

Question 1

Consider $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

- (a) Choose an efficient convergence test and justify the choice.
- (b) Determine whether the series converges.
- (c) If possible, use a known power-series identity to find its exact sum.

Question 1 – Solution

Step 1: Prove convergence efficiently.

For $a_n = n/3^n$,

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3n} \longrightarrow \frac{1}{3} < 1.$$

The Ratio Test is quick and proves convergence.

Step 2: Choose a more informative method for the sum.

Begin with

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Differentiate and multiply by x :

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

At $x = 1/3$,

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \boxed{\frac{3}{4}}.$$

Strategy note.

The Ratio Test classifies the series; generating-function recognition gives both convergence and the exact value.

Question 2

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converges or diverges. Explain why the Integral Test is especially well suited to this expression.

Question 2 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = 1/(x \log x)$ for $x \geq 2$. It is positive and continuous. Also,

$$f'(x) = -\frac{\log x + 1}{x^2(\log x)^2} < 0,$$

so f is decreasing.

Step 2: Evaluate the improper integral.

The integrand contains the differential $dx/x = d(\log x)$, suggesting $u = \log x$:

$$\int_2^b \frac{dx}{x \log x} = \int_{\log 2}^{\log b} \frac{du}{u} = \log(\log b) - \log(\log 2).$$

As $b \rightarrow \infty$, this tends to ∞ .

Conclusion.

The series **diverges** by the Integral Test. The substitution exposes the antiderivative $\log(\log x)$ immediately, which is why this test is more natural than a ratio or root test here.

Question 3

Classify $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ as absolutely convergent, conditionally convergent, or divergent. Use separate tests for ordinary and absolute convergence.

Question 3 – Solution

Step 1: Test the alternating series.

Put $b_n = 1/\sqrt{n}$. Then $b_n > 0$, $b_{n+1} < b_n$, and

$$\lim_{n \rightarrow \infty} b_n = 0.$$

Therefore $\sum (-1)^n b_n$ converges by the Alternating Series Test.

Step 2: Test absolute convergence.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}.$$

This is a divergent p -series because $p = 1/2 \leq 1$.

Conclusion.

The original series is **conditionally convergent**. One test establishes convergence; the second shows that convergence depends essentially on cancellation between alternating signs.

Question 4

Analyze $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 4^n}$.

- (a) Show why the Ratio Test is inconclusive.
- (b) Use a standard central-binomial-coefficient asymptotic to reach a conclusion.

Question 4 – Solution

Step 1: Try the natural factorial test.

If $a_n = (2n)!/[(n!)^2 4^n]$, then

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{4(n+1)^2} = \frac{2n+1}{2n+2} \rightarrow 1.$$

The Ratio Test is therefore inconclusive.

Step 2: Replace it with asymptotic comparison.

Stirling's formula yields the standard estimate

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

Consequently,

$$a_n = \frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

Equivalently, $a_n/(1/\sqrt{n}) \rightarrow 1/\sqrt{\pi} > 0$.

Conclusion.

Since $\sum n^{-1/2}$ diverges, the given series **diverges** by Limit Comparison. A ratio limit of 1 signaled that a finer growth estimate was needed.

Question 5

Determine whether $\sum_{n=1}^{\infty} \frac{1}{n^2 + \cos n}$ converges. For the tail $n \geq 2$, use only the bound $-1 \leq \cos n \leq 1$; no exact trigonometric values are needed.

Question 5 – Solution

Step 1: Ensure the terms are defined and positive.

For $n = 1$, $\cos 1 > 0$ because $0 < 1 < \pi/2$. For $n \geq 2$, $n^2 + \cos n \geq n^2 - 1 > 0$.

Step 2: Build a simple upper comparison.

For $n \geq 2$,

$$n^2 + \cos n \geq n^2 - 1 \geq \frac{1}{2}n^2.$$

Taking reciprocals reverses the denominator comparison:

$$0 < \frac{1}{n^2 + \cos n} \leq \frac{2}{n^2}.$$

Step 3: Apply the Comparison Test.

The series $\sum 2/n^2$ converges because it is a constant multiple of a p -series with $p = 2 > 1$.

Conclusion.

The given series **converges**. A direct comparison is more efficient than ratio or root tests because the bounded trigonometric perturbation does not change the dominant n^2 denominator.

Question 6

Analyze $\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$.

- (a) Simplify the factorial quotient.
- (b) Decide whether telescoping or comparison is more informative.
- (c) Determine convergence and, if possible, the exact sum.

Question 6 – Solution**Step 1: Cancel factorials.**

$$\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}.$$

This already resembles a telescoping difference.

Step 2: Decompose and form partial sums.

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

Thus

$$\begin{aligned} S_N &= \sum_{n=1}^N \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \\ &= \frac{1}{2} - \frac{1}{N+2}. \end{aligned}$$

Step 3: Pass to the limit.

$$\sum_{n=1}^{\infty} \frac{n!}{(n+2)!} = \lim_{N \rightarrow \infty} S_N = \boxed{\frac{1}{2}}.$$

Strategy note.

Comparison with $1/n^2$ proves convergence, but telescoping is better because it also gives the exact sum.

Question 7

Classify $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ as absolutely convergent, conditionally convergent, or divergent. Use bounded partial sums for the sine factor.

Question 7 – Solution

Step 1: Verify bounded partial sums.

The finite identity

$$\sum_{n=1}^N \sin n = \frac{\sin(N/2) \sin((N+1)/2)}{\sin(1/2)}$$

shows that $\left| \sum_{n=1}^N \sin n \right| \leq 1/|\sin(1/2)|$.

Step 2: Apply Dirichlet's Test.

The sequence $1/n$ decreases to 0, and the partial sums of $\sin n$ are bounded. Hence $\sum \sin n/n$ converges.

Step 3: Check absolute convergence.

Since $|\sin n| \geq \sin^2 n$,

$$\sum \frac{|\sin n|}{n} \geq \sum \frac{\sin^2 n}{n} = \frac{1}{2} \sum \frac{1}{n} - \frac{1}{2} \sum \frac{\cos(2n)}{n}.$$

The cosine series converges by Dirichlet's Test while the harmonic series diverges, so the absolute-value series diverges.

Conclusion.

The series is **conditionally convergent**.

Question 8

Determine whether $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$ converges. Explain why the Root Test is the natural first choice.

Question 8 – Solution

Step 1: Recognize the growing exponent.

Let $a_n = (1 + 1/n)^{-n^2}$. Taking an n th root cancels one power of n :

$$\sqrt[n]{a_n} = \left(1 + \frac{1}{n}\right)^{-n}.$$

Step 2: Use the exponential limit.

Since $(1 + 1/n)^n \rightarrow e$,

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = e^{-1} < 1.$$

Conclusion.

The series **converges** by the Root Test. This method is especially efficient because the summand is already a varying base raised to a multiple of n ; the root exposes its limiting geometric rate directly.

Question 9

Determine whether $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$ converges. Compare the Integral Test with Cauchy Condensation and identify the shorter calculation.

Question 9 – Solution**Step 1: Use the Integral Test.**

The function $f(x) = \log x/x^2$ is positive and continuous for $x \geq 2$. Also,

$$f'(x) = \frac{1 - 2 \log x}{x^3} < 0 \quad (x \geq 2).$$

Integration by parts gives

$$\int_2^\infty \frac{\log x}{x^2} dx = \left[-\frac{\log x + 1}{x} \right]_2^\infty = \frac{\log 2 + 1}{2} < \infty.$$

Thus the series converges.

Step 2: Compare with condensation.

Because the terms decrease,

$$2^k a_{2^k} = 2^k \frac{\log(2^k)}{(2^k)^2} = \frac{k \log 2}{2^k},$$

whose series converges by the Ratio Test. Condensation also works, but the integral is slightly shorter and matches the logarithm-over-power form directly.

Question 10

Determine whether $\sum_{n=1}^{\infty} \frac{3n^2 + 1}{n^3 - 2}$ converges or diverges. Identify the dominant powers and use Limit Comparison with an appropriate benchmark series.

Question 10 – Solution**Step 1: Identify the benchmark.**

The dominant-power quotient is

$$\frac{3n^2}{n^3} = \frac{3}{n},$$

so compare with the harmonic term $b_n = 1/n$. A finite number of initial terms, including the negative $n = 1$ term, does not affect convergence.

Step 2: Compute the limit.

For $n \geq 2$ both sequences are positive, and

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{3n^2 + 1}{n^3 - 2} \cdot n \\ &= \lim_{n \rightarrow \infty} \frac{3 + 1/n^2}{1 - 2/n^3} = 3.\end{aligned}$$

The limit is finite and strictly positive.

Conclusion.

Since $\sum 1/n$ diverges, the given series **diverges** by the Limit Comparison Test. Dominant powers correctly predicted harmonic-size terms.