aligned} \log(1+x) - \log 1 &= \int_0^x \frac{dt}{1+t} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}. \end{aligned}$$

Thus

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}.$$

The lower limit makes the constant 0.

Step 3: Test endpoints.At $x = 1$, the alternating harmonic series converges to $\log 2$. At $x = -1$, the series is $-\sum 1/n$ and diverges.**Conclusion.** $R = 1$ and the interval is $\boxed{(-1, 1]}$.

Question 5

Derive the Maclaurin series for $\arctan x$ by integrating the series for $1/(1+x^2)$. Explain the odd powers and determine the endpoint behavior.

Question 5 – Solution**Step 1: Expand the derivative.**With geometric ratio $-t^2$,

$$\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}, \quad |t| < 1.$$

Step 2: Integrate from 0 to x .

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}, \quad |x| < 1.$$

Integrating the even powers t^{2n} produces only odd powers x^{2n+1} ; the value at 0 fixes the constant.**Step 3: Test endpoints.**At $x = 1$, the alternating series converges to $\pi/4$. At $x = -1$, it converges to $-\pi/4$. Both are conditional.**Conclusion.**The radius is $R = 1$ and the interval is $\boxed{[-1, 1]}$.

Question 6

Use the differentiated geometric identity to find a power series for $x/(1-x)^2$. State the index, radius, and exact interval of convergence.

Question 6 – Solution

Step 1: Differentiate the geometric series.

For $|x| < 1$,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}.$$

Step 2: Multiply by x .

$$\boxed{\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n}.$$

Multiplication by x shifts every exponent upward by one but does not change the radius.

Step 3: Test the boundary.

The radius remains $R = 1$. At $x = 1$, the terms are n ; at $x = -1$, they are $n(-1)^n$. Neither term sequence approaches zero.

Conclusion.

The interval of convergence is $\boxed{(-1, 1)}$.

Question 7

Find a Maclaurin series for $1/(1 - x^2)$ by recognizing a geometric series. Explain why only even powers occur and determine the interval of convergence.

Question 7 – Solution**Step 1: Identify the ratio.**Apply $1/(1-r) = \sum r^n$ with $r = x^2$:

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}.$$

Only even powers occur because each power of the ratio x^2 adds two to the exponent.**Step 2: Translate the geometric restriction.**

$$|x^2| < 1 \iff |x| < 1.$$

Hence $R = 1$.**Step 3: Test the endpoints.**At both $x = 1$ and $x = -1$, every term x^{2n} equals 1. Thus the series diverges at both endpoints.**Conclusion.**The interval is $\boxed{(-1, 1)}$.

Question 8

Find a power series for $F(x) = \int_0^x \frac{dt}{1-t^4}$. Justify term-by-term integration, then determine the radius and exact interval of convergence of the resulting series.

Question 8 – Solution**Step 1: Expand the integrand.**For $|t| < 1$,

$$\frac{1}{1-t^4} = \sum_{n=0}^{\infty} t^{4n}.$$

On every closed subinterval inside $(-1, 1)$, this power series converges uniformly, so it may be integrated term by term.

Step 2: Integrate.

$$F(x) = \sum_{n=0}^{\infty} \int_0^x t^{4n} dt = \boxed{\sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1}}.$$

Step 3: Test the endpoints.

Integration preserves radius $R = 1$. At $x = 1$, the series is $\sum 1/(4n+1)$, which diverges by Limit Comparison with $1/n$. At $x = -1$, it is the negative of the same divergent positive series.

Conclusion.The interval is $\boxed{(-1, 1)}$.

Question 9

Find a Maclaurin series for $1/(2 - x)$ by first factoring the denominator. State the radius and interval of convergence, including endpoint tests.

Question 9 – Solution

Step 1: Put the function in geometric form.

$$\frac{1}{2-x} = \frac{1}{2} \frac{1}{1-x/2}.$$

Using the geometric identity with $r = x/2$ gives

$$\frac{1}{2-x} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}.$$

Step 2: Translate the restriction.

The condition $|x/2| < 1$ becomes $|x| < 2$, so $R = 2$.

Step 3: Test endpoints.

At $x = 2$, each term equals $1/2$. At $x = -2$, the terms alternate between $\pm 1/2$. Neither tends to zero, so both series diverge.

Conclusion.

The interval is $\boxed{(-2, 2)}$.

Question 10

Find a Maclaurin series for $x^2/(1 + x^3)$. Identify the geometric ratio, explain the gaps between exponents, and determine the interval of convergence.

Question 10 – Solution**Step 1: Expand the reciprocal.**Use the geometric ratio $r = -x^3$:

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-1)^n x^{3n}, \quad |x^3| < 1.$$

Step 2: Multiply by x^2 .

$$\boxed{\frac{x^2}{1+x^3} = \sum_{n=0}^{\infty} (-1)^n x^{3n+2}}.$$

The exponents are 2, 5, 8, ... because multiplying by each additional ratio contributes three powers of x .

Step 3: Determine convergence.

The interior condition is $|x| < 1$, so $R = 1$. At $x = 1$, terms are $(-1)^n$; at $x = -1$, every term is 1. Neither tends to zero.

Conclusion.

The interval is $\boxed{(-1, 1)}$.