

Question 1

Find the center, radius of convergence, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n}.$$

Test $x = -3$ and $x = 3$ separately and identify each resulting endpoint series.

Question 1 – Solution

Step 1: Recognize the geometric series.

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n.$$

It converges exactly when $|x/3| < 1$, or $|x| < 3$. Thus the center is 0 and $R = 3$.

Step 2: Test the endpoints.

- At $x = 3$, the series is $\sum 1$, which diverges.
- At $x = -3$, the series is $\sum (-1)^n$. Its terms do not approach 0, so it diverges.

Conclusion.

The interval is $\boxed{(-3, 3)}$. Neither endpoint is included.

Question 2

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{nx^n}{4^n}$. Explain why the polynomial factor n does not change the radius but does determine the endpoint behavior.

Question 2 – Solution**Step 1: Use the Ratio Test.**For $a_n = nx^n/4^n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{n} \frac{|x|}{4} \longrightarrow \frac{|x|}{4}.$$

Convergence requires $|x|/4 < 1$, so $|x| < 4$ and $R = 4$.**Step 2: Test the endpoints.**

- At $x = 4$, the terms are n , which do not approach 0.
- At $x = -4$, the terms are $n(-1)^n$, whose magnitudes grow.

Both endpoint series diverge by the nth-term test.

Conclusion.

The interval is $\boxed{(-4, 4)}$. The factor $(n+1)/n \rightarrow 1$, so n does not alter the radius; at the endpoints it prevents the terms from tending to zero.

Question 3

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} x^n/n$. Identify the harmonic or alternating harmonic series arising at each endpoint.

Question 3 – Solution

Step 1: Find the interior interval.

$$\left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| = |x| \frac{n}{n+1} \longrightarrow |x|.$$

The Ratio Test gives convergence for $|x| < 1$ and divergence for $|x| > 1$, so $R = 1$.

Step 2: Test $x = 1$.

The series becomes $\sum 1/n$, the divergent harmonic series.

Step 3: Test $x = -1$.

The series becomes $\sum (-1)^n/n$, which converges by the Alternating Series Test (but not absolutely).

Conclusion.

The interval is $\boxed{[-1, 1)}$. Convergence at the left endpoint is conditional.

Question 4

Find the radius and interval of convergence of $\sum_{n=1}^{\infty} x^n/n^2$. Explain why its radius matches Question 3 while both endpoints now converge.

Question 4 – Solution

Step 1: Find the radius.

$$\left| \frac{x^{n+1}/(n+1)^2}{x^n/n^2} \right| = |x| \left(\frac{n}{n+1} \right)^2 \rightarrow |x|.$$

Thus the interior condition is $|x| < 1$ and $R = 1$.

Step 2: Test the endpoints.

- At $x = 1$, the series is $\sum 1/n^2$, which converges.
- At $x = -1$, the absolute-value series is again $\sum 1/n^2$, so the series converges absolutely.

Conclusion.

The interval is $\boxed{[-1, 1]}$. Both this series and $\sum x^n/n$ have coefficient roots tending to 1, hence the same radius, but the stronger n^{-2} decay makes both boundary series absolutely convergent.

Question 5

Determine the radius and set of convergence of $\sum_{n=0}^{\infty} n!x^n$. Use the Ratio Test to show what happens for every nonzero x .

Question 5 – Solution

Step 1: Fix $x \neq 0$ and form the ratio.

With $a_n = n!x^n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = (n+1)|x| \rightarrow \infty.$$

The Ratio Test therefore gives divergence for every fixed nonzero x . Indeed, the terms eventually increase in magnitude and cannot tend to zero.

Step 2: Check $x = 0$.

At $x = 0$, the series has only its $n = 0$ term nonzero (using $0^0 = 1$ in the power-series convention), so it converges to 1.

Conclusion.

The radius is $\boxed{R = 0}$ and the set of convergence is the single point $\boxed{\{0\}}$. This is not a nondegenerate interval.

Question 6

Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} x^n/n!$. Explain why the factorial denominator allows every real x .

Question 6 – Solution

Step 1: Apply the Ratio Test for a fixed x .

$$\left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \frac{|x|}{n+1} \rightarrow 0.$$

For every fixed real x , the limit is $0 < 1$.

Step 2: State the convergence set.

The series converges absolutely for all $x \in \mathbb{R}$. Thus

$$\boxed{R = \infty}, \quad \boxed{(-\infty, \infty)}.$$

There are no finite endpoints to test.

Step 3: Interpret the factorial.

Each ratio contains the growing divisor $n+1$, which eventually dominates any fixed $|x|$. The series is the familiar expansion $e^x = \sum x^n/n!$.

Question 7

Find the center, radius, and interval of convergence of $\sum_{n=0}^{\infty} (x-2)^n/5^n$. Verify the behavior at both physical endpoints.

Question 7 – Solution

Step 1: Read the geometric structure.

$$\sum_{n=0}^{\infty} \left(\frac{x-2}{5} \right)^n$$

has center $c = 2$ and converges when

$$\left| \frac{x-2}{5} \right| < 1 \iff |x-2| < 5 \iff -3 < x < 7.$$

Thus $R = 5$.

Step 2: Test the endpoints.

- At $x = 7$, the series is $\sum 1$, which diverges.
- At $x = -3$, the series is $\sum (-1)^n$, which also diverges because its terms do not tend to zero.

Conclusion.

The interval is $\boxed{(-3, 7)}$, symmetric about the center 2.

Question 8

Find the center, radius, and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}.$$

Track the center carefully and determine which physical endpoint produces the alternating harmonic series.

Question 8 – Solution**Step 1: Find the interior condition.**

The series is centered at $c = -1$. For $a_n = (-1)^n(x+1)^n/n$,

$$\left| \frac{a_{n+1}}{a_n} \right| = |x+1| \frac{n}{n+1} \longrightarrow |x+1|.$$

Thus $|x+1| < 1$, so $-2 < x < 0$ and $R = 1$.

Step 2: Test $x = -2$.

Here $x+1 = -1$, so

$$\frac{(-1)^n(-1)^n}{n} = \frac{1}{n}.$$

The harmonic series diverges; $x = -2$ is excluded.

Step 3: Test $x = 0$.

Here $x+1 = 1$, producing $\sum (-1)^n/n$, which converges conditionally.

Conclusion.

The interval is $\boxed{(-2, 0]}$.

Question 9

Find the center, radius, and interval of convergence of $\sum_{n=1}^{\infty} \frac{n^2(x-3)^n}{2^n}$. Explain the different roles of the exponential and polynomial factors.

Question 9 – Solution**Step 1: Apply the Ratio Test.**

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(\frac{n+1}{n} \right)^2 \frac{|x-3|}{2} \rightarrow \frac{|x-3|}{2}.$$

Thus $|x-3| < 2$, giving center $c = 3$, radius $R = 2$, and preliminary interval $1 < x < 5$.

Step 2: Test the endpoints.

- At $x = 5$, the terms reduce to n^2 , so the series diverges.
- At $x = 1$, the terms reduce to $n^2(-1)^n$, whose magnitudes do not approach zero.

Conclusion.

The interval is $\boxed{(1, 5)}$. The exponential coefficient 2^{-n} fixes the radius; the polynomial n^2 has root tending to 1 but makes both boundary terms fail the nth-term test.

Question 10

Find the interval of convergence and sum function of $\sum_{n=0}^{\infty} (x^2)^n$. Treat it as a geometric series in x^2 and test $x = \pm 1$.

Question 10 – Solution**Step 1: Apply the geometric criterion.**

The common ratio is $r = x^2$. Therefore

$$|r| < 1 \iff |x^2| < 1 \iff x^2 < 1 \iff |x| < 1.$$

Step 2: Test the endpoints.

At both $x = 1$ and $x = -1$, we have $x^2 = 1$, so the series becomes $\sum 1$ and diverges.

Step 3: Find the sum.

For $|x| < 1$, the geometric formula gives

$$\sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1 - x^2}.$$

Conclusion.

The interval of convergence is $\boxed{(-1, 1)}$. As a power series in x , only even powers occur, but its center is still 0 and its radius is 1.