

Question 1

Consider

$$\sum_{n=1}^{\infty} \frac{(2n+1)^n}{5^n n^n}.$$

- (a) Rewrite the summand as a single quantity raised to the n th power.
- (b) Apply the Root Test and classify the series.

Question 1 – Solution**Step 1: Rewrite the term.**

Let

$$a_n = \frac{(2n+1)^n}{5^n n^n} = \left(\frac{2n+1}{5n} \right)^n = \left(\frac{2}{5} + \frac{1}{5n} \right)^n.$$

Step 2: Take the n th root.Since $a_n > 0$,

$$\sqrt[n]{|a_n|} = \frac{2n+1}{5n} = \frac{2}{5} + \frac{1}{5n}.$$

Therefore $L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 2/5 < 1$.**Conclusion.**

The series **converges absolutely** by the Root Test. The slowly varying $1/(5n)$ correction disappears in the limit, leaving geometric-scale decay with factor $2/5$.

Question 2

Analyze

$$\sum_{n=1}^{\infty} \left(1 + \frac{(-1)^n}{2} \right)^n.$$

- (a) Examine the even and odd terms separately.
- (b) Apply the nth-term divergence test.
- (c) Confirm the conclusion using the limsup form of the Root Test.

Question 2 – Solution**Step 1: Split by parity.**If $n = 2k$, then

$$a_{2k} = \left(1 + \frac{1}{2}\right)^{2k} = \left(\frac{3}{2}\right)^{2k} \rightarrow \infty.$$

If $n = 2k + 1$, then $a_{2k+1} = (1/2)^{2k+1} \rightarrow 0$.**Step 2: Apply the necessary condition.**

A convergent series must have $a_n \rightarrow 0$. The even subsequence is unbounded, so the full term sequence does not approach zero. Hence the series **diverges**.

Step 3: Check with roots.

$$\sqrt[n]{|a_n|} = 1 + \frac{(-1)^n}{2}, \quad \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \frac{3}{2} > 1.$$

The limsup Root Test gives the same result; an ordinary root limit does not exist.

Question 3

Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} (3^n + 2^n)^{1/n} x^n.$$

Use the Root Test for the interior and test both endpoints separately.

Question 3 – Solution

Step 1: Apply the root to the entire term.

For $a_n = (3^n + 2^n)^{1/n} x^n$,

$$\sqrt[n]{|a_n|} = (3^n + 2^n)^{1/n^2} |x|.$$

Write $3^n + 2^n = 3^n[1 + (2/3)^n]$. Then

$$(3^n + 2^n)^{1/n^2} = 3^{1/n} \left[1 + \left(\frac{2}{3} \right)^n \right]^{1/n^2} \longrightarrow 1.$$

Thus the Root Test limit is $L = |x|$: convergence holds for $|x| < 1$ and divergence for $|x| > 1$.

Step 2: Test the endpoints.

Since $(3^n + 2^n)^{1/n} \rightarrow 3$, the terms at $x = 1$ approach 3, and at $x = -1$ their magnitudes approach 3. Neither endpoint passes the nth-term test.

Conclusion.

The radius is $\boxed{R = 1}$ and the interval is $\boxed{(-1, 1)}$.

Question 4

Use the Root Test to determine whether $\sum_{n=1}^{\infty} \frac{n^5}{4^n}$ converges. Explain explicitly why the polynomial factor does not affect the final root limit.

Question 4 – Solution

Step 1: Take the n th root.

$$\sqrt[n]{\frac{n^5}{4^n}} = \frac{n^{5/n}}{4}.$$

Step 2: Evaluate the polynomial-root factor.

Take logarithms:

$$\log(n^{5/n}) = \frac{5 \log n}{n} \rightarrow 0.$$

Exponentiating gives $n^{5/n} \rightarrow e^0 = 1$.

Step 3: Apply the test.

$$L = \lim_{n \rightarrow \infty} \frac{n^{5/n}}{4} = \frac{1}{4} < 1.$$

Therefore the series **converges absolutely**. Under an n th root, any fixed power n^p becomes $n^{p/n} \rightarrow 1$, so the exponential denominator determines the result.

Question 5

Determine whether

$$\sum_{n=2}^{\infty} \left(\frac{\log n}{n} \right)^n$$

converges or diverges. Compute the Root Test limit and interpret what a limit of zero implies.

Question 5 – Solution

Step 1: Take the n th root.

Since $n \geq 2$,

$$\sqrt[n]{\left| \left(\frac{\log n}{n} \right)^n \right|} = \frac{\log n}{n}.$$

Step 2: Evaluate the limit.

By l'Hopital's Rule for the corresponding continuous quotient,

$$\lim_{x \rightarrow \infty} \frac{\log x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Hence $L = 0 < 1$.

Conclusion.

The series **converges absolutely**. A root limit of zero means that for every fixed $0 < r < 1$, the terms eventually satisfy $a_n \leq r^n$; thus the tail is eventually dominated by a geometric series with any chosen ratio r .

Question 6

Analyze $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n} \right)^n$ using the Root Test. State the root limit and the convergence classification.

Question 6 – Solution

Step 1: Take the n th root.

$$\sqrt[n]{\left|\left(\frac{n+1}{2n}\right)^n\right|} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}.$$

Step 2: Evaluate the limit.

$$L = \lim_{n \rightarrow \infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2} < 1.$$

Conclusion.

The series **converges absolutely** by the Root Test. Although the base varies with n , it approaches $1/2$, so the terms eventually have geometric-scale decay. For example, once $n \geq 5$, the root is at most $3/5$, so $a_n \leq (3/5)^n$.

Question 7

Use the Root Test to determine whether

$$\sum_{n=2}^{\infty} \left(1 - \frac{1}{\sqrt{n}}\right)^{n^2}$$

converges. Use logarithms to evaluate the limit of the n th root.

Question 7 – Solution

Step 1: Take the n th root.

$$r_n = \sqrt[n]{a_n} = \left(1 - \frac{1}{\sqrt{n}}\right)^n.$$

Step 2: Take logarithms.

Since $\log(1 - u) \leq -u$ for $0 < u < 1$,

$$\log r_n = n \log \left(1 - \frac{1}{\sqrt{n}}\right) \leq -\sqrt{n} \longrightarrow -\infty.$$

Thus $0 \leq r_n \leq e^{-\sqrt{n}} \rightarrow 0$. (The expansion $\log(1 - u) = -u - u^2/2 + O(u^3)$ gives the sharper relation $\log r_n = -\sqrt{n} - 1/2 + O(n^{-1/2})$.)

Conclusion.

The root limit is $L = 0 < 1$, so the series **converges absolutely** and does so faster than any fixed geometric rate.

Question 8

Analyze

$$\sum_{n=1}^{\infty} \left(\frac{2 + \sin n}{4} \right)^n.$$

Use the limsup form of the Root Test to handle the oscillating base, and explain why an ordinary root limit is unavailable.

Question 8 – Solution**Step 1: Take the root.**Since $1 \leq 2 + \sin n \leq 3$,

$$\sqrt[n]{|a_n|} = \frac{2 + \sin n}{4} \in \left[\frac{1}{4}, \frac{3}{4} \right].$$

This sequence oscillates rather than approaching one value.

Step 2: Compute the limsup.Integer angles modulo 2π are dense because $1/(2\pi)$ is irrational. Consequently, $\sin n$ comes arbitrarily close to 1, so

$$L = \limsup_{n \rightarrow \infty} \frac{2 + \sin n}{4} = \frac{3}{4} < 1.$$

(For convergence alone, the uniform bound $\sqrt[n]{|a_n|} \leq 3/4$ already suffices.)**Conclusion.**The series **converges absolutely** by the limsup Root Test.

Question 9

Determine whether $\sum_{n=1}^{\infty} n^{-n}$ converges or diverges. Use the Root Test and explain why the exponent n makes this test especially efficient.

Question 9 – Solution

Step 1: Rewrite the term.

We have $a_n = n^{-n} = (1/n)^n$.

Step 2: Take the n th root.

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{1}{n}\right)^n} = \frac{1}{n}.$$

Therefore

$$L = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1.$$

Conclusion.

The series **converges absolutely**. Rather than complicating the calculation, the exponent n is exactly canceled by the n th root. Moreover, for any $0 < r < 1$, eventually $1/n < r$, so $n^{-n} < r^n$ and the tail is dominated by a geometric series.

Question 10

Let $c \in \mathbb{R}$. Determine exactly which values of c make

$$\sum_{n=1}^{\infty} c^{n^2}$$

converge. Apply the Root Test where decisive and inspect the boundary cases separately.

Question 10 – Solution

Step 1: Compute the root expression.

$$\sqrt[n]{|c^{n^2}|} = |c|^n.$$

Step 2: Separate the parameter cases.

- If $|c| < 1$, then $|c|^n \rightarrow 0 < 1$, so the series converges absolutely.
- If $|c| > 1$, then $|c|^n \rightarrow \infty$, and the terms c^{n^2} do not approach zero.
- If $|c| = 1$, the Root Test gives the boundary value 1. Directly, $|c^{n^2}| = 1$, so the terms again fail to approach zero.

Conclusion.

The series converges exactly when $|c| < 1$, equivalently $-1 < c < 1$. Both endpoints diverge.