

**Question 1**

Let  $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$  and  $S_N = \sum_{n=1}^N (-1)^{n-1}/n$ .

- (a) State the Alternating Series Estimation Theorem for  $R_N = S - S_N$ .
- (b) Find the least final index  $N$  for which the next-term bound  $1/(N+1)$  is strictly below  $10^{-3}$ .

**Question 1 – Solution**

**Step 1: Verify the theorem applies.**

The magnitudes  $b_n = 1/n$  decrease to 0, so the alternating series converges and

$$|R_N| \leq b_{N+1} = \frac{1}{N+1}.$$

**Step 2: Solve the strict inequality.**

$$\frac{1}{N+1} < 10^{-3} \iff N+1 > 1000 \iff N > 999.$$

The least integer is  $\boxed{N = 1000}$ .

**Step 3: Check minimality.**

For  $N = 999$ , the displayed non-strict bound gives  $|R_{999}| \leq 1/1000 = 10^{-3}$ , so this bound is not strictly below the tolerance. (The actual remainder is strictly smaller than the next term.) For  $N = 1000$ ,  $|R_{1000}| \leq 1/1001 < 10^{-3}$ .

**Conclusion.**

Use the first 1000 terms, ending at index 1000.

**Question 2**

For  $S = \sum_{n=1}^{\infty} 1/n^2$ , let  $S_N = \sum_{n=1}^N 1/n^2$  and  $R_N = S - S_N$ . Use both sides of the Integral Test remainder estimate to bracket  $R_N$ , then bracket  $S$  in terms of  $S_N$ .

**Question 2 – Solution**

**Step 1: State the two-sided estimate.**

For a positive decreasing function  $f$  with  $a_n = f(n)$ ,

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx.$$

Here  $f(x) = x^{-2}$  is positive, continuous, and decreasing.

**Step 2: Evaluate both integrals.**

$$\int_{N+1}^{\infty} \frac{dx}{x^2} = \frac{1}{N+1}, \quad \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N}.$$

Thus

$$\boxed{\frac{1}{N+1} \leq R_N \leq \frac{1}{N}}.$$

**Step 3: Convert the tail bracket to a sum bracket.**

Since  $S = S_N + R_N$ ,

$$\boxed{S_N + \frac{1}{N+1} \leq S \leq S_N + \frac{1}{N}}.$$

The bracket width is  $1/[N(N+1)]$ .

**Question 3**

Let  $S = \sum_{n=1}^{\infty} 1/n^3$ . Using the Integral Test upper remainder bound, find the least final index  $N$  certified to give absolute error  $|R_N| < \frac{1}{2}10^{-4}$ . Explain whether this tolerance alone guarantees identical rounding to four decimal places.

**Question 3 – Solution**

**Step 1: Obtain an upper tail bound.**

With  $f(x) = x^{-3}$ ,

$$0 < R_N \leq \int_N^{\infty} x^{-3} dx = \left[ -\frac{1}{2x^2} \right]_N^{\infty} = \frac{1}{2N^2}.$$

**Step 2: Impose the absolute-error tolerance.**

The requested error is below  $0.00005 = 5 \times 10^{-5}$ :

$$\frac{1}{2N^2} < 5 \times 10^{-5} \iff N^2 > 10,000 \iff N > 100.$$

Thus the least integer certified by this bound is  $\boxed{N = 101}$ .

**Step 3: Check the strict endpoint.**

At  $N = 100$ , the bound equals  $5 \times 10^{-5}$ , so it does not guarantee a strict error smaller than half a unit in the fourth decimal place. At  $N = 101$ , it does.

This absolute-error tolerance alone does not guarantee identical rounding: two nearby numbers can lie on opposite sides of a rounding threshold.

**Question 4**

Approximate  $e^{-1} = \sum_{n=0}^{\infty} (-1)^n/n!$  using  $S_N = \sum_{n=0}^N (-1)^n/n!$ . Find the least final index  $N$  that guarantees correct rounding through six decimal places.

**Question 4 – Solution****Step 1: Use the alternating remainder bound.**

The magnitudes  $1/n!$  decrease to 0, so

$$|R_N| \leq \frac{1}{(N+1)!}.$$

The tolerance  $|R_N| < 5 \times 10^{-7}$  suggests a candidate; correct rounding also requires a bracket within a single rounding interval.

**Step 2: Find the first qualifying factorial.**

$$9! = 362,880, \quad 10! = 3,628,800.$$

Hence

$$\frac{1}{9!} \approx 2.76 \times 10^{-6} > 5 \times 10^{-7}, \quad \frac{1}{10!} \approx 2.76 \times 10^{-7} < 5 \times 10^{-7}.$$

Therefore  $N+1=10$ , so  $\boxed{N=9}$ .

**Step 3: Certify the rounding.**

The alternating sign gives

$$0.3678791887 < S < S_9 + 1/10! < 0.3678794643.$$

Both endpoints lie in  $[0.3678785, 0.3678795)$  and round to 0.367879. Also  $S_8 \approx 0.3678819444$  and  $S_7 \approx 0.3678571429$ ; the earlier even and odd partial sums lie still farther away. Hence no smaller final index gives this rounded value.

**Conclusion.**

Ten terms (indices 0 through 9) suffice. They give  $S_9 \approx 0.3678791887$ , which rounds to 0.367879, the same six-decimal value as  $e^{-1}$ .

**Question 5**

For  $S = \sum_{n=0}^{\infty} (1/3)^n$ , define  $S_N = \sum_{n=0}^N (1/3)^n$ .

- (a) Compute the tail  $R_N = S - S_N$  exactly.
- (b) Compare it with the standard geometric-tail estimate based on the first omitted term.

**Question 5 – Solution**

**Step 1: Identify the first omitted term.**

Because  $S_N$  ends at index  $N$ , the tail begins at  $n = N + 1$ :

$$R_N = \sum_{n=N+1}^{\infty} \left(\frac{1}{3}\right)^n.$$

**Step 2: Sum the geometric tail.**

$$R_N = \frac{(1/3)^{N+1}}{1 - 1/3} = \frac{1}{2 \cdot 3^N}.$$

Thus the exact error is  $\boxed{R_N = 3^{-N}/2}$ .

**Step 3: Compare with the general bound.**

For a geometric tail with ratio magnitude  $r < 1$ ,

$$|R_N| \leq \frac{|a_{N+1}|}{1 - r}.$$

Here  $a_{N+1} = 3^{-(N+1)}$  and  $r = 1/3$ , producing  $1/(2 \cdot 3^N)$ . Because every term is positive and the ratio is exactly constant, the usual “bound” equals the exact tail.

**Question 6**

The Leibniz series is  $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ . If exactly  $M$  terms are used (indices 0 through  $M-1$ ), find the least  $M$  certified by a next-term bound strictly below  $10^{-4}$ .

**Question 6 – Solution**

**Step 1: Identify the first omitted term.**

With  $M$  terms, the final included index is  $M - 1$ , so the first omitted term has index  $M$  and magnitude  $1/(2M + 1)$ .

**Step 2: Apply the alternating estimate.**

$$|R| \leq \frac{1}{2M + 1}.$$

We need

$$\frac{1}{2M + 1} < 10^{-4} \iff 2M + 1 > 10,000 \iff M > 4999.5.$$

Thus the least integer is  $\boxed{M = 5000}$ .

**Step 3: Verify minimality.**

With 4999 terms the bound is  $1/9999 > 10^{-4}$ ; with 5000 terms it is  $1/10001 < 10^{-4}$ . This large count illustrates the slow convergence of the Leibniz series.

**Question 7**

For  $S = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ , derive a two-sided Integral Test estimate for  $R_N = S - S_N$ . Then determine what the upper bound requires to guarantee  $R_N < 0.01$  and interpret the result.

**Question 7 – Solution****Step 1: Verify the setup.**

The function  $f(x) = 1/[x(\log x)^2]$  is positive and decreasing for  $x \geq 2$ .

**Step 2: Integrate the tail.**

With  $u = \log x$ ,

$$\int_A^\infty \frac{dx}{x(\log x)^2} = \int_{\log A}^\infty u^{-2} du = \frac{1}{\log A}.$$

Therefore

$$\boxed{\frac{1}{\log(N+1)} \leq R_N \leq \frac{1}{\log N}}.$$

**Step 3: Solve the requested tolerance.**

$$\frac{1}{\log N} < 0.01 \iff \log N > 100 \iff N > e^{100}.$$

Thus the least certified integer is  $\lfloor e^{100} \rfloor + 1$ , an astronomically large index (about  $2.69 \times 10^{43}$ ).

**Interpretation.**

The lower and upper bounds are asymptotically equal, so this is not merely a loose estimate: the series itself converges extraordinarily slowly.

**Question 8**

Let  $S = \sum_{n=1}^{\infty} ne^{-n}$  and  $R_N = \sum_{n=N+1}^{\infty} ne^{-n}$ . Use the Integral Test remainder theorem to give a two-sided bound for  $R_N$ .

**Question 8 – Solution****Step 1: Check monotonicity.**For  $f(x) = xe^{-x}$ ,

$$f'(x) = e^{-x}(1 - x) \leq 0 \quad (x \geq 1).$$

Thus  $f$  is positive, continuous, and decreasing on the needed interval.**Step 2: Find the antiderivative.**

Integration by parts gives

$$\int_A^\infty xe^{-x} dx = (A + 1)e^{-A}.$$

**Step 3: Apply both remainder bounds.**

$$\int_{N+1}^\infty xe^{-x} dx \leq R_N \leq \int_N^\infty xe^{-x} dx,$$

so

$$\boxed{(N + 2)e^{-(N+1)} \leq R_N \leq (N + 1)e^{-N}}.$$

The upper bound is a decreasing integral majorant for the full omitted tail; no extra first-term adjustment is needed under this standard formulation.

**Question 9**

Let  $S = \sum_{n=1}^{\infty} (-1)^n/n^2$  and  $S_N = \sum_{n=1}^N (-1)^n/n^2$ .

- (a) Give the alternating-series error bound.
- (b) Compare it with an Integral Test bound for the tail of  $\sum 1/n^2$ .
- (c) Evaluate both bounds at  $N = 100$ .

**Question 9 – Solution****Step 1: Use cancellation.**

Since  $1/n^2$  decreases to 0, the alternating estimate gives

$$|S - S_N| \leq \frac{1}{(N+1)^2}.$$

**Step 2: Ignore signs and bound the absolute tail.**

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N}.$$

This also bounds the signed error, but it discards cancellation and is therefore much weaker.

**Step 3: Compare numerically.**

At  $N = 100$ ,

$$\frac{1}{101^2} = \frac{1}{10201} \approx 9.80 \times 10^{-5}, \quad \frac{1}{100} = 10^{-2}.$$

The alternating bound is over 100 times smaller. In general, it is order  $N^{-2}$  rather than  $N^{-1}$ .

**Question 10**

For  $S = \sum_{n=1}^{\infty} 2^{-n^2}$  and  $S_N = \sum_{n=1}^N 2^{-n^2}$ , bound  $R_N = S - S_N$  by a geometric series. Clearly identify its first term and common ratio.

**Question 10 – Solution****Step 1: Reindex the tail.**

Write  $n = N + 1 + k$ , where  $k \geq 0$ . Then

$$(N + 1 + k)^2 = (N + 1)^2 + k(2N + 2) + k^2 \geq (N + 1)^2 + k(2N + 3),$$

because  $k^2 \geq k$  for integers  $k \geq 0$ .

**Step 2: Compare term by term.**

$$2^{-(N+1+k)^2} \leq 2^{-(N+1)^2} \left(2^{-(2N+3)}\right)^k.$$

The comparison series has first term  $2^{-(N+1)^2}$  and ratio  $q = 2^{-(2N+3)}$ .

**Step 3: Sum the geometric majorant.**

$$0 < R_N \leq \frac{2^{-(N+1)^2}}{1 - 2^{-(2N+3)}}.$$

The bound is very close to the first omitted term because  $q$  becomes extremely small.